Available online at www.sciencedirect.com

ScienceDirect

Journal of Combinatorial Theory, Series A 115 (2008) 254–278

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

On the number of pseudo-triangulations of certain point sets ☆

Oswin Aichholzer [a], David Orden [b], Francisco Santos [c], Bettina Speckmann [d]

[a] *Institute for Software Technology, Graz University of Technology, Austria*
[b] *Departamento de Matemáticas, Universidad de Alcalá, Spain*
[c] *Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Spain*
[d] *Department of Mathematics and Computer Science, TU Eindhoven, The Netherlands*

Received 30 January 2006

Available online 28 June 2007

Abstract

We pose a monotonicity conjecture on the number of pseudo-triangulations of any planar point set, and check it on two prominent families of point sets, namely the so-called double circle and double chain. The latter has asymptotically $12^n n^{\Theta(1)}$ pointed pseudo-triangulations, which lies significantly above the maximum number of triangulations in a planar point set known so far.

Keywords: Pseudo-triangulations; Triangulations; Double-circle; Double-chain; Counting

☆ Parts of this work were done while the authors visited the Departamento de Matemática Aplicada II, Universitat Politècnica de Catalunya, the Institut für Softwaretechnologie, Technische Universität Graz and the Departamento de Matemáticas, Universidad de Alcalá.

Research of Oswin Aichholzer is partially supported by Acciones Integradas España–Austria and by the FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung] under grant S09205-N12, FSP Industrial Geometry. Research of David Orden and Francisco Santos is partially supported by Acciones Integradas España–Austria and by grant MTM2005-08618-C02-02 of Spanish Dirección General de Investigación Científica. David Orden is additionally supported by S0505/DPI/0235-02.

E-mail addresses: oaich@ist.tugraz.at (O. Aichholzer), david.orden@uah.es (D. Orden), santosf@unican.es (F. Santos), speckman@win.tue.nl (B. Speckmann).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.06.002

1. Introduction

Pseudo-triangulations, also called geodesic triangulations, are a generalization of triangulations which has found multiple applications in Computational Geometry in the last ten years. They were originally introduced in the context of visibility [20,21] and ray shooting [8,14], but recently have also been applied in kinetic collision detection [1,17], and guarding problems [27], among others. They also have surprising relations to rigidity [15,19,29,30] and locally convex functions [3]. See the recent survey [24] for more information.

A *pseudo-triangle* is a planar polygon that has exactly three convex vertices with internal angles less than π. These vertices are called *corners* and the three inward convex polygonal chains joining them are called *pseudo-edges* of the pseudo-triangle. A *pseudo-triangulation* for a set A of n points in the plane is a partition of $\mathrm{conv}(A)$ into pseudo-triangles whose vertex set is exactly A. Although pseudo-triangulations can be studied for general point sets [18], in this paper we will consider only point sets in general position. A vertex is *pointed* if it has an incident angle greater than π. A *pointed pseudo-triangulation* is a pseudo-triangulation where every vertex is pointed. See, for example, Fig. 1—here, and in Fig. 2, pointed vertices are dark, non-pointed vertices are light. Note that the vertices of $\mathrm{conv}(A)$ are always pointed.

The set of all pseudo-triangulations of a point set has somewhat nicer properties than that of all triangulations. For example, pseudo-triangulations of a point set with n elements form the vertex set of a certain polytope of dimension $3n-3$ whose edges correspond to flips [18]. The diameter of the graph of pseudo-triangulations is $O(n\log n)$ [4] versus the $\Theta(n^2)$ diameter of the graph of triangulations of certain point sets. Also, for standard triangulations, it is not known which sets with a given number of points have the fewest or the most triangulations, but it was shown in [2] that sets in convex position minimize the number of pointed pseudo-triangulations among all point sets with a given number of vertices (hence the number of all pseudo-triangulations, since in convex position all pseudo-triangulations are pointed).

Let A be a point set and let A_I be the subset of its interior points. Let $\mathcal{PT}(A)$ be the set of pseudo-triangulations of A. This set can naturally be stratified into 2^{A_I} sets, one for each possible subset of A_I. More precisely, for each subset $W \subseteq A_I$ we denote by $\mathcal{PT}_W(A)$ the set of pseudo-triangulations of A in which the points of W are pointed and those of $A_I \setminus W$ are non-pointed. For example, $\mathcal{PT}_\emptyset(A)$ is the set of triangulations of A, which we abbreviate as $\mathcal{T}(A)$. Similarly, $\mathcal{PT}_{A_I}(A)$ is the set of pointed pseudo-triangulations of A, that we abbreviate as $\mathcal{PPT}(A)$. In [22], the following inequality is proved: for every $W \subseteq A_I$ and every $p \in W$,

$$3\left|\mathcal{PT}_{W\setminus\{p\}}(A)\right| \geqslant \left|\mathcal{PT}_W(A)\right|. \tag{1}$$

The main goal of this paper is to explore the relation between the numbers of triangulations, pointed pseudo-triangulations, and everything in between, for several specific point sets. In par-

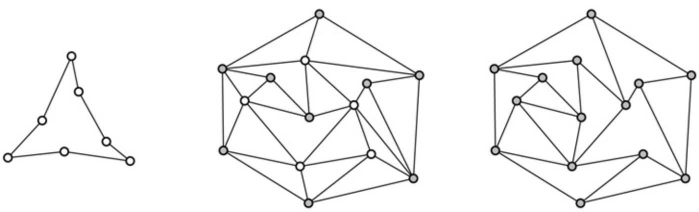


Fig. 1. A pseudo-triangle (left), a pseudo-triangulation (middle), a pointed pseudo-triangulation (right).

