On the K-Uniform Rotund and the Fully Convex Banach Spaces

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Let X be a real Banach space. If X is uniformly convex then it is known that X is 2-uniformly rotund [5] and fully 2-convex [2]. Furthermore, if a Banach space X is either k-uniformly rotund or fully k-convex for some k, then X is reflexive. In this paper, we show that if X is strictly convex and k-uniformly rotund, then X is fully \((k + 1)\)-convex. However, there exists a superreflexive space which is fully 2-convex but is not 2-uniformly rotund and for each \(k \geq 2\), there exists a strictly convex space which is k-uniformly rotund but is not fully k-convex. Thus for each \(k \geq 2\), there exist fully k-convex Banach space which is not fully \((k - 1)\)-convex.

For \(x_1, x_2, ..., x_{k+1}\) in X, let

\[
V(x_1, x_2, ..., x_{k+1}) = \sup \left\{ \left| \begin{array}{c} 1, \ldots, 1 \\ f_1(x_1), \ldots, f_1(x_{k+1}) \\ \vdots \\ f_k(x_1), \ldots, f_k(x_{k+1}) \end{array} \right| : f_i \in X^*, \| f_i \| \leq 1, i = 1, 2, ..., k \right\}.
\]

DEFINITION. Let \(k \geq 1\) be an integer. A Banach space X is said to be k-uniformly rotund (k-UR) if for any \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\) such that for any \(x_i \in X, \| x_i \| \leq 1, \ i = 1, 2, ..., k + 1, \) with \(\| 1/(k + 1) \sum_{i=1}^{k+1} x_i \| \geq 1 - \delta(\epsilon)\) then \(V(x_1, ..., x_{k+1}) < \epsilon\).
Definition. Let \( k \geq 2 \) be an integer. A Banach space \( X \) is said to be fully \( k \)-convex (\( kR \)) if for any sequence \( \{x_i\} \) in \( X \) such that \( \lim_{n \to \infty} \|x_{n+k} - x_n\| = 1 \), then \( \{x_i\} \) is a Cauchy sequence in \( X \). \( X \) is said to be fully convex if it is fully \( k \)-convex for some \( k \geq 2 \).

Lemma 1. Let \( \{x_i\} \) be a sequence in a \( k \)-UR space \( X \) such that \( \{x_i\} \) converges weakly to an element \( x \) in \( X \). If \( \lim_{n \to \infty} \|1/(k+1) \sum_{i=1}^{k+1} x_n\| = 1 \), then \( \|x\| = 1 \) and \( \lim_{i \to \infty} \|x_i - x\| = 0 \).

Proof. Observe that \( \lim_{i \to \infty} \|x_i\| = 1 \). Since \( X \) is \( k \)-UR and \( \lim_{n \to \infty} \|1/(k+1) \sum_{i=1}^{k+1} x_n\| = 1 \), it follows that \( \lim_{n \to \infty} \|x_n + x\| = 0 \). Let \( y_i = x_i - x, i = 1, 2, \ldots \). Suppose that \( \lim_{i \to \infty} \|y_i\| \neq 0 \). By choosing a subsequence if necessary, there exists \( \varepsilon > 0 \) such that \( \|y_i\| \geq \varepsilon, i = 1, 2, \ldots \). Since \( \{y_i\} \) converges weakly to 0, again by choosing a subsequence if necessary, we may assume that \( \{y_i\} \) is a basic sequence in \( X \). Let \( K \) be the basis constant of \( \{y_i\} \) and let \( f_i \in X^* \) such that \( f_i(y_j) = \delta_{i,j}, i, j = 1, 2, \ldots \). Then \( \|f_i\| \leq 2K/e, i = 1, 2, \ldots \). Hence for any \( n_1, \ldots, n_{k+1} \), we have \( V(y_{n_1}, \ldots, y_{n_{k+1}}) \geq (e/2K)^k \). Thus

\[
(e/2K)^k \leq \lim_{n_1, \ldots, n_{k+1} \to \infty} V(y_{n_1}, \ldots, y_{n_{k+1}}) = \lim_{n_1, \ldots, n_{k+1} \to \infty} V(x_{n_1}, \ldots, x_{n_{k+1}})
\]

which is impossible. Q.E.D.

Theorem 2. Every strictly convex \( k \)-UR space is \((k + 1)R\).

