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The structure of some linear transformations

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ABSTRACT

Let $\mathbb{F}$ denote an algebraically closed field and let V denote a finite-dimensional vector space over $\mathbb{F}$. Recently Ito and Terwilliger considered a system of linear transformations A_+, A_-, A_+^*, A_-^* on V which generalizes the notions of a tridiagonal pair and a q-inverting pair. In their paper they mentioned some open problems about this system. In this paper we solve Problem 1.2 with the following results. Let $\{V_i\}_{i=0}^d$ denote the common eigenspaces of A_+, A_- and let $\{V_i^*\}_{i=0}^d$ denote the common eigenspaces of A_+^*, A_-^*. We show that each of A_+, A_-, A_+^*, A_-^* is determined up to affine transformation by the sequences $\{V_i\}_{i=0}^d$; $\{V_i^*\}_{i=0}^d$. We also show that the following are equivalent: (i) there exists a nonzero bilinear form $\langle\ ,\ \rangle$ on V such that $\langle A_+u, v\rangle = \langle u, A_+v\rangle$ and $\langle A_+^*u, v\rangle = \langle u, A_+^*v\rangle$ for all $u, v \in V$; (ii) there exist scalars $\alpha, \alpha^*, \beta, \beta^*$ in $\mathbb{F}$ with α, α^* nonzero such that $A_- = \alpha A_+ + \beta I$ and $A_-^* = \alpha^* A_+^* + \beta^* I$; and (iii) both A_+, A_+^* and A_-, A_-^* are tridiagonal pairs.

1. Introduction

Throughout the paper $\mathbb{F}$ denotes an algebraically closed field and V denotes a vector space over $\mathbb{F}$ with finite positive dimension.

By a *decomposition* of V, we mean a sequence $\{V_i\}_{i=0}^d$ consisting of nonzero subspaces of V such that $V = \sum_{i=0}^d V_i$ (direct sum). For notational convenience we set $V_{-1} := 0$, $V_{d+1} := 0$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V. By the *shape* of this decomposition we mean the sequence $\{\rho_i\}_{i=0}^d$, where ρ_i is the dimension of V_i for $0 \leqslant i \leqslant d$.

By a *linear transformation* on V, we mean an $\mathbb{F}$-linear map from V to V. Let End(V) denote the $\mathbb{F}$-algebra consisting of all linear transformations on V.

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Let A denote a linear transformation on V. By an *eigenspace* of A, we mean a nonzero subspace W of V of the form

$$W = \{v \in V | Av = \theta v\},$$

where $\theta \in \mathbb{F}$. In this case, we call θ the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [5, Definition 9.1]**.** Let $\{V_i\}_{i=0}^{d}$ denote a decomposition of V. For $0 \leqslant i \leqslant d$ let $E_i : V \to V$ denote the linear transformation that satisfies

$$(E_i - I)V_i = 0, \quad E_iV_j = 0, \quad \text{if } i \neq j \ (0 \leqslant j \leqslant d).$$

We observe that E_i is the projection from V onto V_i. We note that $E_iV = V_i$ and

$$I = \sum_{i=0}^{d} E_i, \quad E_iE_j = \delta_{ij}E_i \quad \text{for } 0 \leqslant i, j \leqslant d. \tag{1}$$

Therefore, the sequence $\{E_i\}_{i=0}^{d}$ is a basis for a commutative subalgebra $\mathcal{D}$ of $\mathrm{End}(V)$.

Ito and Terwilliger proposed the following problem.

Problem 1.2 [5, Problem 9.2]**.** *Let $\{V_i\}_{i=0}^{d}$ and $\{V_i^*\}_{i=0}^{\delta}$ denote decompositions of V. Let $\mathcal{D}$ and $\mathcal{D}^*$ denote the corresponding commutative algebras from Definition 1.1. Investigate the case in which* (i)–(v) *hold below:*

(i) *$\mathcal{D}$ has a generator A_+ such that*

$$A_+V_i^* \subseteq V_0^* + \cdots + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta). \tag{2}$$

(ii) *$\mathcal{D}$ has a generator A_- such that*

$$A_-V_i^* \subseteq V_{i-1}^* + \cdots + V_\delta^* \quad (0 \leqslant i \leqslant \delta). \tag{3}$$