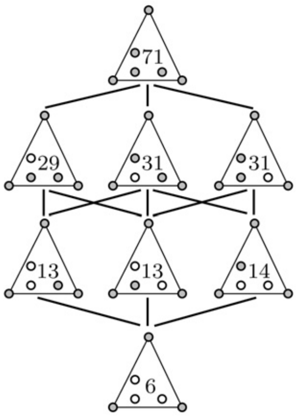


Fig. 2. Hasse diagram of the subsets W of A_I for a set of 6 points.

ticular, we test the following conjecture, which is implicit in previous work, but stated here explicitly for the first time:

Conjecture 1. *For every point set A in general position in the plane, the cardinalities of* $\mathcal{PT}_W(A)$ *are monotone with respect to W. That is to say, for any subset W of A's interior points and for every* $p \in W$*, one has*

$$\left|\mathcal{PT}_W(A)\right| \geqslant \left|\mathcal{PT}_{W\setminus\{p\}}(A)\right|.$$

The conjecture is consistent with the following result, also from [22]: If A has a single interior point and $n-1$ boundary points, then $|\mathcal{PPT}(A)|$ is greater than $|\mathcal{T}(A)|$. Actually, the difference is always equal to the Catalan number C_{n-2}, no matter where the interior point is, while $|\mathcal{T}(A)|$ ranges from $C_{n-2} - C_{n-3} \simeq \frac{3}{4}C_{n-2}$ when the interior point is near the boundary to essentially C_{n-2} when it is near the center.

Conjecture 1 does not imply that the number of pseudo-triangulations of A with, say, k pointed vertices is greater than the number of them with $k-1$ pointed vertices. For example, Fig. 2 shows the eight possibilities of $|\mathcal{PT}_W(A)|$ for a set of six points, three of them interior, displayed in the Hasse diagram of subsets of A_I. The numbers satisfy Conjecture 1, but there are less pointed pseudo-triangulations (71) than pseudo-triangulations with one non-pointed and two pointed vertices ($29 + 31 + 31 = 91$).

Actually, formula (1) says that the same will happen for any point set with at least four interior points. Applied with $W = A_I$ and taking the different possibilities for $v \in A_I$, the formula gives

$$\sum_{v\in A_I} 3\left|\mathcal{PT}_{A_I\setminus\{v\}}(A)\right| \geqslant |A_I| \cdot \left|\mathcal{PT}_{A_I}(A)\right|.$$

In other words, the ratio of pseudo-triangulations with exactly one non-pointed vertex to pointed pseudo-triangulations is at least $|A_I|/3$.

Similarly, the monotonicity is conjectured with respect to the sets W and not only their cardinalities: There exists a set A of 10 points, 7 of them interior, and two subsets W and W' of four and three interior points, respectively, with $|\mathcal{PT}_W(A)| < |\mathcal{PT}_{W'}(A)|$. There are no examples like this with less than 10 points.

As initial evidence for Conjecture 1 we have computed the numbers of triangulations and of pointed pseudo-triangulations for all order types of planar point sets in general position with 10 points or less. This has been done using the order type database in [7]. One implication of Conjecture 1 is that every point set should have at least as many pointed pseudo-triangulations as triangulations, and this is actually the case up to 10 points. Even more, the natural expectation is that the ratio between those two numbers grows exponentially with the number of interior points i, the base of the exponent being between 1 (by Conjecture 1) and 3 (by inequality (1)). In Tables 1 and 2 we show the maximum and the minimum values of the ratio $(|\mathcal{PPT}(A)|/|\mathcal{T}(A)|)^{1/i}$ obtained for each value of the total number of points n and of interior points i.

It is interesting to observe that rows (fixed number of boundary points) and columns (fixed number of interior points) are monotone in both tables, while diagonals (fixed total number of points) are not always monotone in Table 2 (see diagonal $n = 6$ and $n = 9$).

From Table 3 below we can derive the (asymptotic) value of the same parameter $(|\mathcal{PPT}(A)|/|\mathcal{T}(A)|)^{1/i}$ for certain families of planar point sets which are the main object in this paper: double circle, single chain, and double chain. The results are $7/3 \approx 2.333$, 2 and 1.5, respectively. Also, the results in [22] say that if A has a single interior point then

$$\frac{7}{3} \simeq 1 + \frac{C_{n-2}}{C_{n-2} - C_{n-3}} \geqslant \left|\mathcal{PPT}(A)\right|/\left|\mathcal{T}(A)\right| \geqslant 1 + \frac{2C_{n-2}}{C_{n-1} - (n-1)C^2_{(n-2)/2}} \simeq 2,$$

with equality on the left when the interior point is close to the boundary, and on the right when the interior point is at the center of a regular $(n-1)$-gon, with n even.

