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Lumpability and Observability of Linear Systems

PAMELA G. COXSON*

Department of Mathematical Sciences,
University of Delaware, Newark, Delaware 19711

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1. INTRODUCTION

Large scale dynamic system models are plagued by problems of dimensionality and limitations of measurement. It is common practice in applications to lump similar system state components together to get a reduced number of aggregate states. An example of this approach is the PONA scheme for modelling catalytic cracking of oil. The thousands of chemical species involved in this process are lumped into four groups: paraffins, olefins, naphthenes, and aromatics (PONA). For a survey of the use of lumping in the cracking problem, the reader is referred to Weekman [14, 7].

Attention has focused on conditions under which such lumping preserves structural properties of the system dynamics. In particular, when do the lumped states of a linear system satisfy linear dynamics? If this occurs, the system is said to be exactly lumpable. For finite dimensional systems, this issue has been resolved [2, 15], but in practice the conditions specified are almost never satisfied.

In this paper, we show that it is fruitful to consider the lumped variables as observations of the original system. To our knowledge, this viewpoint has not been taken previously. We establish a relationship between lumpability and observability of the lumped observation system for both finite and infinite dimensional linear systems. This relationship allows us to deduce the existence of a minimal size system which gives a true representation of the dynamics of the lumped variables. Definitions of observability and lumpability are given in Sections 2 and 3, including extensions to infinite dimensional systems. The relationship between these two concepts is discussed and we establish the main result concerning lumpability in Section 4 in the case of finite dimensional systems. In Section 5 this discussion is carried over to infinite dimensional systems. A summary and discussion follow in Section 6.

* Present address: Laboratory for Information and Decision Systems, M.I.T., Cambridge, Massachusetts 02139.

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Notation and Conventions

Briefly, $R(T)$ and $N(T)$ will denote the range and nullspace, respectively, of an operator T. $L(X, Y)$ is the space of linear maps $T: X \to Y$, where X and Y are vector spaces; $B(X, Y)$ is the space of continuous linear operators. $X = U \oplus V$ indicates that U and V are topological complements, i.e., U and V are closed, $U + V = X$, and $U \cap V = \{0\}$. $T^{\dagger}$ denotes the topological generalized inverse of T. Specifically, if X and Y are Banach spaces, T in $B(X, Y)$ is surjective, and if $N(T) \oplus U = X$ (for some U), then $T^{\dagger}: Y \to X$ is continuous and is characterized as the unique solution of

$$TT^{\dagger}T = T,$$

$$T^{\dagger}T = I_X - P,$$

$$TT^{\dagger} = I_Y,$$

where P is the projector ($P^2 = P$) on $N(T)$, relative to U (see Nashed [10, p. 59]).

2. Observability

An unforced linear dynamic system is specified by the pair (A, C) of linear transformations representing the time invariant differential system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad x(t_0) = x_0, \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

Here $x(t)$ is in X and $y(t)$ is in Y for each t; X and Y are vector spaces, called the state and observation spaces, respectively. $A: X \to X$ and $C: X \to Y$. The system is n dimensional if $X = R^n$.

A system is said to be completely observable if any initial state x_0 can be reconstructed from knowledge of the observations $Cx(t)$ on a finite interval $[t_0, T(x_0)]$. A state x_0 is nonobservable if $y(t)$ is identically zero when $x(t_0) = x_0$. The following result is well known (e.g., [8, 9]).

THEOREM 2.1. *An n-dimensional linear time invariant system* (1) *is completely observable if and only if its observability matrix $\mathcal{O}$ has rank n, where*

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

The nonobservable states form a subspace of X which coincides with the null space of $\mathscr{O}$.

The observability criterion of Theorem 2.1 has been generalized to the case where X, and hence the system (A, C), is infinite dimensional. The rank condition on the observability matrix $\mathscr{O}$ of a finite dimensional system is equivalent to a statement about the null spaces $N(CA^k)$,

$$\operatorname{rank} \mathscr{O} = n \Leftrightarrow \bigcap_{k=0}^{n-1} N(CA^k) = \{0_X\}.$$

The extended result given below is due to Triggiani [12]. Further generalizations are available in [13].

THEOREM 2.2 (Triggiani). *Let X be a Banach space. Let $A: X \to X$ and $C: X \to Y$ be bounded linear operators. The linear time invariant system (1) is completely observable if and only if*

$$\bigcap_{k=0}^{\infty} N(CA^k) = \{0_X\}.$$

If the system is not completely observable, then $\bigcap_{k=0}^{\infty} N(CA^k)$ is the set of nonobservable states.

