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Splitting theorem, Poincaré–Hopf theorem and jumping nonlinear problems

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Dedicated to Professor Haim Brezis on the occasion of his 60th birthday

Abstract

In this paper, we establish a Gromoll–Meyer splitting theorem and a shifting theorem for $J \in C^{2-0}(E, R)$ and by using the finite-dimensional approximation, mollifiers and Morse theory we generalize the Poincaré–Hopf theorem to $J \in C^1(E, R)$ case. By combining the Poincaré–Hopf theorem and the splitting theorem, we study the existence of multiple solutions for jumping nonlinear elliptic equations.

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Keywords: The Poincaré–Hopf theorem; The splitting theorem; Multiple solutions; Jumping nonlinear elliptic equations

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1. Introduction

The Gromoll–Meyer splitting theorem and the Poincaré–Hopf theorem are very fundamental tools in critical point theory. However, they usually hold under strong assumption that $J \in C^2(E, R)$ (see [2,7]). When we study the existence of multiple solutions for jumping nonlinear elliptic equations, the potential functional $J(u)$ for such an equation belongs only to $C^{2-0}(E, R)$. So the usual critical point theorems such as the splitting theorem, the shifting theorem, the Poincaré–Hopf theorem cannot be used in this case. In order to attack the existence of multiple solutions for jumping nonlinear elliptic equations, it is necessary to establish these theorems at least for $J \in C^{2-0}(E, R)$.

Consider the following problem:

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in R^n.

Suppose that $f \in C^1\left(\overline{\Omega} \times R^1 \setminus \{0\}\right)$ and $\lim_{t \to 0_+} \frac{f(x,t)}{t} = b_0$, $\lim_{t \to 0_-} \frac{f(x,t)}{t} = a_0$, uniformly in $x \in \overline{\Omega}$. Let $J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x, u)\, dx$, where $F(x, t) = \int_0^t f(x, s)\, ds$. Take $E = H_0^1(\Omega)$ and $X = C_0^1\left(\overline{\Omega}\right)$. It is well known that $J \in C^{2-0}(E, R)$. Assume that $u_0 \neq 0$ is a critical point of $J(u)$. By partial regularity of the zero set of the solution of linear and super-linear elliptic equations, we can prove that $J' \in C^1(D, E)$ and $d^2 J(u_0)$ is a bounded linear operator from X to E, where D is a neighborhood of u_0 in X topology. By using the bootstrap argument we can still prove a splitting theorem and the shifting theorem for $J \in C^{2-0}(E, R)$. We can prove that $C_q(J, u_0) \cong \delta_{q1} G$ and $\text{ind}(df, u_0) = -1$ for $J \in C^{2-0}(E, R)$, where u_0 is a mountain pass point of $J(u)$.

The paper is organized as follows. In Section 2 we build up a Gromoll–Meyer splitting theorem and shifting theorem for $J \in C^{2-0}(E, R)$. By using the finite-dimensional approximation, mollifiers and Morse theory we generalize the Poincaré–Hopf theorem to C^1 case in Section 3. Finally, Section 4 is devoted to the existence of multiple solutions for jumping nonlinear elliptic equations.

2. Gromoll–Meyer theory for $J \in C^{2-0}(E, R)$

The Gromoll–Meyer splitting theorem and shifting theorem are very fundamental tools in critical point theory. However, they usually hold under strong assumption that $J \in C^2(E, R)$ (see [2,7]). Let E be a Hilbert space and $X \subset E$ be a Banach space densely embedded in E. Let $J \in C^{2-0}(E, R)$, which implies that $J'(u)$ is local Lipschitz in E. Assume that u_0 is the only critical point of J in a neighborhood D of u_0 in X. Here $J' \in C^1(D, E)$ and $A = d^2 J(u_0)$ is a bounded linear operator from X to E. The kernel N of A is finite dimensional. Let $u - u_0 = w + v$ be the corresponding decomposition of $u - u_0 \in E$. Let $K = K(J) = \left\{u \in E | J'(u) = 0\right\}$. Assume that J satisfies the following property (J):

$V : E \to E$ is a pseudo-gradient vector field of J, $V(x) = x - KG(x)$, where K and G satisfy the following assumptions.

(1) There are two sequences of Banach spaces

$$E_N \hookrightarrow E_{N-1} \hookrightarrow \cdots \hookrightarrow E_1 \hookrightarrow E_0,$$

$$X_{N-1} \hookrightarrow X_{N-2} \hookrightarrow \cdots \hookrightarrow X_0$$

such that

$$E \hookrightarrow E_0, E_N \hookrightarrow X.$$

Denote $\| \|_i = \|\cdot\|_{E_i}$. It is no loss of generality to assume that $\| \|_i \leqslant \| \|_{i+1}$, $i = 0, 1, \ldots, N-1$.

