Note on Frobenius extensions and restricted Lie superalgebras

Rolf Farnsteiner
Department of Mathematics, University of Wisconsin, Milwaukee, WI 53201, USA

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Abstract
This article extends techniques concerning Frobenius extensions in order to study enveloping algebras of restricted Lie superalgebras. In particular, we determine those restricted Lie superalgebras whose enveloping algebras are of finite representation type. In contrast to the non-graded case these algebras are not necessarily serial.

1. Introduction
Recent work on representations of modular Lie algebras [11,12] and Lie superalgebras [2] has elicited new applications of the theory of Frobenius extensions. In this note we extend the general theory in order to study enveloping algebras of restricted Lie superalgebras.

Motivated by the formal similarities of results of [11,12], we consider in Section 2 the question when the extension defined by factor algebras of a Frobenius extension \( R : S \) is Frobenius. Our main result, which can be considered complementary to those by Nakayama and Tsuzuku [19,20], displays a class of ideals of \( R \) for which this is the case.

In Section 3 we employ this approach in order to show that restricted Lie superalgebras give rise to Frobenius extensions. The main emphasis is hereby the computation of the Nakayama automorphism of the relevant Frobenius algebras. The importance of these automorphisms primarily rests on their significance for representation theory: given a Frobenius algebra \( A \) with Nakayama automorphism \( \mu \), the Auslander–Reiten translate \( D\text{Tr} \) is the composite of the square of the Heller operator \( \Omega \) with the operator...
obtained by twisting the action of a given $A$-module by the inverse of $\mu$. Thus, by identifying $\mu$, Theorem 3.1, for example, shows that the notions of $\Omega$-periodicity and $DTr$-periodicity coincide for modules of restricted Lie superalgebras.

In generalization of [22,13,10], we classify in the concluding section those restricted Lie superalgebras whose enveloping algebras have finite representation type. By using Theorem 3.1, we show that, in contrast to the non-graded case, these algebras are not necessarily Nakayama algebras.

2. Extensions of factor algebras

Throughout this paper $F$ is assumed to be a field. Let $S$ be an $F$-algebra. For an $S$-module $M$ and an automorphism $\alpha \in Aut(S)$, we let $_{z}M$ denote the $S$-module with underlying $F$-space $M$ and operation defined via $s θ m := \alpha(s)m \ \forall s \in S$ and $m \in M$.

An extension $R : S$ of $F$-algebras together with an automorphism $\alpha \in Aut(S)$ is called an $\alpha$-Frobenius extension if

(a) $R$ is a finitely generated, projective left $S$-module and

(b) there exists an isomorphism $\phi : R \rightarrow Hom_{S}(R, \ _{z}S)$ of $(R, S)$-bimodules.

The map $\pi := \alpha^{-1} \circ \phi(1)$ is customarily referred to as the Frobenius homomorphism of the extension $R : S$. Note that $\pi(rs) = s\pi(r)$ and $\pi(rs) = \pi(r)\alpha^{-1}(s) \ \forall r \in R, s \in S$.

According to [2, (1.1)] every Frobenius extension with Frobenius homomorphism $\pi : R \rightarrow S$ admits a so-called dual projective pair $(\{x_1, \ldots, x_n\}; \{y_1, \ldots, y_n\})$ satisfying

$$r = \sum_{i=1}^{n} y_i (\alpha \circ \pi)(x_i r) = \sum_{i=1}^{n} \pi(r y_i) x_i \ \forall r \in R.$$ 

Let $R : S$ be an $\alpha$-Frobenius extension, $I \subset R$ a two-sided ideal. In the sequel we shall be concerned with the question when $R/I : S/(S \cap I)$ is a Frobenius extension. Our solution to this problem, which was also studied by Nakayama and Tsuzuku in [20, Section 9] from a different point of view, provides a uniform treatment of recent results (cf. [11,12]) pertaining to enveloping algebras of modular Lie algebras. A new application will be given in Section 3. We begin by collecting some technical properties that characterize the ideals we shall be interested in.

Lemma 2.1. Let $\pi : R \rightarrow S$ be a Frobenius homomorphism and assume that (a) $\pi(I) \subset I \cap S$ and (b) $\alpha(I \cap S) = I \cap S$.

Then the following identities hold:

1. $\{r \in R; \pi(tr) \in I \cap S \ \forall t \in R\} = I = \{r \in R; \pi(rt) \in I \cap S \ \forall t \in R\},$
2. $I = R(I \cap S) = (I \cap S)R.$

Proof. Let $(\{x_1, \ldots, x_n\}; \{y_1, \ldots, y_n\})$ be a dual projective pair relative to $\pi$, i.e.

$$r = \sum_{i=1}^{n} y_i (\alpha \circ \pi)(x_i r) = \sum_{i=1}^{n} \pi(r y_i) x_i \ \forall r \in R.$$
The left-hand equation in conjunction with conditions (a) and (b) readily implies that \( I = R(I \cap S) \). Moreover, if \( \pi(tr) \in I \cap S \ \forall t \in R \), we obtain \( r \in R(I \cap S) - I \). Consequently, \( \{ r \in R; \ \pi(tr) \in I \cap S \ \forall t \in R \} = I \). By the same token, condition (a) implies \( I = (I \cap S)R \) and \( I = \{ r \in R; \ \pi(rt) \in I \cap S \ \forall t \in R \} \). 

Suppose that \( \pi(I) \subseteq I \cap S \) and \( \pi(I \cap S) = I \cap S \). If \( R : S \) is also a \( \beta \)-Frobenius extension with Frobenius homomorphism \( \pi \), then \cite[Proposition 21]{20} guarantees the existence of an element \( u \in R \) such that

(a) \( \beta \circ \pi^{-1}(s) = usu^{-1} \ \forall s \in S \), and

(b) \( \pi(r) = \pi(ru) \ \forall r \in R \).

Consequently, \( \pi(I) \subseteq \pi(I) \subseteq I \cap S \) and \( \beta(I \cap S) = \beta(\pi^{-1}(I \cap S)) = u(I \cap S)u^{-1} = I \cap S \).

As a result, conditions (a) and (b) are intrinsic properties of the ideal \( I \).

**Definition.** Let \( R : S \) be an \( x \)-Frobenius extension, \( I \subseteq R \) an ideal. We say that \( I \) is **admissible** if

(a) there exists a Frobenius homomorphism \( \pi : R \rightarrow S \) such that \( \pi(I) \subseteq I \cap S \), and

(b) \( \alpha(I \cap S) = I \cap S \).

For certain Frobenius extensions admissible ideals possess a simple characterization. Recall that a finitely generated projective \( S \)-module \( M \) is an **\( S \)-progenerator** if there exist \( m_1, \ldots, m_k \in M \) and \( f_1, \ldots, f_k \in \text{Hom}_S(M, S) \) such that \( \sum_{i=1}^{k} f_i(m_i) = 1 \).

**Proposition 2.2.** Suppose that \( R : S \) is an \( x \)-Frobenius extension such that \( R \) is an \( S \)-progenerator.

1. Let \( I \subseteq R \) be an ideal. Then \( I \) is admissible if and only if \( I = R(I \cap S) = (I \cap S)R \).

2. If \( I \) and \( J \) are admissible ideals, so is \( IJ \).

3. If \( (I_q)_{q \in Q} \) is a family of admissible ideals, then \( \sum_{q \in Q} I_q \) is admissible.

4. If \( I \subseteq R \) is an ideal, then there exists a unique maximal admissible ideal \( \mathcal{A}(I) \) that is contained in \( I \).