3. LUMPABILITY

Consider the partially determined system $(K, -)$,

$$\dot{x}(t) = Kx(t), \qquad x(t_0) = x_0. \tag{2}$$

When it is not possible or desirable to observe the entire state vector $x(t)$, there may be a collection of measurement operators $\{M: X \to Y_M\}$ which are feasible. We will assume that $\dim Y_M < \dim X$ for each M in the collection. For a particular choice of measurement operator M, we introduce Definition 3.1.

DEFINITION 3.1. The system $(K, -)$ is exactly lumpable by the linear transformation $M: X \to Y_M$ if the lumped vector $y(t) = Mx(t)$ satisfies an unforced linear differential equation

$$\dot{y}(t) = \hat{K}y(t),$$

for some $\hat{K}: Y_M \to Y_M$.

Wei and Kuo give necessary and sufficient conditions for exact lumpability.

THEOREM 3.2 (Wei and Kuo [15]). *An n-dimensional system $(K,-)$ is exactly lumpable by M if and only if there exists a matrix $\hat{K}$ such that $MK=\hat{K}M$.*

The equation $MK=\hat{K}M$ expresses the rows of MK as linear combinations of the rows of M. This simple observation leads to an equivalent formulation of Theorem 3.2, which seems to have been overlooked in the existing literature.

THEOREM 3.3. *An n-dimensional system $(K,-)$ is exactly lumpable by M if and only if*

$$\operatorname{rank}\begin{pmatrix} M \\ MK \end{pmatrix} = \operatorname{rank} M. \tag{3}$$

This condition clearly suggests that there is an intimate relationship between the notions of lumpability and observability. We return to this point in Section 4. We remark here only that this rank condition is particularly easy to apply when the matrix M has columns which are canonical unit vectors (e.g., $(010)^{\mathrm{T}}$). Such a matrix is called a proper lumping matrix. It corresponds to the case where the components of the state vector are partitioned into groups and each new component is simply the sum of the values in one group. The PONA scheme mentioned in the Introduction represents a proper lumping. Other examples arise in economics, where proper lumping matrices are called grouping mappings (Chipman [5]). The following examples illustrate the application of Theorem 3.3.

EXAMPLE 3.4.a.

$$M=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K=\begin{pmatrix} -2 & 1 & 0.5 \\ 1 & -3 & 0.5 \\ 1 & 2 & -1 \end{pmatrix}.$$

The system $(K,-)$ is exactly lumpable by M if and only if the row vectors of MK have the form $(a\ \ a\ \ b)$;

$$MK=\begin{pmatrix} -1 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Clearly the system is not exactly lumpable.

EXAMPLE 3.4.b.

$$M=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K=\begin{pmatrix} -2 & 5 & 0.6 \\ 1 & -6 & 1.4 \\ 1 & 1 & -2 \end{pmatrix}, \qquad MK=\begin{pmatrix} -1 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

This time the rows have the right format and the system is shown to be exactly lumpable.

The algebraic criterion of Theorem 3.2 is easily seen to be necessary for exact lumpability in an infinite dimensional system, but it does not guarantee the existence of a continuous system operator $\hat{K}$. We shall show that a continuous map $\hat{K}$ can be found whenever M satisfies suitable hypotheses. In particular, we require that $R(M)$ is closed and $N(M)$ is complemented in X. Both conditions are satisfied if Y is finite dimensional, i.e., if the number of measurements at each time t is finite, which is the situation in many applications. It can be assumed without loss of generality, that $Y = R(M)$.

THEOREM 3.5. *If X and Y are Banach spaces, K is in $B(X, X)$, and if M in $B(X, Y)$ satisfies* (i) *$R(M) = Y$, and* (ii) *$N(M)$ has a topological complement in X, then the following are equivalent*:

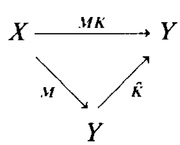
(a) *$(K, -)$ is exactly lumpable by M.*