(2) $G_i : E_i \to X_i$ is bounded and continuous; it satisfies the local Lipschitzian. For each neighborhood U in E_i there exists $M_i = M_U^i$ such that

$$\|G(x) - G(y)\|_{X_i} \leqslant M_i \|x - y\|_{E_i}, \quad \forall x, y \in U.$$

$K : X_i \to E_i$ is a linear bounded operator. We denote $N_i = \|K\|_{L(X_i, E_i)}$, $i = 0, 1, \ldots, N-1$.

(3) The critical set K of J is in X.

Now we have the following.

Theorem 2.1 (*Splitting theorem*). *Under the above assumptions there exists a ball $B_\delta(u_0)$ in X, $\delta > 0$ centered at u_0, a u_0-preserving local homeomorphism h from $B_\delta(u_0)$ into D and a C^1 mapping $g : B_\delta(0) \cap N \to N^\perp \cap X$ such that*

$$J(h(u)) = \frac{1}{2}(Av, v) + J(u_0 + w + g(w)). \tag{2.1}$$

Proof. Let $P : E \to E$ be the orthogonal projection onto $N^\perp$. By the implicit function theorem, there is a mapping $g : B_\delta(0) \cap N \to N^\perp \cap X$ such that $g(0) = 0$, $g'(0) = 0$ and

$$PJ'(u_0 + w + g(w)) = 0. \tag{2.2}$$

Let us define $\widetilde{J}$ on $B_\delta(0) \cap N$ by

$$\widetilde{J}(w) = J(u_0 + w + g(w)).$$

From (2.2)

$$\widetilde{J}'(w) = (I - P)J'(u_0 + w + g(w))$$

and

$$\widetilde{J}''(w) = (I-P)\, J''(u_0 + w + g(w))\left(I + g'(w)\right).$$

In particular,

$$\widetilde{J}'(0) = (I-P)\, J''(u_0) = 0$$

and

$$\widetilde{J}''(0) = (I-P)\, J''(u_0) = (I-P)\, A = 0.$$

Let us define, near $[0,1] \times \{u_0\}$ the function

$$F(t,v,w) = (1-t)\left(\widetilde{J}(w) + \frac{1}{2}\langle Av, v\rangle\right) + t \cdot J(u_0 + v + w + g(w))$$

and the vector field

$$\Phi(t,v,w) = \begin{cases} 0 & \text{if} \quad v = 0, \\ -F_t(t,v,w) \cdot F_v(t,v,w) / \|F_v(t,v,w)\|^2 & \text{if} \quad v \neq 0, \end{cases}$$

where $\|\cdot\|$ denotes the norm in E. By a standard argument (see [2,7]) we can prove that the Cauchy problem

$$\begin{cases} \frac{d\eta}{dt} = \Phi(t,\eta,w), \\ \eta(0) = v,\ v \in B_\delta(0), \end{cases} \tag{2.3}$$

has a solution $\eta(t,v,w)$ for $v \in X$. In fact, by direct computation we know that $\Phi(t,v,w)$ is with the form $v - KG(v)$ and K, G satisfy the property (J). In particular, from the property (J) we have $\eta(t,v,w) \in D$ as $\delta > 0$ small enough (see [3]). It is easy to see that

$$\begin{aligned} \frac{d}{dt} F(t,\eta(t),w) &= F_t(t,\eta(t),w) + \left\langle F_v(t,\eta(t),w), \frac{d\eta}{dt}\right\rangle \\ &= 0 \end{aligned}$$

and then

$$\begin{aligned} \widetilde{J}(w) + \frac{1}{2}\langle Av, v\rangle &= F(0,v,w) \\ &= F(1,\eta(1,v,w),w) \\ &= J(u_0 + \eta(1,v,w) + w + g(w)). \end{aligned}$$

The local homeomorphism h is given by

$$h(u) = h(v, w) = u_0 + w + g(w) + \eta(1, v, w).$$

The local invertibility of h follows from the local invertibility of $\eta(1, \cdot, w)$. The proof is complete. □

We can also establish the shifting theorem. Note that if (W, W_-) is a Gromoll–Meyer pair in E with respect to the negative gradient vector field of $-\nabla J(x)$, where $J(x)$ satisfies all assumptions in Theorem 2.1, then $(W \cap X, W_- \cap X)$ is also a Gromoll–Meyer pair in X. Similar to the case $J \in C^2(E, R)$ we call u_0 a nondegenerate critical point if A has a bounded inverse from E to X. We call the dimension of the negative space of A the Morse index of u_0.