(iii) *$\mathcal{D}^*$ has a generator A_+^* such that*

$$A_+^*V_i \subseteq V_0 + \cdots + V_{i+1} \quad (0 \leqslant i \leqslant d). \tag{4}$$

(iv) *$\mathcal{D}^*$ has a generator A_-^* such that*

$$A_-^*V_i \subseteq V_{i-1} + \cdots + V_d \quad (0 \leqslant i \leqslant d). \tag{5}$$

(v) *There does not exist a subspace $W \subseteq V$ such that $\mathcal{D}W \subseteq W$ and $\mathcal{D}^*W \subseteq W$, other than $W = 0$ and $W = V$.*

Remark 1.3 [5, Note 9.3]**.** Let A, A^* denote a tridiagonal pair on V, as in [4, Definition 1.1]. Then the conditions (i)–(v) of Problem 1.2 are satisfied with

$$A_+ = A, \quad A_- = A, \quad A_+^* = A^*, \quad A_-^* = A^*.$$

Remark 1.4 [5, Note 9.4]**.** Let K, K^* denote a q-inverting pair on V, as in [5, Definition 4.1]. Then the conditions (i)–(v) of Problem 1.2 are satisfied with

$$A_+ = K^{-1}, \ \ A_- = K, \quad A_+^* = K^*, \quad A_-^* = K^{*-1}.$$

Referring to Problem 1.2, we have $d = \delta$ [3, Proposition 2.3], we call this common value the *diameter* of A_+, A_-, A_+^*, A_-^*. For $0 \leqslant i \leqslant d$, the dimensions of V_i and V_i^* coincide [3, Theorem 3.6]; we denote this common value by ρ_i. The sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal, i.e. $\rho_i = \rho_{d-i}$ for $0 \leqslant i \leqslant d$ and $\rho_{i-1} \leqslant \rho_i$ for $1 \leqslant i \leqslant d/2$ [3, Theorems 3.6 and 3.12]. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A_+, A_-, A_+^*, A_-^*.

In this paper we solve Problem 1.2 with the following results. We show that each of A_+, A_-, A_+^*, A_-^* is determined up to affine transformation by the sequences $\{V_i\}_{i=0}^d$; $\{V_i^*\}_{i=0}^d$. We also show that the following are equivalent: (i) there exists a nonzero bilinear form $\langle\ ,\ \rangle$ on V such that $\langle A_+u, v\rangle = \langle u, A_+v\rangle$ and $\langle A_+^*u, v\rangle = \langle u, A_+^*v\rangle$ for all $u, v \in V$; (ii) there exist scalars $\alpha, \alpha^*, \beta, \beta^*$ in $\mathbb{F}$ with α, α^* nonzero such that $A_- = \alpha A_+ + \beta I$ and $A_-^* = \alpha^* A_+^* + \beta^* I$; and (iii) both A_+, A_+^* and A_-, A_-^* are tridiagonal pairs.

2. The split decompositions

Referring to Problem 1.2, we note that both A_+, A_- are diagonalizable on V with eigenspaces $\{V_i\}_{i=0}^d$ and both A_+^*, A_-^* are diagonalizable on V with eigenspaces $\{V_i^*\}_{i=0}^d$. For $0 \leqslant i \leqslant d$, let θ_i (resp. ξ_i) denote the eigenvalue of A_+ (resp. A_-) associated with V_i and let θ_i^* (resp. ξ_i^*) denote the eigenvalue of A_+^* (resp. A_-^*) associated with V_i^*. Assume that E_i (resp. E_i^*) is the projection from V onto V_i (resp. V_i^*) for $0 \leqslant i \leqslant d$. By elementary linear algebra, we have the following equations:

$$E_i = \prod_{0 \leqslant j \leqslant d, j \neq i} \frac{A_+ - \theta_j I}{\theta_i - \theta_j} = \prod_{0 \leqslant j \leqslant d, j \neq i} \frac{A_- - \xi_j I}{\xi_i - \xi_j}; \tag{6}$$

$$E_i^* = \prod_{0 \leqslant j \leqslant d, j \neq i} \frac{A_+^* - \theta_j^* I}{\theta_i^* - \theta_j^*} = \prod_{0 \leqslant j \leqslant d, j \neq i} \frac{A_-^* - \xi_j^* I}{\xi_i^* - \xi_j^*}. \tag{7}$$