In the rest of the paper we consider three families of point sets in the plane: "double circles," "double chains," and what we call "single chains." See Fig. 3 for examples, the exact definitions are given in the respective sections. The double circle is conjectured to be the point set with asymptotically the smallest number of triangulations, for a fixed number of points [7]. The

Table 1
Maximum values of $(\frac{|\mathcal{PPT}(A)|}{|\mathcal{T}(A)|})^{1/i}$ for order types with at most 10 points

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
$n - i = 3$	3.00000	2.54951	2.50665	2.38010	2.31659	2.26583	2.23025
$n - i = 4$	2.66667	2.51661	2.47042	2.44151	2.35995	2.30562	
$n - i = 5$	2.55556	2.48151	2.44824	2.42734	2.41308		
$n - i = 6$	2.50000	2.45607	2.43210	2.41625			
$n - i = 7$	2.46667	2.43763	2.41980				
$n - i = 8$	2.44444	2.42384					
$n - i = 9$	2.42857						

Table 2
Minimum values of $(\frac{|\mathcal{PPT}(A)|}{|\mathcal{T}(A)|})^{1/i}$ for order types with at most 10 points

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
$n - i = 3$	3.00000	2.54951	2.11791	2.00415	1.88343	1.80952	1.75590
$n - i = 4$	2.66667	2.29129	2.01550	1.91670	1.82364	1.76240	
$n - i = 5$	2.27273	2.08637	1.91798	1.84371	1.77002		
$n - i = 6$	2.16667	1.99211	1.83797	1.78048			
$n - i = 7$	2.03937	1.91361	1.80419				
$n - i = 8$	1.98621	1.86445					
$n - i = 9$	1.92318						

Table 3
Asymptotic number of triangulations, pointed pseudo-triangulations and pseudo-triangulations, for special point sets

	double circle	single chain	double chain
$\lvert\mathcal{T}(A)\rvert$	$\sqrt{12}^n$	4^n	8^n
$\lvert\mathcal{PPT}(A)\rvert$	$\sqrt{28}^n$	8^n	12^n
$\lvert\mathcal{PT}(A)\rvert$	$\sqrt{40}^n$	12^n	20^n
Conjecture 1	Holds	Holds	Holds

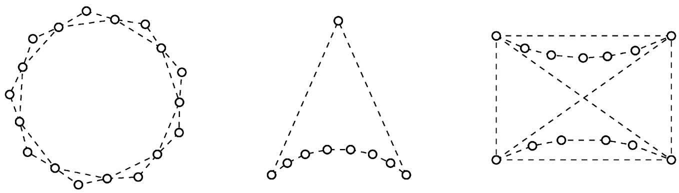


Fig. 3. A double circle (left), a single chain (middle), a double chain (right).

double chain has been the example with (asymptotically) the biggest number of triangulations known ($\Theta^*(8^n)$, see [25]), until a new structure was found recently, the so-called double zig-zag chain, with $\Theta^*(\sqrt{72}^n)$ triangulations [5]. (Here and in the rest of the paper, the notation Θ^* means that a polynomial factor is neglected.) We studied single chains originally as a step to analyze double chains, but it turns out that the number of pseudo-triangulations of single chains also has very interesting combinatorial properties (see Theorem 14).

Our interest in the number of pseudo-triangulations of these point sets is twofold. On the one hand, we prove that Conjecture 1 holds in these three cases. On the other hand, we are interested in how many pseudo-triangulations a general point set in the plane can have. The study of these point sets, which have very many or very few triangulations, should give an indication of it. Even if the minimum number of pseudo-triangulations is achieved by the convex n-gon (as mentioned above), it may well be that the double circle (or, more generally, the point sets in "almost convex position" studied in Section 2) minimize the numbers of pseudo-triangulations for fixed numbers of boundary and interior points.

Our main results are summarized in Table 3, where we show only the global number of all pseudo-triangulations and the extremal cases of triangulations and pointed pseudo-triangulations. In all cases n is assumed to be the total number of points, and a factor polynomial in n has been neglected. The double circle has $n/2$ interior points and the single and double chains have $n-3$ and $n-4$ interior points, respectively.

The paper is organized as follows: Section 2 studies the number of pseudo-triangulations of so-called point sets in almost convex position, among which the double circle is the extremal case. The next three sections are devoted to the single chain. Section 3 gives approximations, within a factor of four, for the numbers of pseudo-triangulations $|\mathcal{PT}(A)|$ and pointed pseudo-triangulations $|\mathcal{PPT}(A)|$ of the single chain. The proof of the crucial result that gives the asymptotics, Theorem 6, is given separately in Section 4. Section 5 uses a different approach to provide a much better approximation of $|\mathcal{PT}(A)|$ and $|\mathcal{PPT}(A)|$ for the single chain. Finally, Section 6 is devoted to the double chain, whose study is based on that of the single chain and, in particular, on the aforementioned Theorem 6.

