Solving Systems of Algebraic Equations by a General Elimination Method

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A method of factorisation of a U-resultant into linear factors is given. Using this method we can obtain solutions and their multiplicities of a system of algebraic equations, provided the system of algebraic equations has finitely many solutions. We directly calculate a matrix \( \tilde{A} \) which gives all solutions of the system by using a Gröbner basis of the ideal generated by the polynomials of the system of algebraic equations.

I. Introduction

It is known that a system of algebraic equations can be solved by using the general elimination method, which we now briefly introduce.

Let \( f_1, f_2, \ldots, f_n \) be polynomials in \( n \) variables \( x_1, x_2, \ldots, x_n \), with coefficients in the field of rational numbers \( \mathbb{Q} \). Then there exists a polynomial \( D(U_0, U_1, \ldots, U_n) \) called the U-resultant of the system, which is known to be a product of linear factors:

\[
D(U_0, U_1, \ldots, U_n) = \prod_{j=0}^{N} (a_{0j}U_0 + a_{1j}U_1 + \ldots + a_{nj}U_n),
\]

and

\[\{(a_{1j}/a_{0j}, a_{2j}/a_{0j}, \ldots, a_{nj}/a_{0j}) | a_{0j} \neq 0, 1 \leq j \leq N\}\]

are the solutions of the system. Here, \( a_{ij} \)'s are complex numbers or elements of a properly chosen finite algebraic extension field of \( \mathbb{Q} \), and the coefficients of \( D(U_0, U_1, \ldots, U_n) \) are rational numbers. Hence, we can obtain the solutions of the system of algebraic equations by calculating the U-resultant of the system and factorising it into linear factors.

In Van Der Waerden (1936), an algorithm of calculation of the U-resultant was introduced, but it was almost impossible to actually execute the calculation before an effective method was presented by Lazard (1981). In Lazard (1981), a method to construct the solutions of the system of algebraic equations from the U-resultant was given, but the method is different from the one which we present in this paper.

In this paper, we present two algorithms:

(A1) calculating the U-resultant by using a Gröbner basis,
(A2) factorising the U-resultant into linear factors,

which we now explain.
Lazard's algorithm treats a large matrix $\phi$ (see Lazard, 1981, p. 84) and from this matrix $\phi$, a matrix $\Lambda$ is obtained by a Gaussian elimination process. Another Gaussian elimination process converts $\Lambda$ to
\[
\begin{pmatrix}
U_0 \cdot \Lambda_0 + \Lambda_1 & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{pmatrix}
\]
and the determinant of the square matrix $U_0 \cdot \Lambda_0 + \Lambda_1$ is a product of all factors of the $U$-resultant containing $U_0$. Our algorithm (A1) gives $\tilde{\Lambda} = U_0 \cdot \Lambda_0 + \Lambda_1$ directly using a Gröbner basis of the ideal generated by polynomials in the system of algebraic equations. This $\tilde{\Lambda}$ gives all solutions of the system of algebraic equations.

Algorithm (A2) is a method of factorisation of the $U$-resultant into linear factors by solving an algebraic equation of a single variable, and from this factorisation the solutions are obtained.

Our idea is based on the fact that a tangent space to a hyperplane is the hyperplane itself. Since $D(U_0, U_1, \ldots, U_n) = 0$ defines a set of hyperplanes in the projective space $P^n(C)$, to obtain linear factors, we have only to find the intersection points of $D(U_0, U_1, \ldots, U_n) = 0$ and a properly chosen line, because the tangent spaces at intersection points are easily calculated.

Roughly, linear factors of the $U$-resultant are tangent spaces to $D = 0$ at these intersection points.

By this method, if a line is chosen properly, the algebraic equation of a single variable inherits the information of multiplicities of solutions carried by the $U$-resultant.

It is desirable to construct a finite algebraic extension of the field of rational numbers in which the algebraic equation deciding the intersection points split completely. Such an algorithm was presented by Trager (1976); but, at present, we cannot efficiently construct a splitting field of a high degree polynomial, so we solve the algebraic equation numerically. Our algorithm is valid if the solutions of the algebraic equation deciding intersection points are given by any method.

In section 2, we discuss how to determine the bases of $A^D$ and $B_{D-1}$. In section 3, we give an algorithm to construct the matrices $\Lambda$ and $\tilde{\Lambda}$ directly. In section 4, we give the algorithm to factorise the $U$-resultant.

In the appendix, we summarise our algorithms and we present some examples of solutions. We also show some results of comparison between the two methods of construction of $\tilde{\Lambda}$:

1. construction via the matrix $\phi$,
2. construction via the Gröbner basis.

We note that the similarity of Gaussian elimination and the method of Gröbner bases is discussed in Lazard (1983). We are inspired by his idea and by his private communication to complete our algorithm (A1).

2. Bases of $A^D$ and $B_{D-1}$

Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in $n$ variables with coefficients in $Q$. We call a polynomial $F(X_0, X_1, \ldots, X_n)$ a homogenisation of $f$ if
\[
F(X_0, X_1, \ldots, X_n) = X_0^{\deg(f)} f(X_1/X_0, X_2/X_0, \ldots, X_n/X_0),
\]
where $\deg(f)$ is the total degree of $f$. 
Let
\[ f_1(x_1, x_2, \ldots, x_n) = 0, \]
\[ f_2(x_1, x_2, \ldots, x_n) = 0, \]
\[ \ldots \]
\[ f_k(x_1, x_2, \ldots, x_n) = 0, \]
be a system of algebraic equations, and let \( F_i(x_0, x_1, \ldots, x_n) \) be homogenisation of \( f_i \) for \( i = 1, 2, \ldots, k \). Let \( \mathbb{R} \) (resp. \( \mathbb{P} \)) be the ring of polynomials \( \mathbb{Q}[x_0, x_1, \ldots, x_n] \) (resp. \( \mathbb{Q}(x_1, x_2, \ldots, x_n) \)). We see \( \mathbb{R} \) is a graded ring with natural grading and we denote by \( \mathbb{R}^d \) the submodule of \( \mathbb{R} \) generated by the homogeneous polynomials of degree \( d \).