(b) *There exists $\hat{K}$ in $B(Y, Y)$ such that $MK = \hat{K}M$.*

(c) *There exists $\hat{K}$ in $L(Y, Y)$ such that $MK = \hat{K}M$.*

(d) *$N(M)$ is contained in $N(MK)$.*

Proof. (a) is equivalent to (b) follows the proof of Theorem 3.2; (b) implies (c) is immediate. To show (c) implies (b), let $M^{\dagger}$ be a topological generalized inverse of M. Existence of $M^{\dagger}$ and uniqueness up to the choice of complement of $N(M)$ follow from the hypotheses (Nashed [10, p. 59]). As indicated in the Introduction, $M^{\dagger}$ is continuous and satisfies among other things, $MM^{\dagger} = I_Y$. If $MK = \hat{K}M$, then $MKM^{\dagger} = \hat{K}MM^{\dagger} = \hat{K}$ and $\hat{K}$ is seen to be continuous. (c) $\Leftrightarrow$ (d); there exists $\hat{K}$ in $L(X, Y)$ such that $MK = \hat{K}M$ if and only if the following commutes:



By the induced function theorem [6], $\hat{K}$ exists if and only if $N(M)$ is contained in $N(MK)$. ∎

Remark. Since the constraints on M were required only to show (c) $\Rightarrow$ (b), we still have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) when they are omitted. Notice that condition (d) is the analogue in the infinite dimensional setting of the rank condition of Theorem 3.3.

4. Observability and Lumpability in Finite Dimensions

As remarked above, condition (3) of Theorem 3.3, and Theorem 3.5d suggest a fundamental relationship between lumping and observability. In particular, it suggests that we consider the lumped variables as observations of the original system. This is the viewpoint mentioned in the Introduction. Thus we consider the lumped observation system (K, M),

$$\begin{aligned} \dot{x}(t) &= Kx(t), \qquad x(t_0) = x_0, \\ y(t) &= Mx(t). \end{aligned} \tag{4}$$

THEOREM 4.1. *Let M,* rank $M = m < n$, *determine an exact lumping of an n-dimensional system with system matrix K. Then the lumped observation system* (4) *is not observable and its nonobservable subspace coincides with the null space of M, $N(M)$.*

Proof. Let $\hat{K}$ satisfy $MK = \hat{K}M$. Multiplying the equation by K^{j-1} on the right gives $MK^j = \hat{K}MK^{j-1} = \hat{K}^j M$. Thus,

$$\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{pmatrix} M \\ MK \\ \vdots \\ MK^{n-1} \end{pmatrix} = \operatorname{rank} M < n,$$

$N(\mathcal{O}) = N(M)$, and the result follows from Theorem 2.1. ∎

Exact lumpability is clearly more restrictive than nonobservability of (4). In an observable system, the entire state vector $x(t_0)$ can be reconstructed from the measured values $y(t)$, $t \geqslant t_0$. In a nonobservable system, it is only possible to get a partial reconstruction, but the reconstructed values may include components which are not directly observed.

EXAMPLE 4.2.

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} x(t), \qquad y(t) = [1 \;\; 0 \;\; 0]\, x(t).$$

Here, only one component of the state is measured directly.

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix};$$

$N(\mathcal{O})$ is spanned by $(0\ \ 0\ \ 1)^T$, so the only component that cannot be determined is the third component. The state $(0\ \ 1\ \ 0)^T$ is not observed since $Mx = 0$, but it is observable (i.e. not nonobservable).

From Theorem 4.1 we see that for the lumped observation system arising from an exact lumping, a state which is not observed is not observable.

The next result, attributed to Balakrishnan, is cited several places without proof (e.g., Aoki [2]). The version given here is appropriate to the present discussion and, for the sake of completeness, we supply a proof. We will use this result to assert the existence of a reduced order system which is completely observable by the lumped observations Mx.

THEOREM 4.3. *If an n-dimensional linear dynamic system (K, M) is not observable, then the partial system $(K, -)$ is exactly lumpable by a matrix N with rank N equal to the rank of $\mathcal{O}$, the observability matrix.*

Proof. Let $r = \operatorname{rank} \mathcal{O}$. Let N be constructed from r independent rows of $\mathcal{O}$.

$$\mathcal{O} = \begin{pmatrix} M \\ MK \\ \vdots \\ MK^{n-2} \\ MK^{n-1} \end{pmatrix} \quad \mathcal{O}K = \begin{pmatrix} MK \\ MK^2 \\ \vdots \\ MK^{n-1} \\ MK^n \end{pmatrix} = \begin{pmatrix} MK \\ MK^2 \\ \vdots \\ MK^{n-1} \\ M\left(\sum_{i=0}^{n-1} a_i K^i\right) \end{pmatrix}.$$