2. The double circle and its relatives

For any given pair of positive integers $v \geqslant 3$ and $i \leqslant v$, we say that a point set A is *in almost convex position with parameters* (v, i) if it consists of a set of v points forming the vertex set of a convex v-gon and a set of i interior points, placed "sufficiently close" to i different edges of the v-gon. Here, we say that an interior point p is placed *sufficiently close to the edge* (r, q) *of the* v*-gon* if no segment connecting two points of A can separate p from (r, q). The *double circle* is the extremal case with $v = i = n/2$, where there is one interior point close to every boundary edge. It has asymptotically $\Theta(\sqrt{12}^n n^{-3/2})$ triangulations [25] and it is conjectured in [7] that this is the smallest number of triangulations that n points in general position in the plane can have. This conjecture is known to be true for $n \leqslant 11$ [6].

Point sets in almost convex position are a special case of what is called "almost-convex polygons" in [16]. There it is shown that the number of triangulations of such a point set does not depend on the choice of the i edges of the v-gon. Indeed, if we call this number $t(v, i)$, the case $W = \emptyset$ of Lemma 2 below provides the recursive formula

$$t(v, i) = t(v + 1, i - 1) - t(v, i - 1) \tag{2}$$

which allows to compute $t(v, i)$ starting with $t(v, 0) = C_{v-2}$ (Catalan numbers). It is interesting that formula (2) can be applied to generate $t(v, i)$ even for $i > v$. The array obtained by this recursion (difference array of Catalan numbers) appears in Sloane's Online Encyclopedia of Integer Sequences [26] with ID number A059346. The numbers obtained for $i > v$ do not have a meaning as triangulations of point sets, but (for small values of v) they have other combinatorial interpretations. For example, the sequence $M_n := t(3, n)$ forms the Motzkin numbers (number of lattice paths from $(0, 0)$ to $(n, 0)$ with steps $(1, 0)$, $(1, 1)$ or $(1, -1)$ and lying above the horizontal axis, see [10,28]). In the proof of Corollary 3 below we use their asymptotic expression, which appears for example in [12, Section VI.4]:

$$M_n = \sqrt{\frac{3}{4\pi}}\, 3^n \left(n^{-3/2} + O\left(n^{-5/2}\right)\right) \in \Theta\left(3^n n^{-3/2}\right). \tag{3}$$

We now generalize the recursive formula (2) to deal also with pseudo-triangulations. Let p be a specific interior point of a point set A in almost convex position and let (q, r) be the convex hull edge which has p next to it. Let B and C be the point sets obtained respectively by deleting p from A and by moving p to convex position across the edge (q, r) (see Fig. 4).

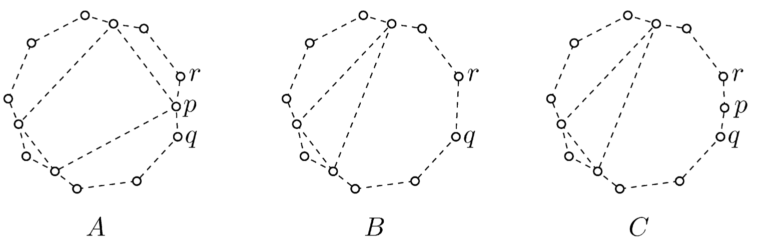


Fig. 4. Almost convex point sets: set A with $v = 9$ and $i = 4$, set B with $v = 9$ and $i = 3$, and set C with $v = 10$ and $i = 3$.

Lemma 2. *For every $W \subseteq A_I$ not containing p (so that W is also a set of interior points of B and C) one has*:

(1) $|\mathcal{PT}_W(A)| = |\mathcal{PT}_W(C)| - |\mathcal{PT}_W(B)|$.
(2) $|\mathcal{PT}_{W\cup\{p\}}(A)| = 2|\mathcal{PT}_W(C)| - |\mathcal{PT}_W(B)|$.
(3) $|\mathcal{PT}_{W\cup\{p\}}(A)| = 2|\mathcal{PT}_W(A)| + |\mathcal{PT}_W(B)|$. *In particular, A satisfies Conjecture* 1.

Proof. It is clear that there are bijections between: (i) pseudo-triangulations of C pointed at W that use the edge (q,r) and pseudo-triangulations of B pointed at W and (ii) pseudo-triangulations of C pointed at W that do not use the edge (q,r) and pseudo-triangulations of A pointed at W. These bijections prove part (1).

To prove part (2), we partition the pseudo-triangulations of A in which p is pointed into three sets: those using the edges (p,q) and (p,r) (and hence having no other edge incident to p), those using (p,q) but not (p,r), and those using (p,r) but not (p,q). The first set is in bijection with the pseudo-triangulations of B. Each of the other two is in bijection with pseudo-triangulations of C that do not use the edge (q,r), that is pseudo-triangulations of C minus those of B. Since the bijections preserve pointedness at interior points (other than p), we get $|\mathcal{PT}_{W\cup\{p\}}(A)| = |\mathcal{PT}_W(B)| + 2(|\mathcal{PT}_W(C)| - |\mathcal{PT}_W(B)|)$, as desired.