Theorem 2.2 (*Shifting theorem*). *Assume that the Morse index of J at u_0 is j. Then we have*

$$C_q(J|_X, u_0) = C_{q-j}\left(\widetilde{J}, 0\right).$$

To prove Theorem 2.2 we need the following:

Lemma 2.3. *Suppose that $E = E_1 \oplus E_2$, $X_i \subset E_i$ is a Banach space densely embedded in E_i, $g_i \in C^{2-0}(E_i, R)$, θ_i is an isolated critical point of g_i, $i = 1, 2$. Assume that g_i satisfies the property (J) and that $\left(W_i \cap X_i, (W_i)_- \cap X_i\right)$ is a Gromoll–Meyer pair of θ_i with respect to the gradient field of g_i in X_i, $i = 1, 2$. Then $(A \times B, (C \times B) \cup (A \times D))$ is a Gromoll–Meyer pair of the function $J = g_1 + g_2$ at $\theta = \theta_1 + \theta_2$ with respect to the gradient vector field of ∇J in $X = X_1 + X_2$, where $A = W_1 \cap X_1$, $B = W_2 \cap X_2$, $C = (W_1)_- \cap X_1$, $D = (W_2)_- \cap X_2$, if θ is an isolated critical point of J.*

This lemma is easy to check.

Lemma 2.4. *Under the hypothesis of Lemma* 2.3 *we have*

$$C_*\left(J|_X, \theta\right) = C_*\left(g_1|_{X_1}, \theta_1\right) \otimes C_*\left(g_2|_{X_2}, \theta_2\right).$$

Proof. Note that

$$C_*\left(J|_X, \theta\right) = H_*(W|_X, W_-|_X)$$

and combining Lemma 2.3 and the Künneth formula we get the lemma. □

Proof of Theorem 2.2. This is a combination of Theorem 2.1 and Lemma 2.4. From Theorem 2.2 and the Palais theorem (see [8]) we have:

Corollary 2.5.

$$C_q(J, u_0) = C_{q-j}\left(\widetilde{J}, u_0\right).$$

Definition 2.6. We call u_0 a mountain pass point if $C_1(J, u_0) \neq 0$.

The following theorem provides more precise information on mountain pass point without the assumption $J \in C^2(E, R)$. It is very useful in semilinear elliptic problems in which the nonlinear term loses the differentiability at some point.

Theorem 2.7. *Assume that $J \in C^{2-0}(E, R)$ and satisfies the assumptions given in Theorem* 2.1. *Assume that u_0 is a mountain pass point and that*

$$\dim \ker(A) = 1$$

if $0 \in \sigma(A)$, where $A = d^2 J(u_0)$ is a bounded linear operator from X to E and $\sigma(A)$ denotes the spectrum of A. Then

$$C_q(J, u_0) \cong \delta_{q1} G.$$

Since we already have Theorem 2.2, the remains of the proof are quite similar to the case $J \in C^2(E, R)$ (see [2]). We omit the proof.

3. Poincaré–Hopf theorem for $f \in C^1(E, R)$

The Poincaré–Hopf theorem shows us the relationship between the indices of a smooth vector field on a manifold M and the Euler Characteristic of the M. If $f \in C^2(E, R)$ the following result is true.

Proposition 3.1 (*see Chang [2]*). *Let E be a real Hilbert space and $f \in C^2(E, R)$ be a function that satisfies the (PS) condition. Assume that*

$$df(u) = u - K(u),$$

where K is a compact mapping and u_0 is an isolated critical point of f. Then we have

$$\begin{aligned} ind(df, u_0) &= \deg(I_d - K, B_\varepsilon(u_0), 0) \\ &= \sum_{q=0}^{\infty} (-1)^q \, rank C_q(f, u_0) \end{aligned} \tag{3.1}$$

for $\varepsilon > 0$ sufficiently small.

We generalize the result as follows.

Theorem 3.2. *The conclusion of Proposition* 3.1 *is still true if* $f \in C^1(E, R)$.

Proof. Let

$$f(u) = \frac{1}{2}\|u\|^2 - G(u) \tag{3.2}$$

and

$$G'(u) = K(u). \tag{3.3}$$

Then

$$df(u) = u - K(u). \tag{3.4}$$

Take $\varepsilon > 0$ small enough such that f only has unique critical point u_0 in $B_{3\varepsilon}(u_0)$ and there exists $\delta > 0$ such that $\|f'(u)\| \geqslant \delta$ as $u \in B_{3\varepsilon}(u_0) \backslash B_\varepsilon(u_0)$. Construct a Gromoll–Meyer pair (W, W_-) by

$$W = f^{-1}[-r + c, r + c] \cap g_\mu, \tag{3.5}$$

$$W_- = f^{-1}(-r + c) \cap W, \tag{3.6}$$

where

$$g_\mu = \{u \in E, g(u) \leqslant \mu\}, \tag{3.7}$$

$$g(u) = \lambda(f(u) - f(u_0)) + \|u\|^2 - \|u_0\|^2, \tag{3.8}$$

$$c = f(u_0) \tag{3.9}$$

and λ, μ, γ are positive numbers to be determined by the following conditions:

$$B_\varepsilon(u_0) \cap f^{-1}\left[-\gamma + c, \gamma + c\right] \subset W \subset B_{2\varepsilon}(u_0) \cap f^{-1}\left[-\rho + c, \rho + c\right], \tag{3.10}$$

$$f^{-1}\left[-\gamma + c, \gamma + c\right] \cap g^{-1}(\mu) \subset B_{2\varepsilon}(u_0) \backslash B_\varepsilon(u_0), \tag{3.11}$$

$$\langle dg(u), df(u)\rangle > 0, \ \forall u \in B_{2\varepsilon}(u_0) \backslash B_\varepsilon(u_0), \tag{3.12}$$

where ρ is small such that $c = f(u_0)$ is the unique critical value of f in $\left[-\rho + c, \rho + c\right]$. It is easy to check that (W, W_-) is a Gromoll–Meyer pair with respect to a negative pseudo-gradient vector field of f (see [2]).