We denote by \( \mathbb{P}_d \) the submodule of \( \mathbb{P} \) generated by polynomials of degree not greater than \( d \). We denote by \( I \) the homogeneous ideal \((F_1, F_2, \ldots, F_k)\) and we denote by \( \overline{I} \) the ideal \((\overline{f_1}, \overline{f_2}, \ldots, \overline{f_k})\).

We denote by \( \mathbb{A} \) the ring \( \mathbb{R}/I \), then \( \mathbb{A} \) is naturally a graded ring \( \mathbb{A} := \sum \mathbb{A}^d \), where \( \mathbb{A}^d = \mathbb{R}^d/\mathbb{P}_d \cap I \). We denote the ring \( \mathbb{P}/\overline{I} \) by \( \mathbb{B} \) and we denote the module \( \mathbb{P}_d/\overline{I}_d \) by \( \mathbb{B}_d \), where \( \overline{I}_d = \mathbb{P}_d \cap I \).

By rearranging, if necessary, we may suppose
\[ d_1 \geq d_2 \geq \ldots \geq d_k, \]
where \( d_i = \deg(F_i) \) for \( i = 1, 2, \ldots, k \), and we put
\[ D = d_1 + d_2 + \ldots + d_n + 1 - n. \]
Here, if \( k < n + 1 \), then we define
\[ D = d_1 + d_2 + \ldots + d_k + 1 - k. \]

The aim of this section is to show algorithms to construct bases of \( \mathbb{A}^D \) and \( \mathbb{B}_{D-1} \) respectively and to show relations between these bases. These two bases are constructed by Gröbner bases which were introduced by Buchberger (1965, 1970). A tutorial introduction into the method of Gröbner bases is given in Buchberger (1985).

**Definition 1.** Let \( Z_0 \) be the set of all positive integers, and let \( Z_0^{n+1} \) be the cartesian product of \( Z_0 \). Let \( A = (a_0, a_1, \ldots, a_n) \) and \( B = (b_0, b_1, \ldots, b_n) \) be elements of \( Z_0^{n+1} \), we define the lexicographic order:
\[ A > B \]
if there is an integer \( i \) (\( 0 \leq i \leq n \)) such that
\[ a_j = b_j \quad \text{for} \quad 0 \leq j < i \quad \text{and} \quad a_i > b_i. \]

**Definition 2.** Let \( F = \sum a_A X^A \) be a non-zero polynomial in \( R \). We define exponents of \( F \), leading exponent of \( F \) and head term of \( F \) abbreviated as \( \text{ex}(F) \), \( \text{lex}(F) \) and \( \text{ht}(F) \) respectively as follows:
\[ \text{ex}(F) = \{ A | a_A \neq 0 \quad \text{in} \quad F = \sum a_A X^A \}, \]
\[ \text{lex}(F) = A_0 \quad \text{such that} \quad A_0 > A \quad \text{for any} \quad A \in \text{ex}(F) \setminus A_0, \]
\[ \text{ht}(F) = a_{A_0} X^{A_0}. \]
DEFINITION 3. A subset $E$ of $\mathbb{Z}_0^{n+1}$ is a monoideal if
\[ E = E + \mathbb{Z}_0^{n+1}. \]

PROPOSITION 1. Let $I = (F_1, F_2, \ldots, F_k)$ be an ideal in $Q[X_0, X_1, \ldots, X_n]$, and let
\[ E = \{ A | A = \text{lex}(F) \text{ for some } F \neq 0 \text{ in } I \}, \]
then $E$ is a monoideal.

PROPOSITION 2. Let $I$ and $E$ be as in Proposition 1, and let \( \{G_1, G_2, \ldots, G_t\} \) be a \Gr\ basis of $I$ with respect to the ordering $\succ$. Then
\[ E = \bigcup_{i=1}^{t} (\text{lex}(G_i) + \mathbb{Z}_0^{n+1}). \]

THEOREM 1. Let $E$ be as in Proposition 2. Let $M_1, M_2, \ldots, M_t$ be monomials generating $\mathbb{R}^D$ such that $\text{ex}(M_i) \in E$ for $i = 1, 2, \ldots, s$ and $\text{ex}(M_j) \in E$ for $j = s+1, \ldots, t$. Then the set of equivalence classes \( \{M_1 + I^D, M_2 + I^D, \ldots, M_t + I^D\} \) in $\mathbb{A}^D$ is a basis of $\mathbb{A}^D$.

PROOF. This is just the homogeneous counterpart to the well-known construction introduced in Buchberger (1965, 1970, Theorem 2.2), see also Buchberger (1985, Lemma 6.7 and Method 6.6).

In Fig. 1, the lattice points on the line not contained in the shaded area determine a basis of $\mathbb{A}^D$.

In practice, it is much more time consuming to calculate a \Gr\ basis of the homogeneous ideal $I$ than to calculate a \Gr\ basis of $\bar{I}$. So we construct a matrix $\bar{A}$ by using a \Gr\ basis of $\bar{I}$, and to construct this $\bar{A}$ we have to construct a basis of $B$.

When we calculate a \Gr\ basis of $\bar{I}$, we use the usual total degree ordering:
\[ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \succ x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \iff i_1 + i_2 + \cdots + i_n > j_1 + j_2 + \cdots + j_n \text{ or } i_1 + i_2 + \cdots + i_n = j_1 + j_2 + \cdots + j_n \text{ and } x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} > x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}. \]

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In practice, it is much more time consuming to calculate a \Gr\ basis of the homogeneous ideal $I$ than to calculate a \Gr\ basis of $\bar{I}$. So we construct a matrix $\bar{A}$ by using a \Gr\ basis of $\bar{I}$, and to construct this $\bar{A}$ we have to construct a basis of $B$.
For each $f$ in $Q[x_1, x_2, \ldots, x_n]$, we interpret $\text{lex}(f)$, $\text{ht}(f)$ with the ordering $\gg$. Here we note that if $f$ is reduced to $\tilde{f}$ by a set of polynomials with respect to the ordering $\gg$, then $\deg(f)$ is not greater than $\deg(\tilde{f})$.