Part (3) is obtained eliminating $|\mathcal{PT}_W(C)|$ from parts (1) and (2). $\square$

This lemma shows that $|\mathcal{PT}_W(A)|$ only depends on the parameters (v,i) of A and the number $|W|$ of points prescribed to be interior. Indeed, let us call $s(v,j,k) = |\mathcal{PT}_W(A)|$ where v is the number of boundary points and $k = |W|$ and $j = i - k$ are the numbers of interior points prescribed to be pointed and non-pointed, respectively. Then, parts (1) and (2) of the lemma translate to

$$s(v, j+1, k) = s(v+1, j, k) - s(v, j, k), \tag{4}$$

and

$$s(v, j, k+1) = 2s(v+1, j, k) - s(v, j, k). \tag{5}$$

From this, all the numbers can be computed recursively: the second formula allows to compute them from the ones with $k = 0$ and the first formula allows to compute those from the numbers $s(v,0,0) = C_{v-2}$. Also, from the fact that $C_{v-2} \in \Theta^*(4^v)$ the formulas lead easily to the guess that $s(v,j,k) \in \Theta^*(4^v 3^j 7^k)$, which we now prove:

Corollary 3. *The number $s(v,j,k) = |\mathcal{PT}_W(A)|$, where A is a point set in almost convex position with v boundary and $j+k$ interior points, and W is a subset of k of them, satisfies*

$$\Omega\big((v+j+k)^{-3/2}\big) \leqslant \frac{s(v,j,k)}{4^v 3^j 7^k} \leqslant O\big(j^{-3/2}\big).$$

Proof. We start with the following slightly nicer versions of the recursions (4) and (5). The first one is a simple rewrite of (4) and the second is obtained eliminating $s(v,j,k)$:

$$s(v+1, j, k) = s(v, j+1, k) + s(v, j, k), \tag{6}$$

$$s(v, j, k+1) = s(v+1, j, k) + s(v, j+1, k). \tag{7}$$

If we now define $r(v,j,k)=\frac{s(v,j,k)}{4^v 3^j 7^k}$, these two recursions translate to:

$$r(v+1,j,k)=\frac{3}{4}r(v,j+1,k)+\frac{1}{4}r(v,j,k), \tag{8}$$

$$r(v,j,k+1)=\frac{4}{7}r(v+1,j,k)+\frac{3}{7}r(v,j+1,k). \tag{9}$$

From this eventually we get

$$\min_{n=j,\dots,v+j+k-3} r(3,n,0)\leqslant r(v,j,k)\leqslant \max_{n=j,\dots,v+j+k-3} r(3,n,0).$$

That is to say,

$$\min_{n=j,\dots,v+j+k-3}\frac{M_n}{4^3 3^n}\leqslant\frac{s(v,j,k)}{4^v 3^j 7^k}\leqslant \max_{n=j,\dots,v+j+k-3}\frac{M_n}{4^3 3^n},$$

where the sequence $M_n=s(3,n,0)$ are the afore mentioned Motzkin numbers. The fact that $M_n\in\Theta(3^n n^{-3/2})$ finishes the proof. □

We believe that a finer use of the asymptotics of the Motzkin numbers would lead to the slightly stronger statement that $s(v,j,k)\in\Theta(4^v 3^j 7^k (v+j+k)^{-3/2})$. Anyway, Corollary 3 implies that

$$\left|\mathcal{PPT}(A)\right|\in\Theta^*\left(4^v 7^i\right),\quad \text{and}\quad \left|\mathcal{PT}(A)\right|\in\Theta^*\left(4^v 10^i\right),$$

where the last formula comes from adding $|\mathcal{PT}_W(A)|=s(v,j,k)$ over all the 2^i values of W:

$$\sum_{k=0}^{i}\binom{i}{k}4^v 3^{i-k}7^k=4^v 10^i.$$

In conclusion, the double circle ($i=v=n/2$) has about $\sqrt{28}^n$ pointed pseudo-triangulations and $\sqrt{40}^n$ pseudo-triangulations in total, modulo a polynomial factor.

We close this section deriving direct recurrences for the total numbers of pseudo-triangulations and of pointed pseudo-triangulations of point sets in almost convex position.

Corollary 4. *Let $pt(v,i)$ and $ppt(v,i)$ denote the numbers of pseudo-triangulations and pointed pseudo-triangulations of a point set in almost convex position with parameters (v,i), respectively. Then*:

(1) $ppt(v,i)=2ppt(v+1,i-1)-ppt(v,i-1)$.
(2) $pt(v,i)=3pt(v+1,i-1)-2pt(v,i-1)$.

Proof. Part (1) is the case $W\cup\{p\}=A_I$ of part (2) of Lemma 2. For part (2) we add parts (1) and (2) of the same lemma over all possible values of W. □

3. The single chain

Throughout this section let A be the point set with the following $l+3$ points: $l+2$ points labeled $0,1,\dots,l,l+1$ forming a convex $(l+2)$-gon, plus a vertex p exterior to this polygon and seeing all edges of it except the edge $(0,l+1)$. A has three convex hull vertices $\{p,0,l+1\}$ and l interior vertices $\{1,\dots,l\}$. We call A a *single chain* and call p the *tip* of A, see Fig. 5 (left).