Set $v = u - u_0$. We could find a finite-dimensional approximation $P_n K(P_n v)$ such that $\forall v \in B_{3\varepsilon}(u_0)$ as n large:

$$\|K(v) - P_n K(P_n v)\|_E < \min\left(\frac{\delta}{6}, \frac{\gamma}{6}\right), \tag{3.13}$$

where E_n is the eigenspace spanned by the eigenfunctions $\varphi_1, \ldots, \varphi_n$ of $K'(u_0)$ and P_n is the projection onto E_n. Define

$$G_n(v) = \int_0^1 P_n K(t P_n v)\, v\, dt + G(0),$$

$$f_n(v) = \frac{1}{2}\|v\|^2 - G_n(v).$$

We have

$$\begin{aligned}
&\sup_{v \in B_{3\varepsilon}(u_0)} |f(v) - f_n(v)| \\
&= \sup_{v \in B_{3\varepsilon}(u_0)} \left|\frac{1}{2}\|v\|^2 - G(v) - \frac{1}{2}\|v\|^2 + G_n(v)\right| \\
&= \sup_{v \in B_{3\varepsilon}(u_0)} \left|\int_0^1 P_n K(t P_n v)\, v\, dt - \int_0^1 K(tv)\, v\, dt\right| \\
&< \min\left(\frac{\gamma}{6}, \frac{\delta}{6}\right).
\end{aligned}$$

By using the mollifier we can find a $\widetilde{K}_n \in C^\infty(B_{3\varepsilon}(u_0) \cap E_n, E_n)$ such that $\forall u \in B_{2\varepsilon}(u_0)$

$$\left\| P_n K(P_n v) - \widetilde{K}_n(P_n v)\right\|_{E_n} < \min\left(\frac{\gamma}{6}, \frac{\delta}{6}\right) \tag{3.14}$$

and therefore

$$\sup_{v \in B_{2\varepsilon}(u_0)} \left|\widetilde{f}(v) - f_n(v)\right| < \min\left(\frac{\gamma}{6}, \frac{\delta}{6}\right), \tag{3.15}$$

where

$$\widetilde{f}(v) = \frac{1}{2}\|v\|^2 - \widetilde{G}_n(v),$$

$$\widetilde{G}_n(v) = \int_0^1 \widetilde{K}_n(t P_n v)\, v\, dt + G(0).$$

Combining (3.13)–(3.15) we get

$$\sup_{v\in B_{2\varepsilon}(u_0)}\left|f(v)-\widetilde{f}(v)\right| < \min\left(\frac{\gamma}{3},\frac{\delta}{3}\right), \tag{3.16}$$

$$\begin{aligned}&\sup_{v\in B_{2\varepsilon}(u_0)}\left\|df(v)-d\widetilde{f}(v)\right\|_E\\ &\quad=\sup_{v\in B_{2\varepsilon}(u_0)}\|K(v)-K_n(P_n v)\|_{E_n} < \frac{\delta}{3}.\end{aligned} \tag{3.17}$$

Since $\|f'(u)\|\geqslant\delta$ as $u\in B_{3\varepsilon}(u_0)\setminus B_\varepsilon(u_0)$, we get that $\widetilde{f}|_{B_{2\varepsilon}(u_0)}$ only has critical points in $B_\varepsilon(u_0)$. As a matter of fact, if it is not true, then there must exist a $\widetilde{u}\in B_{2\varepsilon}(u_0)\setminus B_\varepsilon(u_0)$ such that $\widetilde{f}\,'(\widetilde{u})=0$, but

$$\delta\leqslant\|f'(\widetilde{u})\|\leqslant\|f'(\widetilde{u})-\widetilde{f}'(\widetilde{u})\|+\|\widetilde{f}'(\widetilde{u})\|<\frac{\delta}{3}$$

a contradiction!

By the Smale–Sard theorem we can require $\widetilde{f}$ such that all critical points of $\widetilde{f}$ in $B_{2\varepsilon}(u_0)$ are nondegenerate, say u_j, $j=1,2,\ldots,m$ (see [9,2]).