**Theorem 2.** Let $D$ be the integer defined at the beginning of this section. Suppose the system of algebraic equations (1) has only a finite number of solutions. Then $\mathbb{B}_{D-1}$ is isomorphic to $\mathbb{B}$ as a vector space over $Q$.

**Proof.** Let $t: \mathbb{P}_{D-1}/I_{D-1} \rightarrow \mathbb{P}/I$ be the injection map $t(g + \tilde{I}_{D-1}) = g + \tilde{I}$.

In the proof of Theorem 3 in Lazard (1983, p. 154), we see that $\dim_Q \mathbb{B}_d = \dim_Q \mathbb{B}_d$ for all $d > D - 1$. Suppose we are given an element $g + \tilde{I}$ of $\mathbb{P}/I$. If $\deg(g) \leq D - 1$, then $t(g + \tilde{I}_{D-1}) = g + \tilde{I}$, so we have only to consider the case $r = \deg(g) > D - 1$. Since $\dim_Q \mathbb{B}_d = \dim_Q \mathbb{B}_{D-1}$, there is an element $h + \tilde{I}_{D-1}$ of $\mathbb{B}_{D-1}$ such that $h = g$ modulo $\tilde{I}$, which implies $h = g$ modulo $\tilde{I}$. Q.E.D.

Let $\{g_1, g_2, \ldots, g_\alpha\}$ be a Gröbner basis of $I$ with respect to the order $\gg$, and let $E = \bigcup_{i=1}^\alpha (\text{lex}(g_i) + Z_0)$. 

**Proposition 3.** Let $m_1, m_2, \ldots, m_\alpha$ be all monomials satisfying both

1. $\deg(m_i) < D$ for $i = 1, 2, \ldots, \alpha$,
2. $\text{ex}(m_i) \notin E$.

Then $\{m_1 + \tilde{I}_{D-1}, m_2 + \tilde{I}_{D-1}, \ldots, m_\alpha + \tilde{I}_{D-1}\}$ is a basis of $\mathbb{B}_{D-1}$.

**Proof.** By Lemma 6.7 in Buchberger (1985).

**Theorem 3.** Let $E = \bigcup_{i=1}^\alpha (\text{lex}(g_i) + Z_0)$. 

If the system (1) has only a finite number of solutions, then there are only finitely many monomials $m_1, m_2, \ldots, m_\alpha$ such that $\text{ex}(m_i) \notin E$ for $i = 1, 2, \ldots, \alpha$. These $m_1, m_2, \ldots, m_\alpha$ satisfy the conditions

1. $\deg(m_i) \leq D - 1$,
2. $\{m_1 + \tilde{I}, m_2 + \tilde{I}, \ldots, m_\alpha + \tilde{I}\}$ is a basis of $\mathbb{B}$.

**Proof.** By applying Criterion 4.2 in Buchberger (1970), see also Method 6.9 in Buchberger (1985).

We visualise Method 6.9 of Buchberger (1985) in Fig. 2. All lattice points not contained in $E$ determine a basis of $\mathbb{B}$.
**Proposition 4.** Let $m_1, m_2, \ldots, m_l$ be monomials such that $\deg(m_i) \leq D-1$ for $i = 1, 2, \ldots, l$ and that \{m_1 + \vec{1}, m_2 + \vec{1}, \ldots, m_l + \vec{1}\} is linearly independent over $\mathbb{Q}$. Let $N_i(X_0, X_1, \ldots, X_n) = X_0^D - \deg(m_i) m_i(X_1, X_2, \ldots, X_n)$. Then \{N_1 + I, N_2 + I, \ldots, N_l + I\} is linearly independent in $\mathbb{A}^D$.

**Proposition 5.** Let $m_1, m_2, \ldots, m_l$ be monomials of degree not greater than $D-1$ such that \{m_1 + \vec{1}, m_2 + \vec{1}, \ldots, m_l + \vec{1}\} is a basis of $B$. Let $N_1, N_2, \ldots, N_l$ be the monomials defined in Proposition 4. Given a monomial $M$ in $\mathbb{R}^D$, then there are polynomials $H_1, H_2, \ldots, H_K$ and $H$ in $\mathbb{R}$ such that

\[
M - (a_1 N_1 + a_2 N_2 + \cdots + a_l N_l) = \sum H_i F_i + (X_0 - 1)H,
\]

where $a_1, a_2, \ldots, a_l$ are rational numbers such that $M(1, X_1, X_2, \ldots, X_n) = a_1 m_1 + a_2 m_2 + \cdots + a_l m_l$ modulo $\vec{1}$.

**Proof.** (Compare Section 5 in Buchberger (1970), and Method 6.6 in Buchberger (1985).) Since \{m_1 + \vec{1}, m_2 + \vec{1}, \ldots, m_l + \vec{1}\} is a basis of $B$, there are rational numbers $a_1, a_2, \ldots, a_l$ such that $M(1, X_1, X_2, \ldots, X_n) = a_1 m_1 + a_2 m_2 + \cdots + a_l m_l$ modulo $\vec{1}$. Hence, there are polynomials $h_1, h_2, \ldots, h_K$ such that $M(1, X_1, X_2, \ldots, X_n) - \sum a_i m_i = \sum h_i f_i$.