**Proof.** (1) Necessity follows directly from (2.1). Suppose conversely that \( I = R(I \cap S) = (I \cap S)R \). Let \( \pi : R \rightarrow S \) be a Frobenius homomorphism of the extension \( R : S \). Then we have

\[
(\alpha \circ \pi)(I) = (\alpha \circ \pi)(R(I \cap S)) \subseteq (\alpha \circ \pi)(R)(I \cap S) \subseteq S(I \cap S) \subseteq I \cap S,
\]
as well as

\[
\pi(I) = \pi((I \cap S)R) \subseteq (I \cap S) \pi(R) \subseteq (I \cap S)S \subseteq I \cap S.
\]

Since \( R \) is an \( S \)-progenerator, there exist \( r_1, \ldots, r_n \in R \) and \( f_1, \ldots, f_n \in \text{Hom}_S(R, S) \) such that \( I = \sum_{i=1}^{n} f_i(r_i) \). Let \( t_1, \ldots, t_n \in R \) be elements with \( f_i = t_i \cdot \pi \). \( 1 \leq i \leq n \) and set \( c := \sum_{i=1}^{n} r_i t_i \). Then \( \pi(c) = \sum_{i=1}^{n} (t_i \cdot \pi)(r_i) = \sum_{i=1}^{n} f_i(r_i) = 1 \).

Let \( s \) be an element of \( I \cap S \). Since \( sc \) and \( cs \) belong to \( I \), the above inclusions yield \( \alpha(s) = \alpha(s \pi(c)) = (\alpha \circ \pi)(sc) \in I \cap S \) and \( \alpha^{-1}(s) = \pi(c) \alpha^{-1}(s) = \pi(cs) \in I \cap S \). This proves that \( \alpha(I \cap S) = I \cap S \). Hence, \( I \) is an admissible ideal.
(2) Since \( I \) and \( J \) are admissible, it follows that
\[
IJ = R[(I \cap S)R](J \cap S) = R[R(I \cap S)](J \cap S) \subseteq R(I \cap S)(J \cap S) \subseteq R((IJ) \cap S) \subseteq IJ.
\]
The other identity is obtained analogously. We may now apply (1) to see that \( IJ \) is admissible.

(3) We have \( \sum_{q \in Q} I_q = \sum_{q \in Q} R(I_q \cap S) \subseteq R(\sum_{q \in Q} I_q) \cap S) \subseteq \sum_{q \in Q} I_q. \) Since the identity \( \sum_{q \in Q}(I_q \cap S))R = \sum_{q \in Q} I_q \) also holds, part (1) yields the admissibility of \( I \).

(4) In view of (3) the ideal \( \mathcal{A}(I) \) may be defined to be the sum of all admissible ideals that are contained in \( I \). □

**Remark.** Note that the condition concerning \( R \) is automatically fulfilled if \( R : S \) is a free \( \alpha \)-Frobenius extension.

**Theorem 2.3.** Let \( R : S \) be an \( \alpha \)-Frobenius extension with Frobenius homomorphism \( \pi, I \subset R \) an admissible ideal. Then \( R/I : S/(S \cap I) \) is a \( \beta \)-Frobenius extension with Frobenius homomorphism \( \tilde{\pi} \), where \( \beta(s + S \cap I) = \alpha(s) + S \cap I \) \( \forall s \in S \) and \( \tilde{\pi}(r + I) = \pi(r) + S \cap I \) \( \forall r \in R \).

**Proof.** Since \( \alpha \) stabilizes \( S \cap I \), \( \beta \) is a well-defined automorphism of the algebra \( S/(S \cap I) \). Evidently, \( R/I \) is a finitely generated \( S/(S \cap I) \)-module. Owing to (2.1) the canonical projection \( p : R \to R/I \) induces a natural equivalence \( \text{Hom}_{S/(S \cap I)}(R/I, \cdot) \cong \text{Hom}_{S}(R, \cdot) \) on the category of \( S/(S \cap I) \)-modules. Consequently, \( R/I \) is a projective \( S/(S \cap I) \)-module.

Let \( q : S \to S/(S \cap I) \) denote the natural projection. Since \( (\alpha \circ \pi)(I) \subset I \cap S \) there exists an \( F \)-linear map \( \lambda : R/I \to S/(S \cap I) \) such that \( \lambda \circ p = q \circ \alpha \circ \pi \). Given \( s \in S \) and \( r \in R \) we obtain, writing \( \tilde{\pi} := \alpha \circ \pi \) for ease of notation,
\[
\lambda(q(s) \cdot p(r)) = \lambda(p(sr)) = q(\tilde{\pi}(sr)) = q(\tilde{\pi}(s))q(\tilde{\pi}(r)) = \beta(q(s))\tilde{\lambda}(p(r)),
\]
as well as
\[
\lambda(p(r) \cdot q(s)) = \lambda(p(rs)) = q(\tilde{\pi}(rs)) = q(\tilde{\pi}(r))q(s) = \tilde{\lambda}(p(r))q(s).
\]
Hence \( \tilde{\lambda} \in \text{Hom}_{S/(S \cap I)}(R/I, \beta(S/(S \cap I))) \) and
\[
\Theta : R/I \to \text{Hom}_{S/(S \cap I)}(R/I, \beta(S/(S \cap I))), \quad \Theta(r) = r \cdot \tilde{\lambda}
\]
is a homomorphism of \( (R/I, S/(S \cap I)) \)-bimodules.

Let \( p(r) \) be an element of \( \ker \Theta \). Then \( 0 = (p(r) \cdot \tilde{\lambda})(p(t)) = q((\alpha \circ \pi)(tr)) \forall t \in R, \) whence \( (\alpha \circ \pi)(r) \in I \cap S \) \( \forall r \in R \). Lemma 2.1 now shows that \( r \in I \), so that \( p(r) = 0 \).

If \( \phi \) is an arbitrary element of \( \text{Hom}_{S/(S \cap I)}(R/I, \beta(S/(S \cap I))) \), then \( \phi \circ p \) can be considered an element of \( \text{Hom}_{S}(R, \beta(S/(S \cap I))) \). Since \( R \) is a projective left \( S \)-module,
there exists an $S$-linear map $\psi : R \to S$ such that $q \circ \psi = \varphi \circ p$. Let $r$ be an element of $R$ with $\psi = r \cdot (\alpha \circ \pi)$. For $x \in R$ we thus obtain

$$\varphi(p(x)) = q(\psi(x)) = q((\alpha \circ \pi)(xr)) = \lambda(p(xr)) = (p(r) \cdot \lambda)(p(x)),$$

proving that $\varphi = \Theta(p(r))$.

As a result $R/I : S/(S \cap I)$ is a $\beta$-Frobenius extension with Frobenius homomorphism $\pi = \beta^{-1} \circ \lambda$. 

Frobenius extensions defined by factor algebras can of course also arise when the ideal involved is not admissible. In fact, in [20, Section 9] Nakayama and Tsuzuku studied this problem for ideals $I$ satisfying $I \cap S = (0)$. Their approach generalizes the following situation not covered by (2.3). Let $A$ be a Frobenius algebra over an algebraically closed field $F$. If $\mathcal{H}$ is a maximal ideal of $A$, then $\mathcal{H}$ is not admissible, yet $A/\mathcal{H}$, being isomorphic to a full matrix algebra over $F$, is a symmetric algebra.