For $\widetilde{f}$ we obtain immediately

$$\begin{aligned}W_- = f_{-\gamma+c}\cap W &\subset \widetilde{f}_{-\frac{2}{3}\gamma+c}\cap W\subset f_{-\frac{\gamma}{3}+c}\cap W,\\ &\subset f_{\frac{\gamma}{3}+c}\cap W\subset\widetilde{f}_{\frac{2}{3}\gamma+c}\cap W\subset f_{\gamma+c}\cap W=W.\end{aligned} \tag{3.18}$$

However, there are strong deformation retracts

$$f_{\gamma+c}\cap W\to f_{\frac{\gamma}{3}+c}\cap W$$

and

$$f_{-\frac{\gamma}{3}+c}\cap W\to f_{-\gamma+c}\cap W$$

provided by the Gromoll–Meyer property. We have

$$H_*(W,W_-)=H_*\left(\widetilde{f}_{\frac{2\gamma}{3}+c}\cap W,\ \widetilde{f}_{-\frac{2\gamma}{3}+c}\cap W\right) \tag{3.19}$$

due to the exactness of the homological group sequence. Thus,

$$\operatorname{ind}(df,u_0)=\deg(df,W,0). \tag{3.20}$$

From (3.17) and the homotopy invariance of degree we get

$$\deg(df, W, 0) = \deg\left(d\widetilde{f}, W, 0\right) = \sum_{j=1}^{m} \operatorname{ind}\left(d\widetilde{f}, u_j\right). \tag{3.21}$$

For $\widetilde{f} \in C^{\infty}(E, R)$ we have the local result of the Poincaré–Hopf formula

$$\sum_{j=1}^{m} \operatorname{ind}\left(d\widetilde{f}, u_j\right) = \sum_{j=1}^{m}\sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q\left(\widetilde{f}, u_j\right). \tag{3.22}$$

Since

$$\begin{aligned}\sum_{j=1}^{m}\sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q\left(\widetilde{f}, u_j\right) &= \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} H_q\left(\widetilde{f}_{\frac{2\gamma}{3}+c} \cap W, \widetilde{f}_{-\frac{2\gamma}{3}+c} \cap W\right)\\ &= \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} H_q(W, W_-)\\ &= \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q\left(f, u_j\right).\end{aligned} \tag{3.23}$$

Combining (3.20)–(3.23) we have

$$\operatorname{ind}(df, u_0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(f, u_0).$$

The theorem is proved. □

Using Theorem 2.7 we have the following.

Corollary 3.3. *Assume that* $f \in C^{2-0}(E, R)$ *and satisfies the assumption given in Proposition* 3.1. *If* u_0 *is a mountain pass point and if the smallest* λ_1 *of* $d^2 f(u_0)$ *is simple whenever* $\lambda_1 = 0$, *then* $\lambda_1 \leqslant 0$ *and*

$$ind(df, u_0) = -1.$$

Remark 3.4. For $f \in C^2(E, R)$, Corollary 3.3 has been studied by H. Hofer (see [4]).

From Theorem 3.2 we can generalize the Poincaré–Hopf theorem as follows.

Theorem 3.5. *Suppose that* $f \in C^1(E, R)$ *and that* O *is a bounded domain in* E *on which* f *is bounded and only has isolated critical point in* O.

(a) $O_- \overset{\Delta}{=} \{u \in \partial O | \eta(t, u) \notin O, \forall t > 0\} = O \cap f^{-1}(a)$ *for some* a, *where* $\eta(t, u)$ *is the negative gradient flow of* f *emanating from* u;

(b) *-df points inward at* $\partial O \backslash O_-$, *then we have*

$$\deg(df, O, 0) = \sum_{q=0}^{\infty} (-1)^q \, rank \, H_q(O, O_-).$$

Proof. Note that f only has finite critical points in O, say u_i, $i = 1, 2, \ldots, m$. For each u_i following the argument given in Theorem 3.2 we can construct a Gromoll–Meyer pair $(W_i, (W_i)_-)$ of f and $\widetilde{f_i} \in C^\infty(B_\varepsilon(u_i), R)$ such that

$$\sup_{v \in B_\varepsilon(u_i)} \left| f(v) - \widetilde{f_i}(v) \right| < \min\left(\frac{\gamma}{3}, \frac{\delta}{3}\right), \tag{3.24}$$

$$\sup_{v \in B_\varepsilon(u_i)} \left\| df(v) - d\widetilde{f_i}(v) \right\|_E < \frac{\delta}{3}, \tag{3.25}$$

$$H_*(W_i, (W_i)_-) = H_*\left(\left(\widetilde{f_i}\right)_{\frac{2}{3}\gamma + c} \cap W_i, \left(\widetilde{f_i}\right)_{-\frac{2}{3}\gamma + c} \cap W_i\right), \tag{3.26}$$

where ε, γ, δ were given in Theorem 3.2.