Let $H_i$ be the homogenisation of $h_i$, then

\[
M(X_0, X_1, X_2, \ldots, X_n) - \sum a_i N_i - \sum H_i F_i
\]

is a multiple of $X_0 - 1$. Thus we have a relation

\[
M(X_0, X_1, \ldots, X_n) - \sum a_i N_i - \sum H_i F_i = (X_0 - 1)H.
\]

### 3. Construction of Matrices $\Lambda$ and $\bar{\Lambda}$

Let $U_0, U_1, \ldots, U_n$ be variables algebraically independent from $X_0, X_1, \ldots, X_n$ and $x_1, x_2, \ldots, x_n$ over $\mathbb{Q}$. We denote by $\mathbb{R}_d^d$, $\mathbb{A}_d^d$, $\mathbb{P}_d$ and $\mathbb{B}_d$ respectively the vector spaces

\[
Q(U_0, U_1, \ldots, U_n) \otimes_{\mathbb{Q}} \mathbb{R}^d, \quad Q(U_0, U_1, \ldots, U_n) \otimes_{\mathbb{Q}} \mathbb{A}^d, \\
Q(U_0, U_1, \ldots, U_n) \otimes_{\mathbb{Q}} \mathbb{P}_d \quad \text{and} \quad Q(U_0, U_1, \ldots, U_n) \otimes_{\mathbb{Q}} \mathbb{B}_d.
\]
We define a map
\[ H : \mathbb{R}^D_{D-1} \rightarrow \mathbb{R}^D, \]
with \( L = U_0 X_0 + U_1 X_1 + \cdots + U_n X_n \), and we denote this map by \( L \).

Let \( \rho \) be a natural projection map
\[ \rho : \mathbb{R}^D \rightarrow \mathbb{A}^D, \]
then it is naturally extended to
\[ \rho_U : \mathbb{R}^D_U \rightarrow \mathbb{A}^D_U. \]

We define the map \( \Lambda : \mathbb{R}^D_{D-1} \rightarrow \mathbb{A}^D_U \) as the composition \( \rho_U \cdot L \). Similarly, in the non-homogeneous case, we define the map
\[ \Lambda : \mathbb{P}^D_{D-1} \rightarrow \mathbb{E}_U \]
as the composition
\[ \mathbb{P}^D_{D-1} \rightarrow \mathbb{P}^D_U \rightarrow \mathbb{E}_U, \]
where
\[ \bar{L} = U_0 + U_1 x_1 + \cdots + U_n x_n \quad \text{and} \quad \bar{I}_U = \mathbb{Q}(U_0, U_1, \ldots, U_n) \otimes \bar{I}. \]

The aim of this section is to construct matrix representations of \( \Lambda \) and to construct the essential part \( \tilde{\Lambda} \) of \( \Lambda \).

Let \( M_1, M_2, \ldots, M_s \) be the monomials in \( \mathbb{R}^D \) chosen as in Theorem 1, hence \( \{ M_1 + I^D, m_2 + I^D, \ldots, m_s + I^D \} \) is a basis of \( \mathbb{A}^D \). Let \( X^P \) be a monomial in \( \mathbb{R}^{D-1} \). If we denote the normal form of \( X^P \cdot X_j \) as
\[ a_{1B} M_1 + a_{2B} M_2 + \cdots + a_{sB} M_s, \]
with \( a_{lB} \)'s in \( \mathbb{Q} \), then we see that the normal form of \( X^B \cdot L \) over \( \mathbb{Q}(U_0, U_1, \ldots, U_n) \) is
\[ (\sum U_j a_{1B}^j) M_1 + (\sum U_j a_{2B}^j) M_2 + \cdots + (\sum U_j a_{sB}^j) M_s. \]
Hence, if we put \( \sum a_{lB} = a_{kB} \), then the matrix
\[
\begin{pmatrix}
a_{1B} & a_{2B} & \cdots & a_{sB} \\
a_{2B} & a_{2B} & \cdots & a_{2B} \\
\cdots & \cdots & \cdots & \cdots \\
a_{sB} & a_{sB} & \cdots & a_{sB}
\end{pmatrix}
\]
is a matrix representation of \( \Lambda \). Here \( X^{P_1}, X^{P_2}, \ldots, X^{P_k} \) are all monomials in \( \mathbb{R}^{D-1} \), and \( s \) is the number of solutions of the homogeneous system of algebraic equations
\[ F_1 = F_2 = \ldots = F_k = 0, \]
with multiplicity counted.

Now we construct the matrix \( \overline{\Lambda} \). Let \( \{ m_1, m_2, \ldots, m_l \} \) be the set of monomials of degree not greater than \( D-1 \) such that \( \{ m_1 + \bar{I}, m_2 + \bar{I}, \ldots, m_l + \bar{I} \} \) is a basis of \( \mathbb{E} \), chosen as in Theorem 3. We denote the normal form of \( m_j x_i \) by
\[ \sum b_{kj}^i m_k \]
for \( i = 1, 2, \ldots, l \). Then the normal form of \( \bar{L} \cdot m_j \) is
\[ \sum_k \sum_T (U_i b_{kj}^i) m_k. \]
We put \( \sum U_i b_{kj}^i = b_{kj} \), and we denote the matrix \( (b_{kj})_{1 \leq k, j \leq l} \) by \( \overline{\Lambda} \).
PROPOSITION 6. Let $b_{ij}$'s be the polynomials in $Q[U_0, U_1, \ldots, U_n]$ defined as above. Then

\[ b_{ij} = U_0 + \text{(a linear combination of } U_1, U_2, \ldots, U_n) \text{, if } i \neq j, \ 1 \leq i, j \leq l. \]

\[ b_{ii} = U_0 + \text{(a linear combination of } U_1, U_2, \ldots, U_n), \ 1 \leq i \leq l. \]

PROOF. $m_i$ being chosen as in Theorem 3, $m_i$ is irreducible with respect to the Gröbner basis \{\(g_1, g_2, \ldots, g_n\)\}, so the normal form of

\[ (U_0 + U_1 x_1 + U_2 x_2 + \cdots + U_n x_n) m_i \]

is $U_0 m_i + \sum_j (\text{a linear combination of } U_1, U_2, \ldots, U_n)m_j$.

In the rest of this section, we prove that the determinant of $\tilde{A}$ is the product of all linear factors of the $U$-resultant with non-zero coefficients of $U_0$.