We briefly digress to illustrate how admissible ideals arise in the contexts of finite groups and restricted Lie algebras. Let $R$ be a Hopf algebra with comultiplication $\Delta$, antipode $\eta$, and counit $\varepsilon : R \to F$. The algebra $R$ acts on itself from the left via $r \ast x := \sum_{(r)} r_{(1)} x \eta(r_{(2)})$. The $R$-submodules of $R$ with respect to this action will be referred to as invariant subspaces.

**Lemma 2.4.** Let $R$ be a cocommutative Hopf algebra, $S \subseteq R$ an invariant subalgebra. If $I$ is an ideal of $R$, then $R(I \cap S) = (I \cap S)R$.

**Proof.** Let $r$ be an element of $R$, $s \in I \cap S$. Then we obtain

$$rs = \sum_{(r)} r_{(1)} \varepsilon(r_{(2)}) s = \sum_{(r)} r_{(1)} s \varepsilon(r_{(2)}) = \sum_{(r)} r_{(1)} s \eta(r_{(2)}) r_{(3)}$$

$$= \sum_{(r)} (r_{(1)} \ast s)r_{(2)} \in (I \cap S)R.$$

Consequently, $R(I \cap S) \subseteq (I \cap S)R$. We also have, observing the cocommutativity of $R$,

$$sr = \sum_{(r)} \varepsilon(r_{(1)}) sr_{(2)} = \sum_{(r)} r_{(1)} \eta(r_{(2)}) sr_{(3)} = \sum_{(r)} r_{(1)} \eta(r_{(3)}) s \eta^{2}(r_{(2)})$$

$$= \sum_{(r)} r_{(1)} (\eta(r_{(2)}) \ast s) \in R(I \cap S). \qed$$

A consecutive application of (2.4) and (2.2) immediately yields:

**Corollary 2.5.** Let $R : S$ be a free $\alpha$-Frobenius extension, where $R$ is a cocommutative Hopf algebra and $S$ is an invariant subalgebra. If $I \subseteq R$ is an ideal, then $\mathcal{A}(I) = R(I \cap S)$.

Let $R : S$ be an $\alpha$-Frobenius extension. If $M$ and $N$ are $R$-modules, and $f : M \to N$ is $S$-linear, then the $R$-linear map $Tr_{[R:S]}(f) : M \to N$, $Tr_{[R:S]}(f)(m) := \sum_{i=1}^{n} v_{i} f(x,m)$ is referred to as the trace of $f$. 


We let $C_R(S)_a := \{ a \in R; \text{ as } = a(s)s \forall s \in S \}$ denote the $x$-centralizer of $S$ in $R$. For every $a \in C_R(S)_a$ the map $\ell_a : R \to R; \ell_a(r) = ar$ is $S$-linear, and it follows that $c_{[R,S]}(a) := Tr_{[R,S]}(\ell_a)(1)$ belongs to the center $Z(R)$ of $R$. We shall call the map $c_{[R,S]} : C_R(S)_a \to Z(R)$ the Ikeda-Gaschütz operator of $R : S$. If $x = id_S$, then $c_{[R,S]}(1)$ will be referred to as the Casimir element of the extension $R : S$.

Following Hirata and Sugano [15] we call the extension $R : S$ separable if the map $m : R \otimes_S R \to R; m(x \otimes y) = xy$ is a split surjective map of $R$-bimodules. The following result slightly generalizes [15, (2.18)]. We shall leave the proof, which is similar to the one given in [15], to the interested reader.

Theorem 2.6. Let $R : S$ be an $x$-Frobenius extension. Then $R$ is a separable extension of $S$ if and only if $1 \in c_{[R,S]}(C_R(S)_a)$.

Corollary 2.7. Let $R : S$ be a separable $x$-Frobenius extension, $I \subset R$ an admissible ideal. Then the extension $R/I : S/(S \cap I)$ is separable.

Proof. We adopt the notation of the proof of (2.6) and let ($\{x_1, \ldots , x_n\}, \{ y_1, \ldots , y_n\}$) be a dual projective pair relative to the Frobenius homomorphism $\pi$. Then ($\{ p(x_1), \ldots , p(x_n)\}; \{ p(y_1), \ldots , p(y_n)\}$) is a dual projective pair relative to $\hat{\pi}$. Owing to (2.6) there exists an element $a \in C_R(S)_a$ such that $c_{[R,S]}(a) = 1$. Thus, $p(a) \in C_{[R/I,S/(S \cap I)\beta]}(S/(S \cap I))$ and

$$c_{[R/I,S/(S \cap I)]}(p(a)) = \sum_{i=1}^{n} p(y_i)p(a)p(x_i) = p(c_{[R,S]}(a)) = 1,$$

so that (2.6) yields the separability of the extension $R/I : S/(S \cap I)$.

3. Restricted Lie superalgebras

In the sequel the base field $F$ is assumed to be algebraically closed and of characteristic $p \geq 5$. Let $L = L_0 \oplus L_1$ be a finite-dimensional restricted Lie superalgebra with ordinary universal enveloping algebra $\mathcal{U}(L)$. The restricted enveloping algebra $u(L)$ of $L$ is, by definition, the factor algebra of $\mathcal{U}(L)$ by the ideal $I$ that is generated by the set $J := \{ x^p - x^{[p]} : x \in L_0 \}$. These algebras are known to be Frobenius (cf. [1, p. 163: 3, Corollary 3]), and it will be our first goal to determine the Nakayama automorphism of $u(L)$. Recall that the Nakayama automorphism $\mu$ of a Frobenius algebra $A$ with Frobenius homomorphism $\lambda : A \to F$ is defined by means of

$$\lambda(xy) = \lambda(y \mu(x)) \forall x, y \in A.$$

Let $K = K_0 \oplus K_1$ be a subalgebra of $L$. We write

$$L_0 = \bigoplus_{i=1}^{n} F e_i, \quad L_1 = \bigoplus_{j=1}^{m} F f_j$$
and adopt the multi-index notation of [23, p.51]. According to the Theorem of Poincaré–Birkhoff–Witt [1, (III.2.5)], \( \mathfrak{u}(L) \) is a free left \( \mathfrak{u}(K) \)-module with basis

\[
\{ e^{a f^b}; \ 0 \leq a \leq \tau_0; \ 0 \leq b \leq \tau_1 \},
\]

where \( \tau_0 \) denotes the \( n \)-tuple \((p - 1, \ldots, p - 1)\) and \( \tau_1 \) designates the \( m \)-tuple \((1, \ldots, 1)\). If \( X \subseteq L \) is an \( L_0 \)-invariant subspace, then \( \text{ad}_X : L_0 \to gL(X); \ x \mapsto [x, \cdot] \) is the adjoint representation of \( L_0 \) on \( X \).

**Theorem 3.1.** Let \( L \) be a finite dimensional restricted Lie superalgebra with \( \text{dim}_F L_1 = n \). The algebra \( \mathfrak{u}(L) \) is a Frobenius algebra with Nakayama automorphism \( \mu \) given by \( \mu(x) = x + tr(ad_{L_1}(x))1 - tr(ad_{L_0}(x))1 \ \forall x \in L_0 \) and \( \mu(x) = (-1)^{\tau_1 - 1}x \ \forall x \in L_1 \).