Define

$$\widetilde{f}(v) = \begin{cases} \widetilde{f_i}(v), & v \in B_\varepsilon(u_i), \\ f(v), & v \in \overline{O} \backslash \cup B_{2\varepsilon}(u_i), \end{cases}$$

where $\widetilde{f_i}(v) \in C^\infty(B_\varepsilon(u_i), R)$, $\widetilde{f}(v) \in C^1(O, R)$ and $\widetilde{f}(v)$ only have critical points in $\bigcup_{i=1}^m W_i$. Then we have

$$\begin{aligned} \deg(df, O, 0) &= \deg\left(d\widetilde{f}, \bigcup_{i=1}^m W_i, 0\right) = \sum_{i=1}^m \deg\left(d\widetilde{f}, W_i, 0\right) \\ &= \sum_{i=1}^m \chi(W_i, (W_i)_-) = \sum_{i=1}^m \sum_{q=0}^\infty (-1)^q \operatorname{rank} H_q(W_i, (W_i)_-) \\ &= \sum_{i=1}^m \sum_{q=0}^\infty (-1)^q \operatorname{rank} C_q(f_i, u_i) = \sum_{q=0}^\infty (-1)^q \operatorname{rank} H_q(W, W_-). \end{aligned}$$

The theorem is complete. □

Corollary 3.6. *The conclusion of Theorem* 3.5 *is still true if O is a finite bounded domain in E and f only has finitely many critical points in O.*

4. Applications

We consider the semilinear elliptic boundary value problems of the form

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain in R^n. We make the following assumptions on $f(x,t)$:

(f_1) $\lim_{t\to 0_+} \frac{f(x,t)}{t} = b_0$, $\lim_{t\to 0_-} \frac{f(x,t)}{t} = a_0$, uniformly in $x \in \overline{\Omega}$;

(f_2) $a_0 > \lambda_2$, $b_0 > \lambda_2$, $(a_0, b_0) \in \widehat{A}$, where $\widehat{A} \subset R^2 \backslash \Sigma$ is the connected component of $R^2 \backslash \Sigma$ containing $(\lambda_i, \lambda_{i+1})$, $i = 2, 3, \ldots$, Σ denotes the set of those points $(a, b) \in R^2$ such that

$$\begin{cases} -\Delta u = au^- + bu^+, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{4.2}$$

has a nontrivial solution, where $u^+ = \max\{u(x), 0\}$, $u^- = \min\{u(x), 0\}$.

(f_3) There exist $M_1 > 0$, $M_2 < 0$ such that $f(x, M_1) < 0$, $f(x, M_2) > 0$ for $x \in \Omega$;

(f_4) $f \in C^1\left(\overline{\Omega} \times R^1 \backslash \{0\}\right)$;

(f_5) There exists $\alpha > 0$ and $C_1 > 0$ such that

$$\left|f_t'(x,t)\right| \leqslant C_1\left(1 + |t|^{\alpha-1}\right), \alpha < \frac{n+2}{n-2}, \text{ as } n \geqslant 3, \forall (x,t) \in \Omega \times (R \backslash \{0\}).$$

Theorem 4.1. *Under assumptions* (f_1)–(f_4), (4.1) *admits at least four nontrivial solutions*.

Proof. Take $\varepsilon > 0$ so small that $\varepsilon\varphi_1 < M_1$, $M_2 < -\varepsilon\varphi_1$, and $\{\varepsilon\varphi_1, M_1\}$, $\{M_1, -\varepsilon\varphi_1\}$ are two pairs of sub- and super-solutions of (4.1), where φ_1 is the first eigenfunction of the $-\Delta$ under the Dirichlet boundary value condition. It is well known that there exist u_1^+, u_1^- such that $\varepsilon\varphi_1 < u_1^+ < M_1$, $M_2 < u_1^- < -\varepsilon\varphi_1$, and both u_1^+, u_1^- are local minimizers of the following functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. Consider

$$\widetilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \widetilde{F}(x,u)\,dx,$$

where $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds$ and

$$\widetilde{f}(x,t) = \begin{cases} f(x,M_2), & t < M_2, \\ f(x,t), & M_2 \leqslant t \leqslant M_1, \\ f(x,M_1), & t > M_1. \end{cases}$$

By the Mountain Pass Theorem in Order Intervals (see [6]), $\widetilde{J}$ has a mountain pass point $u_0 \in [M_2, M_1] \setminus \left(\left[M_2, -\varepsilon\varphi_1\right] \cup \left[\varepsilon\varphi_1, M_1\right]\right)$, $u_1^- \ll u_0 \ll u_1^+$, where

$$u \ll v \Leftrightarrow w \stackrel{\Delta}{=} u - v \in \overset{\circ}{P} \stackrel{\Delta}{=} \left\{ w \in C_0^1\left(\overline{\Omega}\right) | w \geqslant 0, \frac{\partial w}{\partial v}|_{\partial\Omega} < 0 \right\}$$

and v is the outward normal direction. From (f_2) we know that $u_0 \neq 0$. Now we claim that

$$C_q\left(\widetilde{J}, u_0\right) \cong \delta_{q1} G.$$

Now we only need to check that $\widetilde{J}$ satisfies all the assumptions of Theorem 2.7:

(1) $\widetilde{J} \in C^{2-0}(E,R)$, $E = H_0^1(\Omega)$;

(2) $\widetilde{J}\,' \in C^1(D,E)$, where D is a neighborhood of u_0 in $X \stackrel{\Delta}{=} C_0^1\left(\overline{\Omega}\right)$ and $A = d^2 f(u_0)$ is a bounded linear operator from $C_0^1\left(\overline{\Omega}\right)$ to $H_0^1(\Omega)$;

(3) $\widetilde{J}$ satisfies the property (J).

In order to prove Theorem 4.1, we also need the following Lemmas 4.2, 4.3 and 4.6:

Lemma 4.2. *Under assumptions* (f_1)–(f_4), $\widetilde{J} \in C^{2-0}(E,R)$.

Proof. Since $\widetilde{f}(x,t)$ is global Lipschitz continuous in t, there exists $L > 0$ such that

$$\left|\widetilde{f}(x,t_1) - \widetilde{f}(x,t_2)\right| < L\,|t_1 - t_2|\,, \forall t_1, t_2 \in R.$$

Therefore,

$$\begin{aligned}\left\|\widetilde{J}'(u_1) - \widetilde{J}'(u_2)\right\|_E &= \left\|u_1 - u_2 - K\widetilde{f}(x,u_1) + K\widetilde{f}(x,u_2)\right\|_E \\ &\leqslant \|u_1 - u_2\|_E + \left\|K\widetilde{f}(x,u_1) - K\widetilde{f}(x,u_2)\right\|_E\end{aligned}$$

$$= \|u_1 - u_2\|_E + \left\| \widetilde{f}(x, u_1) - \widetilde{f}(x, u_2) \right\|_{L^2}$$
$$\leqslant \|u_1 - u_2\|_E + L \|u_1 - u_2\|_{L^2}$$
$$\leqslant C \|u_1 - u_2\|_E .$$

The proof is complete. □

Lemma 4.3. *Under assumptions* $(f_1)-(f_4)$, $\widetilde{J}\,' \in C^1(D, E)$, *where D is neighborhood of* u_0 *in X.*

Proof. Since $u_0 \neq 0$ and u_0 is a solution of (4.1), by the partial regularity of the zero set of the solution of linear and super-linear elliptic equations (see [1]), u_0 only vanishes on a set of measure zero, say Ω_0, and it is sufficient to show that

$$\lim_{\|v\|_X \to 0} \left\langle K\widetilde{f}(x, u_0 + v) - K\widetilde{f}(x, u_0), w \right\rangle_E = \left(\int_{\Omega \backslash \Omega_0} \widetilde{f}'(x, u_0)\, vw\, dx \right)^{\frac{1}{2}}. \quad (4.3)$$

Note that $\widetilde{f}\,'(x, u_0)$ make sense in $\Omega \backslash \Omega_0$ since

$$\left| \widetilde{f}(x, u_0 + v) - \widetilde{f}(x, u_0) \right| < C |v(x)|, \, v(x) \in L^2(\Omega \backslash \Omega_0).$$

By the Lebesgue dominated convergence theorem we have

$$\lim_{\|v\|_X \to 0} \left\langle K\widetilde{f}(x, u_0 + v) - K\widetilde{f}(x, u_0), w \right\rangle_E$$
$$= \lim_{\|v\|_X \to 0} \int_{\Omega \backslash \Omega_0} \left(\widetilde{f}(x, u_0 + v) - \widetilde{f}(x, u_0) \right) w\, dx$$
$$= \int_{\Omega \backslash \Omega_0} \widetilde{f}'(x, u_0)\, vw\, dx. \quad (4.4)$$

To show that $\widetilde{J}\,'$ is continuous, let $u_n \to u_0$ in X. It is easy to see that

$$\left\| \widetilde{J}'(u_n) - \widetilde{J}'(u_0) \right\|_E \leqslant \left\| K\left(\widetilde{f}'(x, u_n) - \widetilde{f}'(x, u_0) \right) \right\|_E + \|u_n - u_0\|_E$$
$$= \left(\int_{\Omega \backslash (\Omega_0 \cup \Omega_n)} \left| \widetilde{f}'(x, u_n) - \widetilde{f}'(x, u_0) \right|^2 \right)^{\frac{1}{2}}$$
$$+ \|u_n - u_0\|_E \to 0, \text{ as } n \to +\infty,$$

where $\Omega_n = \{u_n(x) = 0\}$ is a set of measure zero since $u_n \to u_0$ in $C_0^1\left(\overline{\Omega}\right)$.