Let

\[ N_i(X_0, X_1, \ldots, X_n) = X_0^{d_i - \deg(m_j)} m_j(X_1, X_2, \ldots, X_n) \]

for $i = 1, 2, \ldots, l$. Then by Proposition 4, \{\(N_1 + P_1, N_2 + P_2, \ldots, N_l + P_l\)\} is linearly independent over $Q$. So there is a subset \{\(M_{i_1}, M_{i_2}, \ldots, M_{il-i-1}\)\} of \{\(M_1, M_2, \ldots, M_s\)\} such that

\[ \{N_1 + P_1, N_2 + P_2, \ldots, N_l + P_l, M_{i_1} + P_{i_1}, \ldots, M_{il-i-1} + P_{il-i-1}\} \]

is a basis of $P_0$. By rearranging, if necessary, we may assume

\[ M_{i_1} = M_{i+1}, \ M_{i_2} = M_{i+2}, \ldots, \ M_{il-i-1} = M_s. \]

By this new basis

\[ \{N_1 + P_1, N_2 + P_2, \ldots, N_l + P_l, M_{i_1+1} + P_{i_1+1}, \ldots, M_s + P_s\}, \]

we have another matrix representation of $\tilde{A}$:

\[ \begin{pmatrix} X_0^{B_1}L & X_0^{B_2}L & \cdots & X_0^{B_s}L \\ N_1 & \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1l} \\ c_{21} & c_{22} & \cdots & c_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l1} & c_{l2} & \cdots & c_{ll} \\ \end{pmatrix} \\ \vdots \\ N_{il+1} & \begin{pmatrix} c_{i+11} & c_{i+12} & \cdots & c_{i+1l} \\ \vdots & \vdots & \ddots & \vdots \\ c_{li} & c_{li} & \cdots & c_{ll} \\ \end{pmatrix} \\ \vdots \\ M_{i_1+1} & \begin{pmatrix} c_{i_1+11} & c_{i_1+12} & \cdots & c_{i_1+1l} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1} & c_{i_1} & \cdots & c_{i_1l} \\ \end{pmatrix} \\ \vdots \\ M_s & \begin{pmatrix} c_{s1} & c_{s2} & \cdots & c_{sl} \\ \end{pmatrix} \end{pmatrix} \]

By abuse of notation, we denote this matrix by $A$.

Since $\deg(m_j)$ is not greater than $D-1$,

\[ X_0^{D-1 - \deg(m_j)} m_j(X_1, X_2, \ldots, X_n) \]

is a monomial of degree $D-1$ and by rearranging $X_0^{B_1}, X_0^{B_2}, \ldots, X_0^{B_s}$, we may assume

\[ X_0^{B_i} = X_0^{D-1 - \deg(m_j)} m_j(X_1, X_2, \ldots, X_n) \]

for $j = 1, 2, \ldots, l$.

We have relations

\[ X_0^{B_i}L = \sum_{i=1}^{l} c_{ij}N_i = \sum_{r=1}^{l} c_{ij}M_r = \sum_{r=1}^{s} H_{ij}(X_0, X_1, \ldots, X_n) F_i(X_0, X_1, \ldots, X_n). \]
Since \( \{m_1 + \vec{I}, m_2 + \vec{I}, \ldots, m_t + \vec{I}\} \) is a base of \( B \), we see that \( H(1, X_1, X_2, \ldots, X_n) \) are linear combinations of \( m_1, m_2, \ldots, m_t \) modulo \( \vec{I} \): 

\[
M_r(1, X_1, X_2, \ldots, X_n) = \sum d_{ir} m_i \text{ modulo } \vec{I}, \quad r = l+1, l+2, \ldots, s.
\]

Hence, we obtain relations:

\[
m_i(U_0 + U_1 x_1 + \cdots + U_n x_n) - \sum c_{ij} m_i = \sum c_{ij} d_{ir} m_i \in \vec{I}_i,
\]

for \( i = 1, 2, \ldots, l \). From these relations we see that

\[
c_{ij} - \sum c_{ij} d_{ir} = b_{ij} \quad \text{for } i, j = 1, 2, \ldots, l.
\]

If we put

\[
b_{ij} = c_{ij} - \sum c_{ij} d_{ir}
\]

for \( j = l+1, l+2, \ldots, t \) and \( i = 1, 2, \ldots, l \), then we have a matrix 

\[
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1t} \\
b_{21} & b_{22} & \cdots & b_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
b_{t1} & b_{t2} & \cdots & b_{tt}
\end{pmatrix}
\]

which is a matrix representation of the map \( \Lambda \).

If we put

\[
\Lambda_{21} = (c_{ij})_{1 \leq i \leq l, 1 \leq j \leq s} \quad \text{and} \quad \Lambda_{22} = (c_{ij})_{l+1 \leq i \leq s, l+1 \leq j \leq s},
\]

then the matrices

\[
\Lambda' = \begin{pmatrix} \Lambda_{21} & \Lambda_{22} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} = \begin{pmatrix} \bar{\Lambda} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}
\]

and \( \Lambda \) have the same rank. So we may consider \( \Lambda' \) instead of \( \Lambda \). By Proposition 6, \( \bar{\Lambda} \) is written as

\[
U_0 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \Lambda_{11},
\]

and we can eliminate \( U_0 \) from \( \Lambda_{21} \) by Gaussian elimination. Hence, we may suppose \( U_0 \) does not appear in \( \Lambda_{21} \).

**Proposition 7.** \( U_0 \) does not appear in \( \Lambda_{22} \).

**Proof.** Suppose \( U_0 \) appears in an entry of \( \Lambda_{22} \), then by changing rows and columns, we may suppose \( U_0 \) appears in \( c_{l+1+l+1} \):

\[
c_{l+1+l+1} = aU_0 + \cdots, \quad a \in \mathbb{Q} \quad (a \neq 0).
\]

This implies that the matrix \( \Lambda' \) can be written as

\[
\begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & a & \\
\Lambda_{21} & \Lambda_{22}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & a & \\
\Lambda_{21} & \Lambda_{22}
\end{pmatrix}
\]
and we see that the $U$-resultant is of degree greater than $l$ with respect to $U_0$, which contradicts that the system (1) has at most $l = \dim_\mathbb{F} \mathcal{B}$ solutions [Lazard (1981), Theorem 3.1, p. 81].

This proposition shows that $\det(\bar{\Lambda})$ gives all solutions of the system (1).