Referring to Problem 1.2 and by [1, Definitions 1.1 and 1.4], the pair A_+, A_+^* is irreducible and Hessenberg with respect to the orderings $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$; the pair A_+, A_-^* is irreducible and Hessenberg with respect to the orderings $(\{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$; the pair A_-, A_+^* is irreducible and Hessenberg with respect to the orderings $(\{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$ and the pair A_-, A_-^* is irreducible and Hessenberg with respect to the orderings $(\{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$. For more information on Hessenberg pairs, see [1,2].

For the irreducible Hessenberg pair A_+, A_+^*, define

$$U_i = (V_0 + V_1 + \cdots + V_{d-i}) \cap (V_0^* + V_1^* + \cdots + V_i^*) \tag{8}$$

for $0 \leqslant i \leqslant d$. By [1, Lemma 2.5] the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V, which is called the *split decomposition* of V associated with A_+, A_+^*. Moreover, by [1, Lemmas 2.3 and 3.1] the following hold for $0 \leqslant i \leqslant d$,

$$(A_+ - \theta_{d-i} I)U_i \subseteq U_{i+1}, \quad (A_+^* - \theta_i^* I)U_i \subseteq U_{i-1}; \tag{9}$$

$$U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \quad U_i + \cdots + U_d = V_0 + \cdots + V_{d-i}. \tag{10}$$

3. A subalgebra of End(V)

Referring to Problem 1.2, $\mathcal{D}$ is viewed as the subalgebra of End(V) generated by A_+ (or A_-). In what follows we often view $\mathcal{D}$ as a vector space over $\mathbb{F}$. The dimension of $\mathcal{D}$ is $d+1$ by construction. Moreover, $\{A_+^i | 0 \leqslant i \leqslant d\}$ is a basis for $\mathcal{D}$. There is another basis for $\mathcal{D}$ that is better suited to our purpose. To define it we use the following notation. Let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra of all polynomials in an indeterminate λ that have coefficients in $\mathbb{F}$. For $0 \leqslant i \leqslant d$ we define

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}). \tag{11}$$

We note that η_i is monic with degree i. Therefore $\{\eta_i(A_+) | 0 \leqslant i \leqslant d\}$ is a basis for $\mathcal{D}$. Applying (9) and (11) to the split decomposition $\{U_i\}_{i=0}^d$ of V associated with the irreducible Hessenberg pair A_+, A_+^*, we find

$$\eta_i(A_+)U_0 \subseteq U_i \quad (0 \leqslant i \leqslant d). \tag{12}$$

Extending the argument of [6, Lemma 3.1], we get the following lemma.

Lemma 3.1. *Referring to Problem 1.2, for all nonzero $u \in V_0^*$ and for all nonzero $X \in \mathcal{D}$, we have $Xu \neq 0$.*

Proof. It suffices to show that the vector spaces $\mathcal{D}$ and $\mathcal{D}u$ have the same dimension. We saw earlier that $\{\eta_i(A_+) | 0 \leqslant i \leqslant d\}$ is a basis for $\mathcal{D}$. We show that $\{\eta_i(A_+)u | 0 \leqslant i \leqslant d\}$ is a basis for $\mathcal{D}u$. By (10) and (12) and since $U_0 = V_0^*$, this will hold if we can show $\eta_i(A_+)u \neq 0$ for $0 \leqslant i \leqslant d$. Let i be given and suppose $\eta_i(A_+)u = 0$. We will obtain a contradiction by displaying a subspace W of V that violates Problem 1.2(v). Observe that $i \neq 0$ since $\eta_0 = 1$ and $u \neq 0$. So $i \geqslant 1$. By (11) and since $\eta_i(A_+)u = 0$ we find $u \in V_{d-i+1} + \cdots + V_{d-1} + V_d$. Therefore

$$u \in V_0^* \cap (V_{d-i+1} + \cdots + V_{d-1} + V_d). \tag{13}$$

Define

$$W_r = (V_0^* + V_1^* + \cdots + V_r^*) \cap (V_{d-i+r+1} + \cdots + V_{d-1} + V_d) \tag{14}$$