**Proof.** Owing to [2, (2.2)] the extension \( \mathcal{U}(L) : \mathcal{U}(L_0) \) is a free \( \mathfrak{a} \)-Frobenius extension with winding automorphism \( \pi \) defined by \( \pi(x) = x + tr(ad_{L_1}(x))1 \ \forall x \in L_0 \). Thus, there exists a Frobenius homomorphism \( \pi : \mathcal{U}(L) \to \mathcal{U}(L_0) \) such that

\[
\pi(ux) = \pi(u)x^{-1}(x) \ \forall x \in \mathcal{U}(L_0), \ u \in \mathcal{U}(L).
\]

We let \( I \) be the ideal mentioned in our introductory remarks. Then we have \( I = \mathcal{U}(L)J = J \mathcal{U}(L) \), so that \( I = \mathcal{U}(L)(\mathcal{U}(L_0) \cap I) = (\mathcal{U}(L_0) \cap I) \mathcal{U}(L) \). Thus, a consecutive application of (2.2) and (2.3) identifies \( u(L) : u(L_0) \) as a free Frobenius extension with winding automorphism \( \pi \), satisfying \( \pi(x) = x + tr(ad_{L_1}(x))1 \ \forall x \in L_0 \). Owing to [2, (1.3)] a Frobenius homomorphism \( \omega \) of the Frobenius algebra \( u(L) \) can be written as a composite \( \omega = \gamma \circ \tilde{\pi} \), where \( \gamma \) and \( \tilde{\pi} \) are the Frobenius homomorphisms of \( u(L_0) \) (cf. [5]) and \( u(L) : u(L_0) \), respectively. It is well-known (cf. for example [9]) that the Nakayama automorphism \( \nu \) of \( u(L_0) \) is given by \( \nu(x) = x - tr(ad_{L_0}(x))1 \ \forall x \in L_0 \). For \( s \in u(L_0) \) and \( r \in u(L) \) we thus have

\[
\omega(sr) = \gamma(s\tilde{\pi}(r)) = \gamma(\tilde{\pi}(r)v(s)) = \gamma(\tilde{\pi}(r(x \circ \nu))(s)) = \omega(r(x \circ \nu)(s)),
\]

proving that the Nakayama automorphism \( \mu \) of \( u(L) \) satisfies

\[
\mu(x) = x + tr(ad_{L_1}(x))1 - tr(ad_{L_0}(x))1 \ \forall x \in L_0.
\]

In order to verify the formula for the odd elements, we have to apply more computational techniques. Let \( \{ e_1, \ldots, e_m \} \) and \( \{ f_1, \ldots, f_n \} \) be bases of \( L_0 \) and \( L_1 \), respectively. (Thus, we specialize \( K = (0) \) in our general conventions). For a natural number \( k \), we put \( u(L)_{(k)} := \sum_{|r| \leq k, r \leq \tau_1} u(L_0)f^r \) as well as \( u(L_0)_{(k)} := \sum_{|a| \leq k, a \leq \tau_0} F e^a \). We shall establish the following claims :

(a) \( xe^a \equiv e^a x \mod(u(L_0)_{(|a|-1)L_1}) \ \forall x \in L_1 \),
(b) \( xf^r \equiv (-1)^{|r|} f^r x \mod(u(L)_{(|r|-1)}) \ \forall x \in L_1 \).
(c) \( \omega(u(L_0)_{(|r|-1)L_1}f^r) = 0 \), \( 0 \leq r \leq \tau_1 \).
The proof of (a) proceeds by induction on $|a|$. Let $0 \leq a \leq \tau_0$ be given and put $j := \max\{i : a_i \neq 0\}$. Then we have, observing the inductive hypothesis,
\[
x e^a = x e^{a - \varepsilon_j} e_j = e^{a - \varepsilon_j} x e_j + \sum_{|b| \leq |a| - 2} \sum_{i=1}^n \beta_{b,i} e^b f_i e_j \\
\equiv e^a x + e^{a - \varepsilon_j} \left( \sum_{i=1}^n \alpha_i f_i \right) + \sum_{|b| \leq |a| - 2} \sum_{i=1}^n \beta_{b,i} e^b f_i e_j \\
\equiv e^a x \mod(u(L_0)_{|a|-1} L_1).
\]
The second congruence is a direct consequence of [2, (2.1)]. The same result also shows that
\[
f_j f^r \equiv \pm f^{r + \varepsilon_j} \mod(u(L)_{|r|-1}),
\]
where $f^{r + \varepsilon_j}$ is to be interpreted to be 0 for $r + \varepsilon_j \not\leq \tau_1$. In order to verify (c), we note that $\pi(f^r) = \delta_{r, \tau_1}$ and $\gamma(e^a) = \delta_{a, \tau_0}$. For $a < \tau_0$ we thus obtain
\[
\omega(e^a f_j f^r) = \gamma(e^a \pi(f_j f^r)) = +\gamma(e^a \pi(f^{r + \varepsilon_j})) = +\gamma(e^a \delta_{r, \tau_1}) = 0.
\]
By combining (a), (b) and (c), we therefore conclude the validity of the following identities for $x \in L_1$:
\[
\omega(x e^a f^r) = (\gamma \circ \pi)(x e^a f^r) = \gamma(e^a \pi(x f^r)) = \gamma(e^a \pi((-1)^{|r|} f^r x))
\]
\[
= \gamma(e^a \pi((-1)^{a-1} f^r x)) = \omega(e^a f^r (-1)^{a-1} x).
\]
Consequently, $\mu(x) = (-1)^{a-1} x \ \forall x \in L_1$. □

As a first application we provide the following criteria for the symmetry and unimodularity of $u(L)$. Recall that an augmented Frobenius algebra $(A, \varepsilon)$ is referred to as
unimodular
if every integral of $A$ is a two-sided integral. In what follows, $\varepsilon : u(L) \to F$ denotes the canonical augmentation of $u(L)$.

**Corollary 3.2.** Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra. Then the following statements hold:

1. If $\dim F L_1 \equiv 1 \mod(2)$ and $\tr(ad_{L_0}(x)) = \tr(ad_{L_1}(x)) \ \forall x \in L_0$, then $u(L)$ is symmetric.

2. $u(L)$ is unimodular if and only if $\tr(ad_{L_0}(x)) = \tr(ad_{L_1}(x)) \ \forall x \in L_0$.

**Proof.** (1) Owing to (3.1) the given conditions imply that $\mu(x) = x \ \forall x \in L$. Consequently, $\mu = id_{u(L)}$ so that $u(L)$ is symmetric.

(2) Suppose that $u(L)$ is unimodular. According to [9, (1.1)] there exists a Nakayama automorphism $v : u(L) \to u(L)$ such that $\varepsilon \circ v = \varepsilon$. In virtue of [21, Satz 1] this condition holds for $\mu$ as well, and we therefore obtain
\[
0 = \varepsilon(x) = \varepsilon(\mu(x)) = \tr(ad_{L_1}(x))1 - \tr(ad_{L_0}(x))1 \ \forall x \in L_0.
\]
The converse direction follows immediately from (3.1) and [9, (1.1)] □
For future reference we record two important consequences of (3.1). Given $i \in \{0, 1\}$ we denote the map $\mathcal{K}_0 \to gl(L_i/K_i)$ that is induced by the adjoint representation by $ad_{L_i, K_i}$.