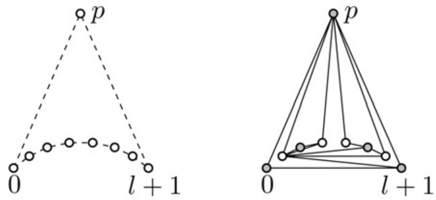


Fig. 5. A single chain A with $l=6$ (left), a pointed pseudo-triangulation in $\mathcal{PPT}_{\{1,3,4,6\}}(A)$ (right).

Besides classifying pseudo-triangulations of A with respect to their sets of pointed vertices, here we need to classify the *pointed* pseudo-triangulations of the single chain according to which interior points are joined to the tip. That is, for each subset $W \subseteq A_I$ we denote by $\mathcal{PPT}_W(A)$ the set of pointed pseudo-triangulations of A in which p is joined to i if and only if $i \in W$, see Fig. 5 (right).

It is easy to realize that from the numbers $|\mathcal{PPT}_W(A)|$ one can recover the numbers $|\mathcal{PT}_W(A)|$, which are our main interest in this section.

Lemma 5. *For every W:*

$$\left|\mathcal{PT}_W(A)\right| = \sum_{W' \subseteq W} \left|\mathcal{PPT}_{W'}(A)\right|.$$

In particular, Conjecture 1 *holds for the single chain.*

Proof. In every pseudo-triangulation of $\mathcal{PT}_W(A)$ the non-pointed vertices $i \in A_I \backslash W$ have to be joined to the tip p. If we delete those edges (p, p_i) for all $i \in A_I \backslash W$, we get an element of a certain $\mathcal{PPT}_{W'}(A)$ with $W' \subseteq W$ (here, W' are the vertices of the pseudo-triangulation which are joined to p but are pointed). This process can clearly be reversed. □

The following theorem is probably the most surprising result in this paper. It says that the sets $\mathcal{PPT}_W(A)$ have the same cardinality as certain subsets of triangulations of a convex $(l+3)$-gon. Apart from its intrinsic interest, this result automatically gives the asymptotics of all the numbers $|\mathcal{PPT}_W(A)|$ (Corollary 7), and hence of all the $|\mathcal{PT}_W(A)|$, too (Corollary 8).

Theorem 6. *Let A be a single chain with l interior vertices, let $W \subseteq A_I$ be a subset of them. Let B be the convex $(l+3)$-polygon with vertex set $A \setminus \{p\} \cup \{q\}$, where q is an extra point on the side opposite to p.*

The pointed pseudo-triangulations of A in which the interior neighbors of p are exactly *the points in W (that is, the elements of $\mathcal{PPT}_W(A)$) have the same cardinality as the triangulations of B in which the interior neighbors of q are* contained *in W.*

See Fig. 6 for an example. To maintain the flow of ideas, we postpone the proof of Theorem 6 to Section 4. Let us remark only that our proof is rather indirect. In particular, it is far from being an explicit bijection between the two sets involved.

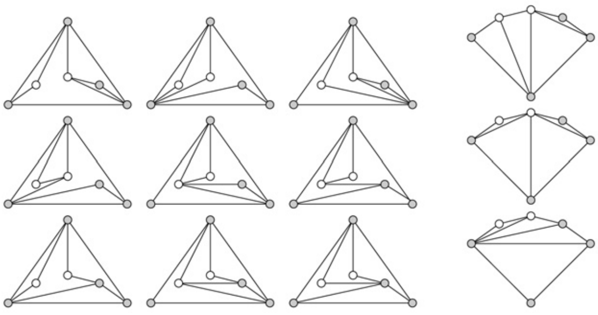


Fig. 6. The nine pointed pseudo-triangulations in $\mathcal{PPT}_W(A)$ and the nine triangulations in which the interior neighbors of q are contained in W. In this example $l = 3$ and $W = \{1, 2\}$.

Corollary 7.

(1) *The numbers $|\mathcal{PPT}_W(A)|$ are strictly monotone with respect to W. That is to say, for every $W \subset \{1, \ldots, l\}$ and every $i \in \{1, \ldots, l\} \setminus W$,*

$$\left|\mathcal{PPT}_{W\cup\{i\}}(A)\right| > \left|\mathcal{PPT}_W(A)\right|.$$

(2) *For any $W \subseteq \{1, \ldots, l\}$, $C_l \leqslant |\mathcal{PPT}_W(A)| \leqslant C_{l+1}$.*

Proof. Part (1) is a direct consequence of Theorem 6. Part (2) follows from part (1) and the facts that $|\mathcal{PPT}_\emptyset(A)| = C_l$ and $|\mathcal{PPT}_{\{1,\ldots,l\}}(A)| = C_{l+1}$ (the latter comes again from Theorem 6, taking $W = A_I$). □

Note that $\mathcal{PPT}_\emptyset(A)$ is in bijection to the set of triangulations of the convex $(l + 2)$-gon with vertices $A \setminus \{p\} = \{0, 1, \ldots, l + 1\}$, hence its cardinality is the Catalan number C_l. Curiously enough, $\mathcal{PPT}_{A_I}(A)$ (that is, the set of pointed pseudo-triangulations in which the tip p is joined to everything), has the cardinality of the next Catalan number C_{l+1}. This follows from Theorem 6, but was first proved in Section 5.3 of [23] (see also the remark and picture on pp. 728–729). There, an associahedron with vertex set $\mathcal{PPT}_{A_I}(A)$ is obtained, and it is regarded as a 1-dimensional analog of the construction of the polytope of pointed pseudo-triangulations for a planar point set. Following this analogy, we regard part (1) of Corollary 7 as a 1-dimensional analog of Conjecture 1.