for $0 \leqslant r \leqslant i-1$ and put

$$W = W_0 + W_1 + \cdots + W_{i-1}. \tag{15}$$

We show W violates Problem 1.2(v). Observe that $W \neq 0$ since the nonzero vector $u \in W_0$ by (13) and since $W_0 \subseteq W$. Next we show $W \neq V$. By (14), for $0 \leqslant r \leqslant i-1$ we have

$$W_r \subseteq V_0^* + V_1^* + \cdots + V_r^* \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^*.$$

By this and (15)

$$W \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^* \subseteq V_0^* + V_1^* + \cdots + V_{d-1}^*.$$

Combining this with the decomposition

$$V = V_0^* + V_1^* \cdots + V_d^* \quad \text{(direct sum)} \tag{16}$$

and using $V_d^* \neq 0$ we find $W \neq V$. We now show $\mathcal{D}W \subseteq W$. Since A_+ is a generator of $\mathcal{D}$, it suffices to show that $(A_+ - \theta_{d-i+r+1}I)W_r \subseteq W_{r+1}$ for $0 \leqslant r \leqslant i-1$, where $W_i := 0$. Let r be given. From the construction we have

$$(A_+ - \theta_{d-i+r+1}I) \sum_{h=d-i+r+1}^{d} V_h = \sum_{h=d-i+r+2}^{d} V_h. \tag{17}$$

By Problem 1.2(i) we have

$$(A_+ - \theta_{d-i+r+1}I) \sum_{h=0}^{r} V_h^* \subseteq \sum_{h=0}^{r+1} V_h^*. \tag{18}$$

Combining (17) and (18) we find $(A_+ - \theta_{d-i+r+1}I)W_r \subseteq W_{r+1}$ as desired. We have shown $\mathcal{D}W \subseteq W$. We now show $\mathcal{D}^*W \subseteq W$. Since A^*_- is a generator of $\mathcal{D}^*$, it suffices to show that $(A^*_- - \xi^*_r I)W_r \subseteq W_{r-1}$ for $0 \leqslant r \leqslant i-1$, where $W_{-1} := 0$. Let r be given. From the construction we have

$$(A^*_- - \xi^*_r I)\sum_{h=0}^{r} V^*_h \subseteq \sum_{h=0}^{r-1} V^*_h. \tag{19}$$

By Problem 1.2(iv) we have

$$(A^*_- - \xi^*_r I)\sum_{h=d-i+r+1}^{d} V_h = \sum_{h=d-i+r}^{d} V_h. \tag{20}$$

Combining (19) and (20) we find $(A^*_- - \xi^*_r I)W_r \subseteq W_{r-1}$ as desired. We have shown $\mathcal{D}^*W \subseteq W$. We have now shown that $W \neq 0$, $W \neq V$, $\mathcal{D}W \subseteq W$, $\mathcal{D}^*W \subseteq W$, contradicting Problem 1.2(v). We conclude $\eta_i(A_+)u \neq 0$ and the result follows. $\square$

4. Each of A_+, A_-, A^*_+, A^*_- is determined up to affine transformation by the eigenspaces

Referring to Problem 1.2, let $\{U_i\}_{i=0}^d$ be the split decomposition of V associated with A_+, A^*_+. In this section we show that each of A_+, A_-, A^*_+, A^*_- is determined up to affine transformation by the eigenspaces V_i and V^*_i $(0 \leqslant i \leqslant d)$.

Extending the argument of [6, Lemma 4.1], we get the following lemma.

Lemma 4.1. *Referring to Problem 1.2, assume that* $d \geqslant 1$. *Then the following* (i)–(ii) *are equivalent for all* $X \in \mathrm{End}(V)$.

(i) $X \in \mathcal{D}$ *and* $XV^*_0 \subseteq V^*_0 + V^*_1$.
(ii) *There exist scalars* r, s *in* $\mathbb{F}$ *such that* $X = rA_+ + sI$.