Proof. Let \( \{x_i\} \) be a sequence in a \( k \)-UR space \( X \) such that \( \lim_{n \to \infty} \|1/(k+1) \sum_{i=1}^{k+1} x_n\| = 1 \). Since every \( k \)-UR space is reflexive, \( \{x_i\} \) has a weak sequential cluster point \( x \) in \( X \). By Lemma 1, it suffices to show that \( x \) is the unique weak sequential cluster point of \( \{x_i\} \). Suppose \( y \) is a weak sequential cluster point of \( \{x_i\} \). Then there exist \( n_1 < m_1 < n_2 < m_2 < \cdots \), such that \( \{x_{m_i}\} \), respectively, \( \{x_{n_i}\} \), converges weakly to \( x \) (resp. to \( y \)). By Lemma 1, \( \|x\| = \|y\| = 1 \) and \( \lim_{i \to \infty} \|x_{n_i} - x\| = \lim_{i \to \infty} \|y_{m_i} - y\| = 0 \). Since \( \lim_{n \to \infty} \|1/(k+1) \sum_{i=1}^{k+1} x_n\| = 1 \), by triangle inequality, it is easy to see that \( \lim_{n \to \infty} \|1/2(x_n + x_m)\| = 1 \). Apply Lemma 1 to \( \{1/2(x_n + x_m)\} \), we conclude that \( \|1/2(x + y)\| = 1 \). However, since \( X \) is strictly convex, it follows that \( x = y \). This completes the proof of Theorem 2. Q.E.D.

Remark. Since every \( k \)-UR space is superreflexive [5], however, there exist \( 2R \) spaces which are not superreflexive [1, 2]. Hence there exist fully convex Banach spaces which are not \( k \)-UR for any \( k \geq 1 \).

Example. There exists a \( 2R \) space \( X \) which is isomorphic to \( l_2 \) but \( X \) is not \( k \)-UR for all \( k \geq 1 \).
Let \( E = (l_2, \| \cdot \|) \), where
\[
\| x \|^2 = \| a_1 \| + (a_2^2 + a_3^2 + \cdots)^{1/2} + \left\{ \frac{(a_2^2)}{2} + \cdots + \left( \frac{(a_n^2)}{n} \right) + \cdots \right\}^2
\]
for all \( x = (a_1, a_2, \ldots) \) in \( E \). In [3], it is proved that \( E \) is 2R. Let \( X = (\Sigma \oplus E)_K \). Then \( X \) is 2R [2]. We claim that \( X \) is not \( k \)-UR for all \( k \geq 1 \). Fix an integer \( k \geq 1 \). Let \( \{ e_i \} \) be the usual unit vector basis of \( l_2 \). For \( n = 1, 2, \ldots \), let
\[
x_i^n = \left( \frac{e_1 + e_n}{2}, e_{n+1}, e_{n+2}, \ldots, e_n, 0, 0, \ldots \right),
\]
where the last nonzero vector is at the \((k+1)\)-coordinate. Then
\[
\lim_{n \to \infty} \| x_i^n \| = \sqrt{k + 1}, \quad i = 1, 2, \ldots, k + 1 \quad \text{and} \quad \lim_{n \to \infty} \| x_1^n + \cdots + x_{k+1}^n \| = (k + 1)^{3/2}.
\]
However, if \( f_1 = (e_1, 0, 0, \ldots), \ f_2 = (0, e_1, 0, \ldots), \ldots, \ f_k = (0, \ldots, 0, e_1, 0, \ldots) \), then \( V(x_1^n, x_2^n, \ldots, x_{k+1}^n) \geq 1/2^{k+1} \| f_1 \| \cdots \| f_k \| > 0 \) for all \( n = 1, 2, \ldots \). This completes the proof that \( X \) is not \( k \)-UR.

The following examples show that in some sense, Theorem 2 is the best possible result. The examples are modifications of the reflexive Banach space given by Smith [4] which is 2R but is not LUR.

**Example 2.** For each \( k \geq 2 \), there exists a strictly convex Banach space \( X_k \), isomorphic to \( l_2 \), which is \( k \)-UR but is not \( kR \).