**Corollary.** The number of lattice points not contained in $\bar{E}$ in Fig. 2 coincides to the number of the solutions of the system of algebraic equations (1).

### 4. Factorisation of the $U$-resultant

As we noted in the introduction, $D(U) = 0$ defines some hyperplanes in the $n$ dimensional projective space. The solutions of the system of algebraic equations (1) are determined by the linear factors of $D(U)$ with non-zero coefficient of $U_0$, which correspond to some of those hyperplanes $D(U) = 0$. If we put $U_0 = 1$ and identify $(1, U_1, U_2, \ldots, U_n)$ with $(U_1, U_2, \ldots, U_n)$, an affine coordinates of $\mathbb{C}^n$, then the hyperplanes of $D(U) = 0$ corresponding to the solutions of the system are those which do not pass through $0 = (0, 0, \ldots, 0) \in \mathbb{C}^n$. So we have only to factorise $\det(\bar{\Lambda})$ instead of the whole $D(U)$.

Hereafter, we denote $\det(\bar{\Lambda})$ by $D(U)$. Let $L$ be a line passing through $0$:

$$(a_1, a_2, \ldots, a_n)t,$$

where $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, and let the non-zero solutions of

$$D(1, ta_1, ta_2, \ldots, ta_n) = 0$$

be $t_1, t_2, \ldots, t_s$ then, in $P^n(\mathbb{C})$, a hyperplane of $D(U) = 0$ on $(1, a_1t_j, a_2t_j, \ldots, a_nt_j)$ is

$$\sum_i \frac{\partial D}{\partial U_i}(1, a_1t_j, a_2t_j, \ldots, a_nt_j)U_i = 0,$$

if $t_j$ is a solution of multiplicity 1.

If the multiplicity $m_j$ of $t_j$ is not 1, then we have two cases:
(1) For each \( n-1 \) linearly independent lines \( L_1, L_2, \ldots, L_{n-1} \), which are also independent from \( L \), a solution of

\[
D(1, b_1t, b_2t, \ldots, b_{n-1}t) = 0 \quad (L_t = (b_1, b_2, \ldots, b_{n-1})t),
\]

near \( t_j \) is of multiplicity \( m_j \). Then \( D \) is of the form

\[
D = H^m \cdot D,
\]

where \( H = 0 \) defines a hyperplane passing through \((1, a_1t_j, a_2t_j, \ldots, a_{n-1}t_j)\). In this case the first coefficient of the solution corresponding to this point is

\[
\frac{\delta^m D/\delta U_0^m \ldots \partial U_1^m(1, a_1t_j, a_2t_j, \ldots, a_{n-1}t_j)}{\delta^m D/\delta U_0^m(1, a_1t_j, a_2t_j, \ldots, a_{n-1}t_j)},
\]

and the other coefficients of the solution are calculated in the same manner.

(2) There is a line \( L' : (a'_1, a'_2, \ldots, a'_n)t \) near \( L \), such that the solutions of

\[
D(1, a'_1t, a'_2t, \ldots, a'_nt) = 0
\]

near \( t_j \) are of multiplicity strictly less than \( m_j \). Then we try it with another line \( L' \) near \( L \). If this second case happens at \( L \), then we can find the above line \( L' \) by at most \( n \) trials by taking independent lines. Figures 3 and 4 illustrate cases 1 and 2 respectively.

NOTES ON THE COMPUTATION

In the rest of this section, we give some notes on computing intersection points and tangents. By our assumption, \( D(U) \) is the determinant of the matrix \( \Lambda \). Expansion of the determinant of \( \Lambda \) is quite time consuming, but we can calculate intersection points and tangents without expanding the full determinant of \( \Lambda \) as follows:

(1) To calculate intersection points, before expanding the determinant, we substitute \( U_0, U_1, \ldots, U_n \) by \( 1, a_1t, a_2t, \ldots, a_nt \) in each entry of the matrix and then expand the determinant of it.

(2) To calculate tangents, we have only to note that

\[
\frac{\partial}{\partial U_0} \det(a_{ij}(U))(A) = \sum_j \det \begin{pmatrix}
a_{11}(A) & \partial a_{1j}/\partial U_0(A) & \ldots & a_{1n}(A) \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}(A) & \partial a_{nj}/\partial U_0(A) & \ldots & a_{nn}(A)
\end{pmatrix},
\]

where \( a_{ij}(U) = a_{ij}(U_0, U_1, \ldots, U_n) \) are polynomials and \( A \) is a point in \( P^n(\mathbb{C}) \).
5. Appendix

Now we summarise our algorithm.

Input: a system of algebraic equations.
Output: their solutions over $C$ (if there are only finitely many solutions otherwise report “infinitely many solutions”).

1. \{\(f_1 = 0, f_2 = 0, \ldots, f_k = 0\)\} a system of algebraic equations with \(f_i \in \mathbb{Q}[x_1, x_2, \ldots, x_n]\) and \(k \geq n\).

2. For \(i = 1\) to \(k\) do \(d_i := \deg(f_i)\).

3. Rearrange \(d_1, d_2, \ldots, d_k\) to satisfy
   \[d_1 \geq d_2 \geq \ldots \geq d_k.\]

4. \(D := d_1 + d_2 + \cdots + d_n - n + 1\).

5. \(\{g_1, g_2, \ldots, g_n\}\): a Gröbner basis of the ideal \(\langle f_1, f_2, \ldots, f_k \rangle\).

6. \text{Finite-number} := \text{true}.

7. For \(i = 1\) to \(u\) do
   
   if no \(g_i\) has a leading monomial of the form \(x_i^a\)
   
   then \text{finite-number} := \text{false}.

8. If not \text{finite-number}, then
   
   \((\text{write “infinitely many solutions”; stop})\)
   
   (see Buchberger, 1985, Method 6.9)

9. \(\{m_1, m_2, \ldots, m_l\}\): monomials of degree \(\leq D - 1\) which are irreducible with respect to
    \(\{g_1, g_2, \ldots, g_n\}\).