Proof. (i)⇒(ii): Assume $X \neq 0$; otherwise the result is trivial. Pick a nonzero $u \in V^*_0$ and note that $u \in U_0$ by (10). We have $Xu \in V^*_0 + V^*_1$ by assumption, so

$$Xu \in U_0 + U_1 \tag{21}$$

in view of (10). Recall $\{\eta_i(A_+) | 0 \leqslant i \leqslant d\}$ is a basis for $\mathcal{D}$. We assume $X \in \mathcal{D}$, so there exists $\alpha_i \in \mathbb{F}(0 \leqslant i \leqslant d)$ such that

$$X = \sum_{i=0}^{d} \alpha_i \eta_i(A_+). \tag{22}$$

We show $\alpha_i = 0$ for $2 \leqslant i \leqslant d$. Suppose not and define $j = \max\{i | 2 \leqslant i \leqslant d, \alpha_i \neq 0\}$. We will obtain a contradiction by showing

$$0 \neq U_j \cap (U_0 + U_1 + \cdots + U_{j-1}). \tag{23}$$

Note that $\eta_j(A_+)u \neq 0$ by Lemma 3.1 and $\eta_j(A_+)u \in U_j$ by (12). Also by (22) we find $\eta_j(A_+)u$ is in the span of Xu and $\eta_0(A_+)u, \eta_1(A_+)u, \ldots, \eta_{j-1}(A_+)u$; combining this with (12) and (21) we find $\eta_j(A_+)u$ is contained in $U_0 + U_1 + \cdots + U_{j-1}$. By these comments $\eta_j(A_+)u$ is a nonzero element in $U_j \cap (U_0 + U_1 + \cdots + U_{j-1})$ and (23) follows. Line (23) contradicts the fact that $\{U_i\}_{i=0}^d$ is a decomposition of V and we conclude $\alpha_i = 0$ for $2 \leqslant i \leqslant d$. Now $X = \alpha_1\eta_1(A_+) + \alpha_0 I$. Therefore $X = rA_+ + sI$ with $r = \alpha_1$ and $s = \alpha_0 - \alpha_1\theta_0$.

(ii)⇒(i): Immediate from Problem 1.2(i). $\square$

Theorem 4.2. *Let A_+, A_-, A^*_+, A^*_- be linear transformations on V satisfying the conditions* (i)–(v) *of Problem 1.2 for the orderings of $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^d$. Let $A'_+, A'_-, A'^*_+, A'^*_-$ also be linear transformations on V satisfying the conditions* (i)–(v) *of Problem 1.2 for the orderings of $\{V'_i\}_{i=0}^d$ and $\{V'^*_i\}_{i=0}^d$. Assume $V_i = V'_i$ and $V^*_i = V'^*_i$ for $0 \leqslant i \leqslant d$. Then* $\text{Span}\{A_+, I\} = \text{Span}\{A'_+, I\}$, $\text{Span}\{A_-, I\} = \text{Span}\{A'_-, I\}$, *and* $\text{Span}\{A^*_+, I\} = \text{Span}\{A'^*_+, I\}$, $\text{Span}\{A^*_-, I\} = \text{Span}\{A'^*_-, I\}$.

Proof. Assume $d \geqslant 1$; otherwise the result is clear. Let $\mathcal{D}$ (resp. $\mathcal{D}'$) denote the subalgebra of $\text{End}(V)$ generated by A_+ (resp. A'_+). Since $V_i = V'_i$ for $0 \leqslant i \leqslant d$ we find $\mathcal{D} = \mathcal{D}'$. So $A'_+ \in \mathcal{D}$. Applying Lemma 4.1 to the linear transformations A_+, A_-, A^*_+, A^*_- on V (with $X = A'_+$), there exist $r, s \in \mathbb{F}$ such that $A'_+ = rA_+ + sI$. Note that $r \neq 0$; otherwise $A'_+ = sI$ has a single eigenspace which contradicts $d \geqslant 1$. It follows that $\text{Span}\{A_+, I\} = \text{Span}\{A'_+, I\}$. Similarly we find the other assertions. □

5. The bilinear forms

Throughout this section let V' denote a vector space over $\mathbb{F}$ such that $\dim V' = \dim V$.