**Corollary 3.3.** The extension $u(L) : u(K)$ is a free Frobenius extension with winding automorphism $\chi$ and Frobenius homomorphism $\pi$ given by

$$
\chi(x) = x + tr(ad_{L_i, K_i}(x))l - tr(ad_{L_o, K_o}(x))l \quad \forall x \in K_0
$$

$$
\chi(x) = (-1)^q x \quad \forall x \in K_1, \quad q = \dim_F L_1/K_1
$$

and

$$
\pi(x^a f^r) = \delta_{a, \tau_0} \delta_{r, \tau_1}, \quad 0 \leq a \leq \tau_0, \quad 0 \leq r \leq \tau_1,
$$

respectively.

**Proof.** Let $\pi_L$ and $\pi_K$ denote the Frobenius homomorphisms of $u(L)$ and $u(K)$, as constructed in the proof of (3.1). The projection $\pi : u(L) \to u(K)$ along the vector $e^a f^r$ is a homomorphism of $u(K)$-modules, which satisfies $\pi_K \circ \pi = \pi_L$. In view of [2, (1.4)] it therefore suffices to verify

$$
\pi(rs) = \pi(r)\pi^{-1}(s) \quad \forall r \in u(L), \ s \in u(K).
$$

Let $\mu_L$ and $\mu_K$ be the Nakayama automorphisms of $u(L)$ and $u(K)$, respectively. By (3.1) we have $\mu_L(u(K)) = u(K)$. Since $\chi = \mu_L \circ \mu_K^{-1}$, the desired identity now follows from the arguments given in [2, p. 414f].

**Corollary 3.4.** Let $K_0 \subset L_0$ be a $p$-ideal containing $[L_1, L_1]$. Then the extension $u(L) : u(K_0 \cap L_1)$ is separable if and only if $L_0/K_0$ is a torus.

**Proof.** Adopting the previously given notation, we see that the Frobenius homomorphism $\pi$ of the extension $u(L) : u(K)$ satisfies $\pi(e^a) = \delta_{a, \tau_0}$. Consequently, $\pi|_{u(L_0)}$ is a Frobenius homomorphism for the extension $u(L_0) : u(K_0)$ and a dual projective pair $\{x_1, \ldots, x_n\} : \{y_1, \ldots, y_n\}$ for the latter is also a dual projective pair for $u(L) : u(K)$.

Suppose that $L_0/K_0$ is a torus. Owing to [10, (2.4)] we have

$$
0 \neq c_{[u(L_0), u(K_0)]}(1) = \sum_{i=1}^n y_i x_i = c_{[u(L), u(K)]}(1).
$$

Thus, (2.6) implies the separability of the extension $u(L) : u(K)$.

To verify the converse direction, let $I := [L_1, L_1]_p$ and put $J := I \cap L_1$. According to (2.2) the ideal $u(L)J$ is admissible, and it follows directly from (2.7) that $u(L)/u(L)J \cdot u(K)/(u(L)J \cap u(K))$ is separable. Since the canonical isomorphism $u(L)/u(L)J \cong u(L_0/I)$ maps $u(K)/(u(L)J \cap u(K))$ onto $u(K_0/I)$, it follows that the extension $u(L_0/I) : u(K_0/I)$ is also separable. Consequently, [10, (2.4)] shows that $L_0/K_0 \cong (L_0/I)/(K_0/I)$ is a torus.
4. Superalgebras of finite representation type

Throughout this section we consider a finite-dimensional restricted Lie superalgebra $L = L_0 \oplus L_1$ whose base field $F$ is algebraically closed and of characteristic $p \geq 7$. We let $N(L_0)$, $\text{rad}_p(L_0)$, and $T(L_0)$ denote the largest nilpotent, $p$-nilpotent, and toral ideal of the restricted Lie algebra $(L_0, [\cdot, \cdot])$, respectively (cf. [23] for the definitions). The purpose of this section is to classify those restricted Lie superalgebras whose enveloping algebras have finite representation type.

Given an associative $F$-algebra $A$ and a finite-dimensional $A$-module $M$, we recall that the complexity $c_A(M)$ is defined to be the rate of growth of a minimal projective resolution $\mathcal{P}$ of $M$. Thus, if $\mathcal{P} := (P_n)_{n \geq 0}$, then

$$c_A(M) = \min \{ c \in \mathbb{N}_0; \exists \lambda > 0, \dim_F P_n \leq \lambda n^{c-1} \forall n \geq 1 \}.$$

We begin by noting the following basic result.

**Lemma 4.1.** Let $V$ be an $F$-vector space with Grassmann algebra $\wedge(V)$. If $c_{\wedge(V)}(F) \leq 1$, then $\dim_F V \leq 1$.

**Proof.** Suppose that $\dim_F V \geq 2$ and let $X \subset V$ be a two-dimensional subspace. Since $\wedge(V)$ is a projective $\wedge(X)$-module, it follows that $c_{\wedge(X)}(F) \leq 1$. Hence, there exists $b \in \mathbb{N}$ such that $\dim_F \text{Ext}^n_{\wedge(X)}(F,F) \leq b$. Direct computation, however, shows that $\dim_F \text{Ext}^n_{\wedge(X)}(F,F) = 2n + 1 \forall n \geq 0$. \qed

An element $x \in L_0$ is said to be toral if $x^{(p)} = x$ and $p$-nilpotent if it is annihilated by some iterate of the $p$-map. If $X \subset L_0$ is a subset, then $X_p$ denotes the $p$-subalgebra of $L_0$ that is generated by $X$.

**Theorem 4.2.** Suppose that $L_1 \neq (0)$. Then the following statements are equivalent:

1. $u(L)$ has finite representation type.
2. There exists a toral element $t_0 \in L_0$, a $p$-nilpotent element $x_0 \in L_0$, and $y_1 \in L_1$ such that $L = L_0 \oplus Fy_1$, $(F_{x_0})_p \subset [L_1, L_1]_p \subset N(L_0)$, $L_0 = N(L_0) + Ft_0$, $N(L_0) = T(L_0) \oplus (F_{x_0})_p$.

**Proof.** (1) $\Rightarrow$ (2): We proceed in several steps.

(a) $u(L_0)$ has finite representation type. Since $u(L)$ has finite representation type, Heller’s Theorem [14] yields the periodicity of the trivial $u(L)$-module $F$. Thus, $c_{u(L)}(F) \leq 1$ and as $u(L)$ is a projective $u(L_0)$-module, we also have $c_{u(L_0)}(F) \leq 1$. Consequently, [13, (2.4)] implies (a).

(b) There exists a toral element $t_0 \in L_0$ and a $p$-nilpotent element $x_0 \in L_0$ such that $L_0 = N(L_0) + Ft_0$, $N(L_0) = T(L_0) \oplus (F_{x_0})_p$. This follows directly from (a) and [10, (4.3)].

(c) Let $T := T(L_0) + Ft_0$. Then $T$ is a maximal torus of $L_0$ and there exists at most one root $\alpha$ relative to $T$. The corresponding root space $(L_0)_\alpha$ has dimension 1.
Since $N(L_0)$ is abelian, the space $I := N(L_0)^{[p]} = T(L_0) + (Fx_0^{[p]})_p$ is contained in the center $C(L_0)$ of $L_0$ and thereby in particular a $p$-ideal. We consider $\mathcal{L}_0 := L_0/I$ as well as the natural projection $\pi : L_0 \to \mathcal{L}_0$ and let $\hat{T} \subset L_0$ be a maximal torus of $L_0$ containing $T$. According to [23, (II.4.5)] $\pi(\hat{T})$ is a maximal torus of $\mathcal{L}_0$. By (b) we have $\mathcal{L}_0 = F \pi(x_0) + F \pi(t_0)$. Thus, $\pi(\hat{T}) \subset \pi(Ft_0)$, whence $\hat{T} \subset T + (Fx_0^{[p]})_p$. This implies $\hat{T} = T$.