The proof is complete. □

Remark 4.4. Under the following assumption: there exists a monotone increasing function $M : R_+^1 \to R_+^1$ and constant $C_2 > 0$ such that

$$M(r) \leqslant C_2 \left(1 + r^{\alpha-1}\right), r \geqslant 0$$

and

$$\left|f(x,t) - f\left(x,t'\right)\right| \leqslant M(r) \left|t - t'\right| \quad \text{as} \quad t, t' \in [-r, r].$$

Lemma 4.2 is still true for $J(u)$ (see [3]).

Remark 4.5. Under assumption (f_5) Lemmas 4.2 and 4.3 are true for $J(u)$.

Lemma 4.6. *Under assumption* (f_5), *or* f *is Lipschitz continuous,* $J(u)$ *satisfies the property* (J).

Proof. From (f_5) J is C^{2-0} on $H_0^1(\Omega)$ and

$$J'(u) = u - Kf(x,u),$$

where $K = (-\Delta)^{-1}$ is an operator with the Dirichlet boundary condition. We may choose $\delta > 0$ such that

$$\alpha < \delta + \frac{n+2}{n-2}\left(1 - \delta\right). \tag{4.5}$$

Define $q_0 = \frac{2n}{n-2}$ $(n \geqslant 3)$ and

$$\frac{1}{q_{i+1}} = \frac{\alpha}{q_i} - \frac{2}{n}, i = 0, 1, 2, \ldots .$$

From (4.5) we have an integer N such that

$$q_0 < q_1 < q_2 < \cdots < q_N, q_N > \alpha n.$$

Denote $P_i = \frac{q_i}{\alpha}$, $E_{i+1} = W^2_{p_{i+1},0}(\Omega)$, $X_i = L^{p_i}(\Omega)$, $i = 0, 1, \ldots, N-1$ and $E_0 = L^{q_0}(\Omega)$. By applying the embedding theorem and assumption (f_5) we have

$$L^{q_i} \overset{f}{\hookrightarrow} L^{p_i} \overset{K}{\hookrightarrow} W^2_{p_i,0}(\Omega) \hookrightarrow L^{q_{i+1}}$$

and

$$E \overset{\Delta}{=} H_0^1 \hookrightarrow E_0, X \overset{\Delta}{=} E_N = W^2_{p_N,0}(\Omega).$$

Thus, K, f satisfy assumptions(1)–(3) of property (J). □

Remark 4.7. $\widetilde{J}$ satisfies the property (J).
This is because $\widetilde{f}$ is Lipschitz continuous.

Now we continue the proof of Theorem 4.1. We have checked that $\widetilde{J}$ satisfies all conditions of Theorem 2.7. Therefore,

$$C_q\left(\widetilde{J}, u_0\right) \cong \delta_{q1} G.$$

From (f_2) we have that

$$C_q\left(\widetilde{J}, 0\right) \cong \delta_{qd_i} G,$$

where d_i is the dimension of the subspace N_i spanned by the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_i$. This implies that $u_0 \neq 0$ and $\mathrm{ind}\left(d\widetilde{J}, 0\right) = (-1)^{d_i}$. If $\widetilde{J}$ only has three critical points in $[M_2, M_1]$, from Theorem 2.7 and Corollary 3.3, we have

$$\begin{aligned} 1 &= \deg\left(\widetilde{J}, [M_2, M_1], 0\right) \\ &= \mathrm{ind}\left(d\widetilde{J}, 0\right) + \mathrm{ind}\left(d\widetilde{J}, u_0\right) + \mathrm{ind}\left(d\widetilde{J}, u_1^+\right) + \mathrm{ind}\left(d\widetilde{J}, u_1^-\right) \\ &= (-1)^{d_i} + (-1) + 1 + 1 \end{aligned}$$

a contradiction! The proof of Theorem 4.1 is complete. □

Now we consider problem (4.1) with jumping nonlinearities both at zero and infinity. We make more assumptions on $f(x, t)$.

(f_6) $\lim_{t\to+\infty} \frac{f(x,t)}{t} = b_\infty$, $\lim_{t\to-\infty} \frac{f(x,t)}{t} = a_\infty$ uniformly in $x \in \overline{\Omega}$;

(f_7) $a_\infty > \lambda_2$, $b_\infty > \lambda_2$, $(a_\infty, b_\infty) \in \widehat{A}$, where $\widehat{A}$ was given in assumption(f_2);

(f_8) $\exists \theta > 2$ and $M > 0$ such that

$$\theta F(x, t) \leqslant t f(x, t), \forall x \in \Omega \quad \text{for} \quad |t| \geqslant M.$$

Theorem 4.8. *Under assumptions* (f_1)–(f_4), (f_6), (f_7), (4.1) *has at least seven nontrivial solutions.*

Theorem 4.9. *Under assumptions* (f_1)–(f_4), (f_5), (f_8), (4.1) *has at least seven nontrivial solutions.*

Since we have Theorem 2.7 and Corollary 3.3, the argument is similar to [5].

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