A map $\langle\, , \,\rangle : V \times V' \to \mathbb{F}$ is called a *bilinear form* whenever the following conditions hold for $u, v \in V$, for $u', v' \in V'$, and for $\alpha \in \mathbb{F}$: (i) $\langle u + v,\ u'\rangle = \langle u,\ u'\rangle + \langle v,\ u'\rangle$; (ii) $\langle \alpha u,\ u'\rangle = \alpha\langle u,\ u'\rangle$; (iii) $\langle u,\ u' + v'\rangle = \langle u,\ u'\rangle + \langle u,\ v'\rangle$; and (iv) $\langle u,\ \alpha u'\rangle = \alpha\langle u,\ u'\rangle$. Let $\langle\, , \,\rangle : V \times V' \to \mathbb{F}$ denote a bilinear form. Then the following are equivalent: (i) there exists a nonzero $v \in V$ such that $\langle v,\ v'\rangle = 0$ for all $v' \in V'$; (ii) there exists a nonzero $v' \in V'$ such that $\langle v,\ v'\rangle = 0$ for all $v \in V$. The form is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. By a *bilinear form on V* we mean a bilinear form $\langle\, , \,\rangle : V \times V \to \mathbb{F}$.

By an $\mathbb{F}$-*algebra anti-isomorphism* from $\text{End}(V)$ to $\text{End}(V')$ we mean an isomorphism of $\mathbb{F}$-vector spaces $\sigma : \text{End}(V) \to \text{End}(V')$ such that $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \text{End}(V)$. By an *anti-automorphism* of $\text{End}(V)$ we mean an $\mathbb{F}$-algebra anti-isomorphism from $\text{End}(V)$ to $\text{End}(V)$.

Let $\langle\, , \,\rangle : V \times V' \to \mathbb{F}$ denote a nondegenerate bilinear form. Then there exists a unique anti-isomorphism $\sigma : \text{End}(V) \to \text{End}(V')$ such that $\langle Xv,\ v'\rangle = \langle v,\ X^\sigma v'\rangle$ for all $v \in V$, $v' \in V'$ and $X \in \text{End}(V)$. Conversely, given an anti-isomorphism $\sigma : \text{End}(V) \to \text{End}(V')$, there exists a nonzero bilinear form $\langle\, , \,\rangle : V \times V' \to \mathbb{F}$ such that $\langle Xv,\ v'\rangle = \langle v,\ X^\sigma v'\rangle$ for all $v \in V$, $v' \in V'$ and $X \in \text{End}(V)$. This form is nondegenerate and uniquely determined by σ up to multiplication by a nonzero scalar in $\mathbb{F}$. We say the form $\langle\, , \,\rangle$ is *associated* with σ. For more information on bilinear forms, see [7].

Lemma 5.1. *Referring to Problem 1.2, let $\langle\, , \,\rangle$ denote a nonzero bilinear form on V that satisfies*

$$\langle A_+u, v\rangle = \langle u, A_+v\rangle, \quad \langle A^*_+u, v\rangle = \langle u, A^*_+v\rangle, \quad \text{for } u, v \in V. \tag{24}$$

Then $\langle\, , \,\rangle$ is nondegenerate.

Proof. It suffices to show that the space $W = \{w \in V | \langle w, V\rangle = 0\}$ is zero. Using (24) and since A_+ generates $\mathcal{D}$ we routinely find $\mathcal{D}W \subseteq W$. Similarly $\mathcal{D}^*W \subseteq W$. Therefore $W = 0$ or $W = V$ in view of Problem 1.2(v). But $W \neq V$ since $\langle\, , \,\rangle$ is nonzero, so $W = 0$ as desired. □

Lemma 5.2. *Referring to Problem 1.2, let $\langle\, , \,\rangle$ denote a nonzero bilinear form on V that satisfies* (24). *Then we have*

$$\langle A_-u, v\rangle = \langle u, A_-v\rangle, \ \langle A^*_-u, v\rangle = \langle u, A^*_-v\rangle, \ \text{for } u, v \in V.$$

Proof. By the equation on the left in (24) and since A_+ generates $\mathcal{D}$ we see $\langle Xu, v\rangle = \langle u, Xv\rangle$ for all $X \in \mathcal{D}$ and all $u, v \in V$. Now taking $X = A_-$ we get the equation on the left in Lemma 5.2. The equation on the right in Lemma 5.2 is similarly proved. □

Lemma 5.3. *Referring to Problem 1.2, let* $\langle\, , \,\rangle$ *denote a nonzero bilinear form on* V *that satisfies* (24). *Then there exist scalars* $\alpha, \alpha^*, \beta, \beta^*$ *in* $\mathbb{F}$ *with* α, α^* *nonzero such that* $A_- = \alpha A_+ + \beta I$ *and* $A_-^* = \alpha^* A_+^* + \beta^* I$.