Let \( k \geq 2 \) be an integer and let \( i_1 < i_2 < \cdots < i_k \). For each \( x = (a_1, a_2, \ldots) \) in \( l_2 \), define \( \| x \|_{i_1, \ldots, i_k} = (\sum_{j=1}^k |a_j|^2 + \sum_{i \neq i_1, \ldots, i_k} a_i^2) \). It is clear that
\[
\| x \|_{i_1, \ldots, i_k} \leq \| x \|_{i_1, \ldots, i_k, \ldots, i_k} \leq \sqrt{k} \| x \|_{i_1, \ldots, i_k} \quad \text{for all} \ x \ \text{in} \ l_2.
\]
Let \( X_{i_1, \ldots, i_k} = (l_2, \| \cdot \|_{i_1, \ldots, i_k}) \). Then \( X_{i_1, \ldots, i_k} \) is clearly isometrically isomorphic to \((l_2 \oplus l_2)_K\). Hence the spaces \( X_{i_1, \ldots, i_k} \) are \( k \)-UR but is not \((k-1)\)-UR for all \( i_1, \ldots, i_k \). Furthermore, the family \( \{ X_{i_1, \ldots, i_k} \} \) has the same module of \( k \)-rotundity, i.e., for each \( \varepsilon > 0 \) the same \( \delta(\varepsilon) \) can be used in \( X_{i_1, \ldots, i_k} \) for all \( i_1, \ldots, i_k \). For \( x \in l_2 \), let \( \| x \|_k = \sup_{i_1 < \cdots < i_k} \| x \|_{i_1, \ldots, i_k} \) and let \( E_k = (l_2, \| \cdot \|_k) \). Then \( \| x \|_{i_1, \ldots, i_k} \leq \| x \|_k \leq \sqrt{k} \| x \|_{i_1, \ldots, i_k} \) for all \( x \in X_{i_1, \ldots, i_k} \). We claim that \( E_k \) is \( k \)-UR but is not \( kR \).

To see that \( E_k \) is not \( kR \), let \( \{ x_i \} \) be the usual unit vector basis of \( l_2 \). Then it is easy to see that \( \| x_i \|_k = 1 \), \( i = 1, 2, \ldots \), \( \lim_{m \to \infty} \| 1/k \sum_{j=1}^k x_{i_j} \|_k = 1 \) and \( \| x_i - x_j \|_k \geq 2 \) for all \( i \neq j \).
To show that $E_k$ is $k$-UR, for $x_1, \ldots, x_{k+1}$ in $l_2$, let $V_k(x_1, \ldots, x_{k+1})$ be the volume determined by $x_1, \ldots, x_{k+1}$ in $E_k$. Similarly, let $V_l(x_1, \ldots, x_{k+1})$, respectively, $V_{i_1, \ldots, i_l}(x_1, \ldots, x_{k+1})$, be the corresponding volume in $l_2$, respectively, in $X_{i_1, \ldots, i_l}$. It is easy to see that for any $x_1, \ldots, x_{k+1}$, $V_{i_1, \ldots, i_l}(x_1, \ldots, x_{k+1}) \leq V_{i_1, \ldots, i_l}(x_1, \ldots, x_{k+1}) \leq V_k(x_1, \ldots, x_{k+1})$. Given $\varepsilon > 0$, let $\delta(\varepsilon/k^{k/2}) > 0$ be such that for any $i_1 < \cdots < i_k$, if $\|x\|_{i_1, \ldots, i_k} \leq \varepsilon / k^{k/2}$, then $V_{i_1, \ldots, i_l}(x_1, \ldots, x_{k+1}) < \varepsilon$. Suppose $\|x\|_{i_1, \ldots, i_k} \leq \varepsilon / k^{k/2}$, then $V_{i_1, \ldots, i_l}(x_1, \ldots, x_{k+1}) < \varepsilon$. This completes the proof that $E_k$ is $k$-UR.

Finally, let $X_k = (l_2, \|\cdot\|)$, where $\|x\|^2 = \|x\|^2 + \sum_{i=1}^\infty \frac{a_i^2}{2^i}$ for all $x = (a_1, a_2, \ldots)$ in $l_2$. It is easy to see that $X_k$ is isomorphic to $l_2$ and is strictly convex. It is straightforward to show that $X_k$ is $k$-UR but is not $kR$. Finally, let us remark that it is easy to show that if $X_i$ is $k_i$-UR, $i = 1, 2$, then $(X_1 \oplus X_2)_{l_2}$, $1 < p < \infty$, is $(k_1 + k_2 - 1)$-UR (however, the $l_2$-sum of two 2-UR spaces need not be 2-UR, see [6] for details).

Remark. By Theorem 2, the space $X_k$ is $(k+1)$-UR but is not $kR$. Furthermore, let $X = (\sum_{k=1}^\infty X_k)_{l_2}$. Then $X$ is isomorphic to $l_2$ but $X$ is not fully convex.

REFERENCES