The rows of $\mathcal{O}K$, and hence the rows of NK, lie in the rowspace of N. Thus the existence of $\hat{K}$ satisfying $NK = \hat{K}N$ is assured. ∎

We assume that M has full rank (otherwise, there are redundant measurements which can be eliminated). If the system (K, M) is exactly lumpable, then $\operatorname{rank} \mathcal{O} = \operatorname{rank} M = m$ and we can let $N = M$ in Theorem 4.3. Even if (K, M) is not exactly lumpable, the rowspace of N still contains the rows of M. In this case, there exists an $m \times r$ matrix L such that $M = LN$.

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Thus we can formulate a new system with state space $Z = NX$, and with observed variables LNX that coincide with the lumped variables MX.

$$\dot{z}(t) = \hat{K}z(t), \qquad z(t_0) = Nx(t_0).$$
$$y(t) = Lz(t).$$

It is easily verified that the system $(\hat{K}, L)$ is completely observable.

System realization procedures such as the Kalman–Ho algorithm [8] or stochastic techniques developed by Akaike [1] can be invoked to identify $\hat{K}$ and L given input data (initial states) and measurements for (K, M). This does not require knowledge of K, which is important since K is generally not available, as indicated in [15]. In the exactly lumpable case where $L = M$, $\hat{K}$ can be identified in a straight forward manner as suggested in Wei and Kuo [15]. The procedures mentioned provide a natural extension of the method of Wei and Kuo to the case where (K, M) is not exactly lumpable.

5. Infinite Dimensional Systems

Infinite dimensional system models have been used to represent systems in which the natural physical state has a very large number of closely packed components. By closely packed we mean that adjacent components are difficult to distinguish by measurements. Seinfeld [11] takes this approach in a model of aerosol dynamics, where the state gives the distribution of particles by size. The ith component $x_i = x(v_i)$ of the discrete state is the number of i-mers (particles composed of i molecules) having volume v_i. For large i, i-mers, and $(i+1)$-mers are very close in size. Seinfeld fixes a $k \gg 1$ after which state components are described by a function of volume $x(v)$, where v ranges over $[v_k, \infty)$.

Aris and Gavalas [3] showed how infinite dimensional systems could be used to model chemical reactions involving large numbers of chemical species. They considered systems $(K, -)$ given by integrodifferential equations,

$$\frac{\partial}{\partial t} x(v, t) = \int_a^b K(u, v)\, x(u, t)\, du. \tag{5}$$

The infinite dimensional state is called a continuous mixture. The use of lumping in continuous mixture models is investigated by Bailey [4].

In this section, we consider dynamic systems (K, M) for which the state space X is a Banach space and the system operators K and M are linear and continuous. Lumped variables are determined by a bounded transformation $M: X \to Y$; $Y = \mathrm{cl}(R(M))$. If $Y = R^n$, $n < \infty$, we will say that M is finite with finite representation $Mx(t) = [M_1 x(t),..., M_n x(t)]^T$. As in the finite dimensional case, the more practical notion is of a proper lumping in which the state is partitioned into a finite number of sets and the components in each set are lumped. If X is a space of integrable functions (e.g., $L^p[a, b]$) we write

DEFINITION 5.1. Let X be a Banach space of functions integrable on Ω and for each fixed t, $x(\cdot, t)$ is in X. An operator $M: X \to R^n$ is a proper lumping of X if $M_k x(t) = \int_{\Omega_k} x(u, t)\, du$, where Ω is the disjoint union of $\{\Omega_i\}$.

Our first result removes the finite state space condition in Theorem 4.1, extending the result to systems with Banach state space.

THEOREM 5.2. *Let M in $B(X, Y)$ satisfy $N(M) \neq \{0\}$. If M determines an exact lumping of the partial system $(K, -)$, then the lumped observation system (K, M) is not observable and its nonobservable subspace coincides with $N(M)$.*

Proof. Exact lumpability assures the existence of a bounded $\hat{K}$ satisfying $MK = \hat{K}M$. It follows that $MK^i = \hat{K}^i M$ and thus $N(MK^i)$ is contained in $N(M)$ for every i. Since $\bigcap_{n=0}^{\infty} N(MK^n) = N(M) \neq \{0\}$, the system (K, M) is not observable and the nonobservable subspace is $N(M)$, as claimed. ∎

Theorem 4.3, which facilitated the identification of a system completely observable by the measurements Mx, can be generalized to include infinite dimensional systems if the space of nonobservable states has a topological complement. Many practical choices of lumping operators M fulfill this requirement. In particular, it holds if X is a Hilbert space or if the nonobservable space has finite codimension.