Proof. Recall that the linear transformations A_+, A_-, A_+^*, A_-^* satisfy the conditions (i)–(v) of Problem 1.2 for the orderings of $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^d$. By Lemma 5.1 the bilinear form $\langle\, , \,\rangle$ is nondegenerate and let σ denote the anti-automorphism of $\mathrm{End}(V)$ associated with $\langle\, , \,\rangle$. Applying [3, Theorem 4.1] to σ and using Lemma 5.2 we see that the linear transformations A_-, A_+, A_-^*, A_+^* satisfy the conditions (i)–(v) of Problem 1.2 for the orderings of $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^d$. We have shown that both A_+, A_-, A_+^*, A_-^* and A_-, A_+, A_-^*, A_+^* satisfy the conditions (i)–(v) of Problem 1.2 for the orderings of $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^d$. Thus there exist scalars $\alpha, \alpha^*, \beta, \beta^*$ in $\mathbb{F}$ with α, α^* nonzero such that $A_- = \alpha A_+ + \beta I$ and $A_-^* = \alpha^* A_+^* + \beta^* I$ by Theorem 4.2. □

Lemma 5.4. *Referring to Problem 1.2, if there exist scalars* $\alpha, \alpha^*, \beta, \beta^*$ *in* $\mathbb{F}$ *with* α, α^* *nonzero such that* $A_- = \alpha A_+ + \beta I$ *and* $A_-^* = \alpha^* A_+^* + \beta^* I$, *then both* A_+, A_+^* *and* A_-, A_-^* *are tridiagonal pairs.*

Proof. Assume the linear transformations A_+, A_-, A_+^*, A_-^* satisfy the conditions (i)–(v) of Problem 1.2. Note that the pair A_+, A_+^* is irreducible and Hessenberg with respect to the orderings $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ and the pair A_-, A_-^* is irreducible and Hessenberg with respect to the orderings $(\{V_{d-i}\}_{i=0}^d, \ \{V_{d-i}^*\}_{i=0}^d)$. Since there exist scalars $\alpha, \alpha^*, \beta, \beta^*$ in $\mathbb{F}$ with α, α^* nonzero such that $A_- = \alpha A_+ + \beta I$ and $A_-^* = \alpha^* A_+^* + \beta^* I$, the irreducible Hessenberg pairs A_+, A_+^* and A_-, A_-^* have the same orderings of eigenspace sequences. So the pair A_+, A_+^* is irreducible and Hessenberg with respect to the orderings of $(\{V_{d-i}\}_{i=0}^d, \ \{V_{d-i}^*\}_{i=0}^d)$. We have shown that the pair A_+, A_+^* is irreducible and Hessenberg with respect to each of $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ and $(\{V_{d-i}\}_{i=0}^d, \ \{V_{d-i}^*\}_{i=0}^d)$. By [1, Proposition 4.4] A_+, A_+^* is a tridiagonal pair. Similarly A_-, A_-^* is a tridiagonal pair. □

From [8, Theorem 1.4] and Lemmas 5.3 and 5.4 we have the following theorem.

Theorem 5.5. *Referring to Problem 1.2, the following* (i)–(iii) *are equivalent:*

(i) *There exists a nonzero bilinear form* $\langle\, , \,\rangle$ *on* V *that satisfies* (24).
(ii) *There exist scalars* $\alpha, \alpha^*, \beta, \beta^*$ *in* $\mathbb{F}$ *with* α, α^* *nonzero such that* $A_- = \alpha A_+ + \beta I$ *and* $A_-^* = \alpha^* A_+^* + \beta^* I$.
(iii) *Both* A_+, A_+^* *and* A_-, A_-^* *are tridiagonal pairs.*

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