Since $N(L_0) = C(L_0) + Fx_0$, there exists at most one one-dimensional root space $(L_0)_x$ relative to $T$.

We decompose the $T$-module $L_1$ into its weight spaces and write $L_1 = \bigoplus_{\lambda \in \mathfrak{h}} (L_1)_\lambda$, where $\mathfrak{h} \subset T^*$ is the set of weights of $L_1$ relative to $T$.

(d) Let $\beta \in \mathfrak{h} \setminus \{0, \frac{1}{2} \lambda \}$. Then $\dim F(L_1)_\beta = 1$, $[(L_1)_\beta, (L_1)_\beta] = (0)$, and $\mathfrak{h} \subset \{0, \frac{1}{2} \lambda, \beta, -\beta\}$ or $\mathfrak{h} \subset \{0, \frac{1}{2} \lambda, \beta, \lambda - \beta\}$. By choice of $\beta$ we have $[(L_1)_\beta, (L_1)_\beta] \subset (L_0)_2 = (0)$. Consequently, $(L_1)_\beta$ is a subalgebra of $L$ with enveloping algebra $\wedge (L_1)_\beta$. We therefore obtain $c_{\mathfrak{h}, (L_1)_\beta}(F) \leq c_{\mathfrak{h}, L}(F) \leq 1$, so that (4.1) yields $\dim F(L_1)_\beta = 1$.

Now let $\gamma \neq \beta$ be another element of $\mathfrak{h} \setminus \{0, \frac{1}{2} \lambda\}$. If $[(L_1)_\beta, (L_1)_\gamma] = (0)$, then $K := (L_1)_\beta \oplus (L_1)_\gamma$ is a subalgebra of $L$ with enveloping algebra $\wedge (K)$ and (4.1) yields a contradiction. In view of (c) we therefore have $\beta + \gamma \in \{0, \frac{1}{2} \lambda\}$. Thus, our claim follows if $\lambda$ is not a weight. Otherwise, note that $[(L_1)_\beta, (L_1)_\lambda] \subset (L_0)_\lambda = (0)$, so that (4.1) shows $\dim F(L_1)_\beta \oplus (L_1)_\gamma \leq 1$. Hence $-\beta$ and $\gamma - \beta$ cannot both be weights.

(e) Suppose that $\mathfrak{h} \setminus \{0, \frac{1}{2} \lambda\} \neq \emptyset$. Then there exists $\gamma \in \mathfrak{h} \setminus \{0, \frac{1}{2} \lambda\}$ such that $[(L_1)_\beta, (L_1)_\gamma] = (0)$. Let $\beta$ be an element of $\mathfrak{h} \setminus \{0, \lambda\}$. If (e) does not hold, we either have $\beta + \lambda, -\beta + \lambda \in \mathfrak{h}$ or $\beta + \lambda, 2\lambda - \beta \in \mathfrak{h}$. Using the fact that $p \geq 7$ in conjunction with (d), a case by case analysis shows that this is impossible.

(f) If $\mathfrak{h} \setminus \{0, \frac{1}{2} \lambda\} \neq \emptyset$, then $L = T(L_0) \oplus Fy_1$, where $Fy_1 = T(L_0)$ and $[y_1, y_1] = 0$. According to (e) there exists $\lambda \in \mathfrak{h} \setminus \{0, \frac{1}{2} \lambda\}$ such that $[(L_0)_\lambda, (L_1)_\lambda] = (0)$. Now let $(L_0)_\lambda = Fx_\lambda$. Since $N(L_0) = C(L_0) + Fx_0 = C(L_0) + Fx_\lambda$, it follows that $x_0 = x_\lambda + z$,

where $z \in C(L_0) \subset (L_0)_0$. This implies $[x_0, (L_1)_\lambda] \subset [z, (L_1)_\lambda] \subset (L_1)_0$. As $x_0$ operates nilpotently on $L_1$, we obtain, observing $\dim F(L_1)_\lambda = 1$, $[x_0, (L_1)_\lambda] = (0)$.

Next, we consider the restricted Lie superalgebra $\mathcal{S} := (Fx_0)_p \oplus (L_1)_\lambda$, whose enveloping algebra is isomorphic to $u((Fx_0)_p) \wedge (L_1)_\lambda$. Since $c_{\mathfrak{h}, \mathcal{S}}(F) \leq 1$, the K"unneth formula readily implies $u((Fx_0)_p) \wedge (L_1)_\lambda \cong F$, so that $x_0 = 0$. This shows that $N(L_0) = T(L_0) = L_0$.

From (d) we now obtain $\mathfrak{h} \subset \{0, \lambda, -\lambda\}$, whence

$L = T(L_0) \oplus (L_0)_{\lambda} \oplus (L_1)_{-\lambda}$.

Consequently, $[(L_1)_0, (L_1)_{\pm \lambda}] \subset (L_0)_{\pm \lambda} = (0)$. Given $x, y \in (L_0)_0$ and $z \in (L_1)_\lambda$ we therefore have

$[[x, y], z] - [x, [y, z]] + [y, [x, z]] = 0$,

proving that $[(L_1)_0, (L_1)_0] \subset \ker \lambda$. 
Next, we consider \( x \in (L_1)_\lambda \), and \( y \in (L_1)_{-\lambda} \). Then
\[
\hat{\lambda}([x, y])x = ([x, y], x) = [x, [y, x]] + [y, [x, x]] = -\hat{\lambda}([y, x])x = -\lambda([x, y])x.
\]
Hence, \( \lambda \) annihilates \([L_1, L_1]_\pm \) and we obtain for the \( p \)-ideal \( X := \ker \hat{\lambda} \) of \( L_0 \) the relations \([X, L_1] = (0)\), and \([L_1, L_1] \subset X\). Consequently, \( \mathcal{L} := L/X \) is a restricted Lie superalgebra such that \([\mathcal{L}, \mathcal{L}'] = (0)\). Since \( u(\mathcal{L}) \) has, as a factor algebra of \( u(L) \), finite representation type, we conclude from [14] that \( c_{u(\mathcal{L})}(F) \leq 1 \). Thus, we also have \( c_{\mathcal{A}(\mathcal{L'})}(F) \leq 1 \), and (4.1) yields
\[
\dim_F L_1 = \dim_F \mathcal{L}_1 \leq 1.
\]
In view of our results above we shall henceforth assume that \( \frac{1}{2} \lambda \) and 0 are the only weights of \( L_1 \) relative to \( T \).