THEOREM 5.3. *If the dynamic system (K, M) is not completely observable, and the set of nonobservable states is complemented in X, then there is a linear transformation T defined on X such that*

(a) *$(K, -)$ is exactly lumpable by T, and*

(b) *the system with states Tx and observations Mx is completely observable.*

Proof. (a) Let T be the canonical projection

$$T: X \to X \Big/ \bigcap_{i=0}^{\infty} N(MK^i).$$

We see that $N(T) \subset N(TK)$ since,

$$\begin{aligned} Tx = 0 \Rightarrow MK^i x = 0 \qquad & \forall i = 0, 1, 2, \ldots, \\ \Rightarrow MK^{i-1}(Kx) = 0 \qquad & \forall i = 1, 2, \ldots, \\ \Rightarrow T(Kx) = 0, \end{aligned}$$

where $N(T)$ is the space of nonobservable states of (K, M); $R(T)$ is closed. By Theorem 3.5, $(K, -)$ is exactly lumpable by T if $N(T)$ is complemented in X.

(b) With T as above, let $\hat{K}$ satisfy $TK = \hat{K}T$. Since $N(M) \subset N(T)$, there exists $L: TX \to Y$ such that $M = LT$. We note that Tx is in $N(L\hat{K}^i)$ $\forall i$ if and only if x is in $N(T)$. Thus the system $(\hat{K}, L)$, with states Tx and observations Mx, is completely observable.

6. Summary and Discussion

The lumped state variable Mx of a system $(K, -)$ which is exactly lumpable by M satisfies a linear differential equation

$$(\dot{Mx}) = \hat{K}(Mx), \qquad (Mx)_0 = Mx(t_0). \tag{6}$$

This reduced system can be used to study (simulate, control, etc.) the behavior of the lumped variables. However, when $Y = MX$ is small relative to X, exact lumpability is unlikely.

If the system $(K, -)$ is not exactly lumpable by M, the lumped variable Mx does not satisfy an equation of the form (6). Its evolution depends on the distribution of the state components within lumps over time. The impracticality of measuring this distribution is one of the motivations for lumping. We clearly do not want to use the original state vector to study the lumped variable. One possible approach is to search for a linear system of dimension $m = \dim R(M)$ which approximates the dynamics of Mx (e.g., let $\hat{K}: MX \to MX$ be defined by $\hat{K} = MKM^{\dagger}$).

In this paper a different approach has been taken. The lumped variable Mx is viewed as the observation vector of the system (K, M). If (K, M) were completely observable, it would be theoretically possible to reconstruct all of the original state values $x(t)$ from the observations $Mx(t)$. However, the technique of lumping is introduced to avoid such a reconstruction in systems with a large number of state variables. The extra effort required to unravel the values of the original state (and, more to the point, to identify the system matrix K) is again subject to the problem of dimensionality. Thus, maximal observability is not the goal. In fact, it was pointed out in Section 4 that exact lumpability is in some sense equivalent to minimal observability. The exactly lumped variable cannot be used to reconstruct any other components of the original state. If (K, M) is not exactly lumpable and not completely observable, Theorems 4.3 and 5.3 guarantee the existence of a reduced order system (K_T, L_T), K_T: $TX \to TX$ and $L_T T = M$, which is completely observable. (K_T, L_T) is the system which is important to the lumping problem. It is the smallest linear system with outputs that coincide with the lumped state Mx of $(K, -)$. If M represents an exact lumping, then $T = M$, $K_T = MKM^{\dagger}$ and $L_T = I_m$.

Usually the choice of M is open, constrained only by the feasibility of measurement and the goals of the model ($M = 0$ is an exact lumping but does not lead to a model; $M = I$ is not feasible). We have not addressed the question of how to choose M. The conventional wisdom that similar physical variables should be lumped together has a basis in theory, but it does not settle the question of what and how much to lump. Once M is fixed, K_T and L_T can be identified by way of a realization scheme, as discussed in Section 4.

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