10. \(m_{ij} := \text{the normal form of } x_i m_j\)

    which is expressed as
    \[\sum_k b_{kj} m_k.\]

11. For \(k = 1\) to \(l\) do
   
   for \(j = 1\) to \(l\) do \(b_{kj} := \sum_i b_{ki} a_i.\)

12. \(D(U) := \det(b_{kj})\).

13. Give a line \((a_1, a_2, \ldots, a_n)\) and make an algebraic equation
    \[D(1, a_1 t, a_2 t, \ldots, a_n t) = 0.\]

14. \(\{t_1, t_2, \ldots, t_N\}\) := the solutions of the equation
    \[D(1, a_1 t, a_2 t, \ldots, a_n t) = 0.\]

---

### Table 1. Comparison of the two methods tested

<table>
<thead>
<tr>
<th>Number of examples</th>
<th>Number of variations, degree of polynomials</th>
<th>Gaussian† ms</th>
<th>Gröbner† ms</th>
<th>Ga/Gr ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2, (2, 2)</td>
<td>244</td>
<td>93</td>
<td>2.6</td>
</tr>
<tr>
<td>100</td>
<td>2, (2, 3)</td>
<td>1035</td>
<td>224</td>
<td>4.6</td>
</tr>
<tr>
<td>100</td>
<td>2, (3, 3)</td>
<td>5596</td>
<td>664</td>
<td>8.4</td>
</tr>
<tr>
<td>90</td>
<td>3, (2, 2)</td>
<td>30976</td>
<td>5280</td>
<td>5.8</td>
</tr>
</tbody>
</table>

(HITAC M-680H, memory = 3072kb)

† Mean value taken to construct the matrix $\bar{A}$. 
15. For \( i = 1 \) to \( N \) do
\[
\langle\langle P_i := (a_1 t_i, a_2 t_i, \ldots, a_N t_i)\rangle\rangle;
\]
\[A_i := \left( \frac{\partial D/\partial U_1(P_i)}{\partial D/\partial U_0(P_i)}, \frac{\partial D/\partial U_2(P_i)}{\partial D/\partial U_0(P_i)}, \ldots, \frac{\partial D/\partial U_N(P_i)}{\partial D/\partial U_0(P_i)} \right)\).

16. \( \varepsilon := \) a small positive number that depends on the system
(\text{In our case } 10^{-6}).
17. Good-solution := true.
18. For \( i = 1 \) to \( N \) do
for \( j = 1 \) to \( k \) do
if absolute value of \( f_j(A_i) > \varepsilon \), then
  good-solution := false.
19. If good-solution then \( \langle\langle \text{return } \{A_1, A_2, \ldots, A_N\}; \text{stop} \rangle\rangle \) else go to 13 and give another line which is linearly independent to the lines given previously.
20. (The case that no line gives good solutions.)
  Choose among the given lines the line \( L = (a_1, a_2, \ldots, a_n)t \) such that the number of distinct solutions is greatest.
21. Try 20 again with respect to the \( n \) linearly independent lines which are very near to the line chosen in 20.
22. Calculate the tangent cone (which is equal to a hyperplane with multiplicity greater than 1) [see Griffiths & Harris (1978)] at each intersection point \( P_t \) of the line chosen in 22 and return the tangent cones and their multiplicities.
23. End.

Before showing examples, we give some notes on the program executed. All steps except step 14 are written in RLISP and REDUCE-2 over Cambridge LISP. Step 14 is written in FORTRAN, and in this step, we use the Durand-Kerner-Aberth method (Kerner, 1966; Aberth, 1973) to solve algebraic equations of a single variable.

Examples 1, 2 and 3 show the results of comparison between the two methods of the construction of \( \tilde{A} \) (see also Table 1). Our experiment shows that the Gröbner basis method is more effective than Gaussian elimination to obtain the matrix \( \tilde{A} \) by using REDUCE-2 (or RLISP). We note that we presented the time taken to construct the matrix \( \tilde{A} \), because the two methods (Gaussian and Gröbner) use the same process after the construction of \( \tilde{A} \).

Because we expand the full determinant of the matrix \( \tilde{A} \), our method does not work for large-scale problems. For example, we cannot solve problems of three variables with degree 2, 3, 3 respectively (though the matrix \( \tilde{A} \) can be constructed).

Example 4 shows not good solutions. The given line passes through a singular point of \( D(U) = 0 \).

At a singular point \( P \),
\[
\frac{\partial D}{\partial U_0}(P) = a_0, \quad \frac{\partial D}{\partial U_1}(P) = a_1, \ldots, \frac{\partial D}{\partial U_n}(P) = a_n
\]
must all be equal to 0, but \( P \) being obtained numerically, the absolute value of all these numbers are small but it happens that these numbers are not precisely zero. Hence,
\[
(a_1/a_0, a_2/a_0, \ldots, a_n/a_0)
\]
becomes a not good solution.

Example 5 shows good solutions to the same system of algebraic equations as in Example 4, obtained by choosing another line.
Example 1.

Construction by Gaussian elimination.

ENTER ALGEBRAIC EQUATIONS:

\[ \text{ALGEO}(1) := X_1^2 + X_2^2 - 2 \]
\[ \text{ALGEO}(2) := X_1 \times X_2 - 1 \]

* CONSTRUCTION OF LAMBDA COMPLETED *=
TIME: 162 MS*

\[ U_{\text{RESULT}} := -U_0 + 2 \times U_0 \times U_1 + 4 \times U_0 \times U_1 \times U_2 + 2 \times U_0 \times U_2 - U_1 - 4 \times U_1 \times U_2 - 6 \times U_1 \times U_2 + 4 \times U_1 \times U_2 \]

Construction using Groebner basis.

\[ \text{ALGEO}(1) := X_1^2 + X_2^2 - 2 \]
\[ \text{ALGEO}(2) := X_1 \times X_2 - 1 \]

*CONSTRUCTION OF LAMBDA COMPLETED*=
TIME: 69 MS

\[ U_{\text{RESULT}} := -U_0 + 2 \times U_0 \times U_1 + 4 \times U_0 \times U_1 \times U_2 - 2 \times U_0 \times U_2 + U_1 + 4 \times U_1 \times U_2 + 3 \times U_2 + 6 \times U_1 \times U_2 + 4 \times U_1 \times U_2 + U_2 \]

1) The examples were computed at the Computer Center, University of Tokyo, by HITAC M-680H with 3000 KB of memory.
2) Time being taken to construct \( \Lambda \).