(g) \([N(L_0), L_1] = (0)\). We first note that, on account of our present assumption, we have
\[
[(L_0)_x, L_1] \subset [(L_0)_x, (L_1)_x] + [(L_0)_x, (L_1)_0] \subset (L_1)_x + (L_1)_z = (0).
\]
As before, we write \( (L_0)_x = Fx_0 \). Since \( C(L_0) = T(L_0) + C_{(0)} \), \( Fx_0 \) it follows that
\[
x_0 = x_2 + t + \sum_{i \geq 1} \gamma_i x_0^{[p]^i}.
\]
where \( t \in T(L_0) \) and \( \gamma_i \in F \). Observing \( x(T(L_0)) = (0) \), we note that \( T(L_0) \) annihilates \( L_1 \), so that \((ad x_0)(L_1) \subset (ad x_0)^p(L_1)\). Since \( x_0 \) operates nilpotently on \( L_1 \), this entails \([x_0, L_1] = (0)\). Thus, \([F x_0, L_1] = (0)\) and \([N(L_0), L_1] = (0)\).

(h) \([L_1, L_1]_p \subset N(L_0) \) and \( \dim_F L_1 = 1 \). It follows from (g) that \([N(L_0), [L_1, L_1]] = (0)\), proving that \([L_1, L_1] \subset N(L_0) \) in \( L_0 \). By (b), this space coincides with \( N(L_0) \). Consequently, \( N(L_0) \) is an ideal of \( L \) and the enveloping algebra of \( \mathcal{L} := L/N(L_0) \), being isomorphic to \( u(L)/u(L)N(L_0) \), has finite representation type. We let \( \bar{t}_0 \) denote the image of \( t_0 \) under the canonical projection. Then \( \mathcal{L} = \bar{F} t_0 \oplus \mathcal{L}_1 \) and \([\mathcal{L}_1, \mathcal{L}_1] = (0)\). Consequently, \( \mathcal{L}_1 \) is a subalgebra of \( L \) and since \( u(\mathcal{L}) \) is projective over \( u(\mathcal{L}_1) \), it follows that \( c_{\mathcal{A}(\mathcal{L}_1)}(F) \leq 1 \). Now (4.1) yields \( 1 \leq \dim_F L_1 = \dim_F \mathcal{L}_1 \leq 1 \).

(i) \((F x_0)_p \subset [L_1, L_1]_p \). We put \( L_1 := F y_1 \) and \( v := [y_1, y_1] \). Owing to (g) and (h), \([L_1, L_1]_p = (F v)_p \) is an ideal of \( L \). Let \( \mathcal{L} := L/(F v)_p \) and consider the natural projection \( \pi : L \to \mathcal{L} \). Then \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) is a restricted Lie superalgebra whose enveloping algebra \( u(\mathcal{L}) \) is a quotient of \( u(L) \) and thereby has finite representation type. Since \([\mathcal{L}_1, \mathcal{L}_1] = (0)\) and \((F x_0)_p \) operates trivially on \( L_1 \), it follows that the subalgebra \( u((F \pi(x_0)) \oplus \mathcal{L}_1) \) is isomorphic to \( u((F \pi(x_0)) \otimes_F F[X]/(X^2)) \). Since the trivial module of this algebra has complexity \( \leq 1 \), we readily obtain from the Küneth formula that \( u((F \pi(x_0)) \otimes_F F[X]/(X^2)) = F \). Consequently, \((F x_0)_p \subset (F v)_p \).

(2) \( \Rightarrow \) (1). We put \( L_1 := F y_1 \), \( v := [y_1, y_1] \) and consider the ideal \( H := [L_1, L_1]_p \). Let \( L_1 = (F v)_p \oplus F y_1 \), of \( L_1 \). Since \( v = 2y_1^2 \), we have \( u(H) = F[y_1] \). By decomposing \( \frac{1}{2} v \) into its semisimple and \( p \)-nilpotent part, we see that there exist mutually distinct elements...
\(x_1, \ldots, x_r\) of \(F\) and a natural number \(n\) such that the polynomial \(f := \prod_{i=1}^r (X - x_i)^{p^n} \in F[X]\) annihilates \(y_i^2\). Let \(\zeta_i \in F\) be elements such that \(\zeta_i^2 = x_i\), \(1 \leq i \leq r\). Then \(y_i\) is a root of \(g := \prod_{i=1}^r (X - \zeta_i)^{p^n} \prod_{i=1}^r (X + \zeta_i)^{p^n}\), implying that \(u(H)\) is a factor algebra of \(F[X]/(g)\). Owing to the Chinese Remainder Theorem the latter algebra is isomorphic to

\[
\bigoplus_{i=1}^r (F[X]/((X - \zeta_i)^{p^n})) \oplus (F[X]/((X + \zeta_i)^{p^n})) \cong \bigoplus_{i=1}^r F[X]/(X^{p^n}).
\]

Since each constituent has finite representation type, \(F[X]/(g)\), and its factor algebra \(u(H)\) also have this property.

From the inclusions \((Fx_0)_p \subset (Fx)_p \subset N(L_0)\) and (3.4) it now follows that the extensions \(u(L) : u(N(L_0) \oplus L_1)\) and \(u(N(L_0) \oplus L_1) : u(H)\) are separable. A two-fold application of [18, Theorem 4] thus ensures that \(u(L)\) has finite representation type.

The representation finite universal enveloping algebras of restricted Lie algebras are known to be Nakayama algebras (cf. [10,13]). Our next result shows that this is not the case for Lie superalgebras.

**Theorem 4.3.** Let \(L = L_0 \oplus L_1\) be a restricted Lie superalgebra such that \(L_1 = F y_1 \neq (0)\); \([x, y_1] = \lambda(x)y_1\ \forall x \in L_0\). Then the following statements are equivalent:

1. \(u(L)\) is a Nakayama algebra,
2. \(u(L)\) has finite representation type and \(\lambda \neq \frac{1}{2} x\).

**Proof.** Note that \(\lambda : L_0 \longrightarrow F\) induces an automorphism \(\varphi_\lambda\) of \(u(L_0)\) by means of

\[
\varphi_\lambda(x) = x + \lambda(x)1 \ \forall x \in L_0.
\]

Given a \(u(L_0)\)-module \(M\) we shall denote the module \(\varphi_\lambda M\) by \(\lambda M\). We begin with a general observation concerning irreducible \(u(L)\)-modules.

(a) Let \(V\) be an irreducible \(u(L_0)\)-module. Then \(\hat{V} := u(L) \otimes_{u(L_0)} V\) is irreducible if and only if \(\text{Hom}_{u(L_0)}(V, \hat{V}) = (0)\).

For any \(u(L_0)\)-module \(M\), we have an isomorphism

\[
u(L) \otimes_{u(L_0)} M \cong M \oplus \lambda M
\]

of \(u(L_0)\)-modules. Consequently, Frobenius reciprocity yields

\[
\text{Hom}_{u(L_3)}(\hat{V}, \hat{V}) \cong \text{Hom}_{u(L_0)}(V, \hat{V}) \cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.

(1) \(\Rightarrow\) (2): If \(u(L)\) is a Nakayama algebra, then \(u(L)\) has finite representation type. Suppose that \(\hat{\lambda} = \frac{1}{2} \lambda\). Let \(\mu : u(L_0) \longrightarrow u(L_0)\) be the Nakayama automorphism of

\[
\text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong \text{Hom}_{u(L_0)}(V, V) \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V})
\]

\[
\cong F \oplus \lambda \text{Hom}_{u(L_0)}(V, \hat{V}),
\]

so that (a) follows directly from Schur's Lemma.
$u(L_0)$. By virtue of (3.1) and (4.2) we have

$$
\mu(x) = x - \frac{1}{2} x \mathbf{1} & \forall x \in L_0.
$$

We consider the one-dimensional $u(L_0)$-modules $F$, $\mathbf{1}F$, $2\mathbf{1}F$ as well as the induced modules $\bar{\mathbf{1}F}; \gamma \in \{id_{u(L_0)}, \psi_\gamma\}$. According to (a) these modules are irreducible, and (*) in conjunction with Frobenius reciprocity now implies

$$
\text{Ext}^1_{u(L_0)}(\bar{\mathbf{1}F}, \mathbf{1}F) \cong \text{Ext}^1_{u(L_0)}(F, \mathbf{1}F) \cong \text{Ext}^1_{u(L_0)}(F, \mathbf{1}F) \oplus \text{Ext}^1_{u(L_0)}(F, 2\mathbf{1}F).
$$

Since $\mu^{-1} = \psi_\gamma$ [10, (3.2)] implies that $\dim \text{Ext}^1_{u(L_0)}(\bar{\mathbf{1}F}, \mathbf{1}F) = 1$. By the same token, we obtain $\dim \text{Ext}^1_{u(L_0)}(\bar{\mathbf{1}F}, \mathbf{1}F) = 1$. Owing to (*) we have $\bar{\mathbf{1}F} \neq \mathbf{1}F$, so that an application of [17, Theorem 9] shows that $u(L)$ is not a Nakayama algebra.

(2) $\Rightarrow$ (1): Suppose first that $\lambda \neq 0$. The proof of (4.2) then shows that $L = \mathcal{T}(L_0) \oplus F y_1$ as well as $[y_1, y_1] = 0$. Thus, $u(L_0)$ is semisimple and hence decomposes into a direct sum

$$
u(L_0) = \bigoplus \gamma \mathbf{1}F
$$

of one-dimensional modules. Hence, we obtain

$$
u(L) \cong \bigoplus \gamma (u(L) \otimes_{u(L_0)} \mathbf{1}F),
$$

a direct sum of projective $u(L)$-modules. Consequently, the dimension of each projective indecomposable $u(L)$-module is bounded by 2, forcing all these modules to be uniserial. As a result, $u(L)$ is a Nakayama algebra.

We finally assume $\lambda = 0$. Then $y_1$ belongs to the center $\mathcal{Z}(u(L))$ of $u(L)$ and Schur’s Lemma implies that every irreducible $u(L)$-module is irreducible when considered a $u(L_0)$-module. Moreover, if $V$ is an irreducible $u(L_0)$-module on which $y_1$ acts by the scalar $\gamma \in F$, then $V$ obtains the structure of a $u(L)$-module by letting $y_1$ act by a root of $X^2 - \gamma \in F[X]$.

We write a set of representatives of the isomorphism classes of irreducible $u(L_0)$ as the disjoint union $\text{Irr}(u(L_0)) = \mathcal{S}_0 \cup \mathcal{S}_1$ of modules on which $y_1^2$ acts trivially and invertibly, respectively. According to (4.2), each $V \in \mathcal{S}_0$ is one-dimensional so that there results a decomposition

$$
u(L_0) = \bigoplus \gamma P(V) \oplus \bigoplus \gamma (\dim F V)P(V).
$$

with principal indecomposable modules $P(V)$. Writing $\hat{P}(V) := u(L) \otimes_{u(L_0)} V$, we obtain

$$
u(L) \cong \bigoplus \gamma \hat{P}(V) \oplus \bigoplus \gamma (\dim F V)\hat{P}(V).
$$

By our earlier observations there correspond two irreducible $u(L)$-modules to each $V \in \mathcal{S}_1$. Since the principal indecomposable $u(L)$-module of $V$ is a direct summand of
\( \hat{P}(V) \), it follows that \( \hat{P}(V) \) is a direct sum of two principal indecomposables. Owing to (\( \ast \)) these are also principal indecomposable \( u(L_0) \)-modules. In view of [13, (2.4)] or [10, (3.2)], \( u(L_0) \) is a Nakayama algebra, so that the summands are uniserial \( u(L) \)-modules.

It remains to consider the case, where \( V \in \mathcal{I}_0 \). Then \( \hat{P}(V) \) is a principal indecomposable \( u(L) \)-module and the projection \( \pi : P(V) \rightarrow V \) induces a surjective map \( \tilde{\pi} : \hat{P}(V) \rightarrow V \); \( u \otimes p \mapsto u.\pi(p) \). Direct computation shows that

\[
\ker \tilde{\pi} = (F_{\lambda_1} \otimes_F P(V)) \oplus (F 1 \otimes_F \ker \pi).
\]

Suppose that \( W \) is an irreducible \( u(L) \)-module such that \( \Ext^1_{u(L)}(V, W) \neq (0) \). Owing to [8, (2.1)] \( y^2 \) acts trivially on \( W \), so that the \( u(L_0) \)-module \( W \) belongs to \( \mathcal{I}_0 \). Consequently,

\[
\Ext^1_{u(L)}(V, W) \cong \Hom_{u(L)}(\ker \tilde{\pi}, W) \cong \Hom_{u(L_0)}(\ker \pi, W)
\]

\[
\cong \Ext^1_{u(L_0)}(V, W),
\]

and [10, (3.2)] implies \( \dim F \Ext^1_{u(L)}(V, W) = 1 \) as well as the fact that the \( u(L_0) \)-module \( W \) is isomorphic to \( \mu^{-1}V \). By applying [17, Theorem 9] we now conclude that the block of \( u(L) \) containing \( V \) is a Nakayama algebra.

**Corollary 4.4** (cf. [1, p.167f; 4]). The algebra \( u(L) \) is semisimple if and only if \( L_0 \) is a torus and \( L_1 = (0) \).

**Proof.** Suppose \( u(L) \) to be semisimple. Then \( F \) is a projective \( u(L_0) \)-module and Hochschild’s Theorem [16] shows that \( L_0 \) is a torus. According to (4.2) we have \( \dim_F L_1 \leq 1 \) and \( \langle [L_1, L_1], L_1 \rangle = (0) \). It follows that the universal enveloping algebra of \( \mathcal{L} := L/[L_1, L_1] \) is semisimple. Since \( \langle \mathcal{L}_1, \mathcal{L}_1 \rangle = (0) \), \( F \) is a projective \( \wedge (\mathcal{L}_1) \)-module, whence \( \dim_F L_1 = \dim_F \mathcal{L}_1 = 0 \).

The reverse direction is an immediate consequence of [16].

In his paper [24] Voigt characterized those restricted Lie algebras, whose restricted enveloping algebras are of tame representation type (cf. [7, p. 9] for the definition). In particular, the only simple restricted Lie algebra with this property is \( \mathfrak{sl}(2) \). The following example shows that in the more general context of restricted Lie superalgebras, we have to allow at least one more class.

**Example.** The algebra \( u(\mathfrak{osp}(1,2)) \) has tame representation type.

**Proof.** According to [2, p. 423], \( \mathcal{U}(\mathfrak{osp}(1,2)) : \mathcal{U}(\mathfrak{sl}(2)) \) is a Frobenius extension of first kind whose Casimir element is the identity. By applying (2.6) and (2.7) successively, we find that \( u(\mathfrak{osp}(1,2)) : u(\mathfrak{sl}(2)) \) is a separable Frobenius extension. Since \( u(\mathfrak{sl}(2)) \) is tame, [6, Proposition 2] implies that \( u(\mathfrak{osp}(1,2)) \) is of tame or finite type. However, owing to (4.2) the former alternative does not apply.
References