Example 2.

Construction by Gaussian elimination.

ENTER ALGEBRAIC EQUATIONS:

\[ \text{ALGEO}(1) := X_1^2 + X_2^2 + X_3^2 - 1 \]
\[ \text{ALGEO}(2) := X_1 \times X_2 + X_1 + X_2 + 2 \]
\[ \text{ALGEO}(3) := X_1 \times X_2 \times X_3 - 1 \]
CONSTRUCTION OF LAMBDA COMPLETED

TIME: 16179 MS

U-RESULT := (UO - 6*U0*U1 - 6*U0*U2 + 13*U0*U1*U2 + 4*U0*U1*U3 + 6*U0*U2*U3 + 6*U0*U3

- ... + 21*U2*U3 + 6*U2*U3 + U3 )/8

Example 3.

Construction by Gaussian elimination.

ENTER ALGEBRAIC EQUATIONS:

ALGEO(1):=X1**3 + X1*X2 - X2**2 + X3
ALGEO(2):=X1**2 + X3**2 + X1 + X2
ALGEO(3):=X2**2 + X3 + X1

CONSTRUCTION OF LAMBDA COMPLETED

TIME: 14480 MS

Expansion of the U-resultant was not completed within the time limit (3 min.).

Construction using Groebner base.

To the same problem.

TIME: 1674 MS

U-RESULT := U0*(U0 + 4*U0*U1 - 6*U0*U1*U2 - 16*U0*U1*U3 - 3*

2*U0*U2*U3 - 4*U0*U3 - 2*U0*U1 + 20*U0*U1 - 3*

7*U2*U3 + 3*U2*U3 + 22*U2*U3 + 5*U2*U3 - 15*

3*U2 - 7*U2*U3 + 4*U2*U3 + 5*U3)
Example 4.

\[ f_1 = x_1 x_2^2 - x_3^2, \]
\[ f_2 = x_1^2 + x_2^2 - x_3^2, \]
\[ f_3 = x_1 x_2 + x_1^3 + x_2^3. \]

Line: \( t(1,0,0). \)

A point \((0,0,0)\) is a solution of multiplicity 8.

Other solutions:

\[
\begin{align*}
X_1 &= 1.69E-5 + 2.3537494 \\
X_2 &= -1.72E-5 + 2.0229495 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= 1.2471492 + 1.057771E-1 \\
X_2 &= 5.0500746 - 8.176292E-1 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= -1.69E-5 + 2.3538114 \\
X_2 &= 1.72E-5 + 2.0230128 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= 1.2471548 + 1.057925E-1 \\
X_2 &= 5.0501396 + 8.178325E-1 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= -1.69E-5 + 2.3538114 \\
X_2 &= 1.72E-5 + 2.0230128 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= 1.2471492 + 1.057889E-1 \\
X_2 &= 5.0500896 + 8.178294E-1 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= -1.69E-5 + 2.3538114 \\
X_2 &= 1.72E-5 + 2.0230128 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= -9.07564E-2 + 5.089023E-1 \\
X_2 &= 7.052947E-1 + 1.936408E-1 \\
X_3 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
X_1 &= -9.08224E-2 + 5.088868E-1 \\
X_2 &= 7.054385E-1 + 1.936786E-1 \\
X_3 &= 0 \\
\end{align*}
\]
Example 5.

The system of algebraic equations is the same as in Example 4.

line: t(1,1,1).
A point (0,0,0) is a solution of multiplicity 6.
Other solutions:

\[ X_1 = -1.247152 \times 10^{-2} - 1.857848 \times 10^{-1} \]
\[ X_2 = 5.050106 \times 10^{-1} + 8.178309 \times 10^{-1} \]
\[ X_3 = -1.065832 \times 10^{-1} + 1.701125 \times 10^{-1} \]

... ...

\[ X_1 = 2.3537804 \]
\[ X_2 = (-2.0229811) \]
\[ X_3 = 3.1036648 \]

... ...

\[ X_1 = 1.247152 \times 10^{-2} - 1.857848 \times 10^{-1} \]
\[ X_2 = 5.050106 \times 10^{-1} + 8.178309 \times 10^{-1} \]
\[ X_3 = -1.065832 \times 10^{-1} + 1.701125 \times 10^{-1} \]

... ...

\[ X_1 = 9.07894 \times 10^{-2} + 5.088946 \times 10^{-1} \]
\[ X_2 = 7.053664 \times 10^{-1} + 1.934597 \times 10^{-1} \]
\[ X_3 = 4.928936 \times 10^{-1} + 1.034044 \times 10^{-1} \]

... ...

\[ X_1 = 1.247152 \times 10^{-2} - 1.857848 \times 10^{-1} \]
\[ X_2 = 5.050106 \times 10^{-1} + 8.178309 \times 10^{-1} \]
\[ X_3 = 1.065832 \times 10^{-1} + 1.701125 \times 10^{-1} \]

... ...

Not good solutions!
\[
X_1 = 9.07894 \times 10^{-2} + 5.08894 \times 10^{-1}
\]
\[
X_2 = -7.05366 \times 10^{-1} + 1.93659 \times 10^{-1}
\]
\[
X_3 = -4.92893 \times 10^{-1} + 1.83404 \times 10^{-1}
\]

\[
X_1 = 2.35378
\]
\[
X_2 = -2.02298
\]
\[
X_3 = -3.10366
\]

We thank Professor D. Lazard and Professor B. Buchberger for their advice and Dr T. Sasaki for valuable discussions. We thank Mr H. Murao for helping us to install our program on the computer at the Computer Centre, University of Tokyo. We thank Mr S. Moritsugu for allowing us to use his program to calculate Gröbner bases.

References