Corollary 8.

(1) *For every $W \subseteq \{1, \ldots, l\}$, $2^{|W|}C_l \leqslant |\mathcal{PT}_W(A)| \leqslant 2^{|W|}C_{l+1}$.*
(2) $3^l C_l \leqslant |\mathcal{PT}(A)| \leqslant 3^l C_{l+1}$.
(3) *In particular, $|\mathcal{PT}_W(A)| \in \Theta(2^{|W|}4^l l^{-\frac{3}{2}})$, $|\mathcal{PPT}(A)| \in \Theta(8^l l^{-\frac{3}{2}})$ and $|\mathcal{PT}(A)| \in \Theta(12^l l^{-\frac{3}{2}})$.*

Proof. Part (1) comes from applying part (2) of Corollary 7 to each summand in the expression $|\mathcal{PT}_W(A)| = \sum_{W' \subseteq W} |\mathcal{PPT}_{W'}(A)|$ of Lemma 5. Part (2) comes from adding the inequalities in part (1) for all the subsets $W \subseteq \{1, \ldots, l\}$. Finally, part (3) follows from $C_l \in \Theta(4^l l^{-3/2})$. □

Since $C_{l+1}/C_l < 4$, parts (1) and (2) of this corollary approximate the numbers $|\mathcal{PT}(A)|$ and $|\mathcal{PPT}(A)|$ within a factor of four. In Section 5 we show how to obtain much better approximations.

4. Proof of Theorem 6

Let us recall the statement we want to prove:

Theorem 6. *Let A be a single chain with l interior vertices, let $W \subseteq A_I$ be a subset of them. Let B be the convex $(l+3)$-polygon with vertex set $A \setminus \{p\} \cup \{q\}$, where q is an extra point "on the side opposite" to p.*

Then, the pointed pseudo-triangulations of A in which the interior neighbors of p are exactly *the points in W (that is, the elements of $\mathcal{PPT}_W(A)$) are in bijection to the triangulations of B in which the interior neighbors of q are* contained *in W.*

Let us denote with $\mathcal{T}_W(B)$ the set of triangulations of B mentioned in the statement. The way we prove that $|\mathcal{T}_W(B)|$ and $|\mathcal{PPT}_W(A)|$ are the same number is by showing that both families of numbers satisfy the same recursive formula. To this end, let W be a non-empty subset of $\{1,\dots,l\}$ and choose an element $v \in W$. Let $W_1 = \{w \in W\colon\ w < v\}$ and $W_2 = \{w \in W\colon\ w > v\}$ be the sets of elements of W on both sides of v. Moreover, let:

- A_1 and A_2 be the "single chains" having as vertices $\{p,0,1,\dots,v\}$ and $\{p,v,v+1,\dots,l+1\}$, respectively.
- B_1 and B_2 be the convex polygons having as vertices $\{q,0,1,\dots,v\}$ and $\{q,v,v+1,\dots,l+1\}$, respectively.

Then:

Lemma 9. *The following recurrence holds*:

$$\left|\mathcal{T}_W(B)\right| - \left|\mathcal{T}_{W\setminus\{v\}}(B)\right| = \left|\mathcal{T}_{W_1}(B_1)\right| \cdot \left|\mathcal{T}_{W_2}(B_2)\right|.$$

Proof. The difference in the left-hand side coincides with the triangulations of B that use the edge (q,v) and have the (other) interior neighbors of q contained in W. Clearly, those triangulations can be obtained by triangulating B_1 and B_2 independently. □

In the rest of this section we prove that the same recursion holds for the numbers $|\mathcal{PPT}_W(A)|$, except we do it under the assumption that v is the first element in W. This assumption is enough for our purposes because knowing that

$$\left|\mathcal{PPT}_W(B)\right| - \left|\mathcal{PPT}_{W\setminus\{v\}}(B)\right| = \left|\mathcal{PPT}_{W_1}(B_1)\right| \cdot \left|\mathcal{PPT}_{W_2}(B_2)\right|$$

for any particular v, together with the inductive hypothesis that $|\mathcal{PPT}_{W'}(A')|$ and $|\mathcal{T}_{W'}(B')|$ coincide whenever $|W'| < |W|$ and the base case $|\mathcal{PPT}_\emptyset(A)| = |\mathcal{T}_\emptyset(B)|$, implies that $|\mathcal{PPT}_W(A)| = |\mathcal{T}_W(B)|$. That is, in order to prove Theorem 6 it is enough to prove:

Proposition 10. *For every $W \subseteq \{1,\dots,l\}$ and for $v = \min(W)$,*

$$\left|\mathcal{PPT}_W(A)\right| - \left|\mathcal{PPT}_{W\setminus\{v\}}(A)\right| = \left|\mathcal{PPT}_{W_1}(A_1)\right| \cdot \left|\mathcal{PPT}_{W_2}(A_2)\right|.$$

Our first observation is that:

Lemma 11. *If $v = \min(W)$ then $|\mathcal{PPT}_{W_1}(A_1)| \cdot |\mathcal{PPT}_{W_2}(A_2)|$ equals the number of elements of $\mathcal{PPT}_W(A)$ that use the edge $(v, l+1)$.*

Proof. The edges (p, v) and $(v, l+1)$ separate the triangle $\mathrm{conv}(A)$ into two regions, so that we can count their number of pointed pseudo-triangulations independently. The region on the right is the convex hull of A_2. Hence, it only remains to show that the region on the left, let us denote it A_L, has the same number of pointed pseudo-triangulations that join W_1 to p as A_1 has. Note that $v = \min(W)$ implies $W_1 = \emptyset$. That is, $|\mathcal{PPT}_{W_1}(A_1)|$ is just the number of triangulations of the $v+1$ points in convex position $\{0, \dots, v\}$. For A_L, we know in addition that none of the vertices $\{0, \dots, v-1\}$ can be connected to v, or v would be non-pointed in the pseudo-triangulation of A under consideration otherwise. Thus, $|\mathcal{PPT}_{W_1}(A_L)|$ equals the number of triangulations of the $v+1$ points $\{0, \dots, v-1, l+1\}$, too. $\square$

Let $\mathcal{PPT}_W(A)^*$ denote the elements of $\mathcal{PPT}_W(A)$ that do not use the edge $(v, l+1)$. The above lemma implies that Proposition 10 is equivalent to:

Proposition 12. *For every $W \subseteq \{1, \dots, l\}$ and for $v = \min(W)$,*

$$\left|\mathcal{PPT}_W(A)^*\right| = \left|\mathcal{PPT}_{W\setminus\{v\}}(A)\right|.$$

We will prove this via an explicit (although complicated) bijection. For it, we classify the elements of $\mathcal{PPT}_W(A)$ and $\mathcal{PPT}_{W\setminus\{v\}}(A)$ via the following parameters.

Definition. Let $W = \{v_1, \dots, v_k\}$ and $v = \min(W) = v_1$.

(1) For an element T of $\mathcal{PPT}_W(A)$, we call *end-point vector* of T the vector $(x_1, \dots, x_k)$ of length $|W|$ and with entries taken from $\{0, \dots, l+1\}$, defined as follows: For every i, there is a single pseudo-edge in T having v_i as a reflex vertex. Since $v_i \in W$ and $T \in \mathcal{PPT}_W(A)$, one of the two corners joined by this pseudo-edge is the tip p. We define x_i to be the other corner.

(2) Similarly, the *end-point vector* of an element T of $\mathcal{PPT}_{W\setminus\{v\}}(A)$ is the vector $(x_1, \dots, x_k)$ of length $|W|$ and with entries in $\{0, \dots, l+1\}$ such that:
   - x_1 is the third corner of the triangle below the edge $(v-1, v)$. This is well-defined because neither $v-1$ nor v belong to $W \setminus \{v\}$, hence the edge $(v-1, v)$ is in T and both $v-1$ and v are corners of the pseudo-triangle below it. Moreover, this pseudo-triangle must necessarily be a triangle.
   - For every $i > 1$, x_i is defined as in part (1).

In the rest of this section we denote by $\mathcal{PPT}_{W,(x_1,\dots,x_k)}$ the subset of $\mathcal{PPT}_W(A)$ consisting of elements with end-point vector $(x_1, \dots, x_k)$. Similarly, we denote by $\mathcal{PPT}_{W\setminus\{v\},(x_1,\dots,x_k)}$ the elements of $\mathcal{PPT}_{W\setminus\{v\}}(A)$ with end-point vector $(x_1, \dots, x_k)$. With this notation, the main part of the proof is to show the following bijections:

Lemma 13. *Let $(x_1, \dots, x_k)$ be a vector of length k with entries in $\{0, \dots, l+1\}$.*

(1) *If $x_1 > v$, then $|\mathcal{PPT}_{W,(x_1,\dots,x_k)}| = |\mathcal{PPT}_{W\setminus\{v\},(x_1,\dots,x_k)}|$.*

(2) *If* $0 < x_1 < v$, *then* $|\mathcal{PPT}_{W,(x_1,\dots,x_k)}| = |\mathcal{PPT}_{W\setminus\{v\},(x_1-1,x_2^*,\dots,x_k^*)}|$, *where* x_i^* *equals* x_i (*resp., equals* v) *if* $x_i \neq x_1$ (*resp.,* $x_i = x_1$).
(3) *If* $x_1 = 0$, *then* $|\mathcal{PPT}_{W,(x_1,\dots,x_k)}| = |\mathcal{PPT}_{W\setminus\{v\},(l+1,x_2^*,\dots,x_k^*)}|$, *where* x_i^* *equals* x_i (*resp., equals* v) *if* $x_i \neq x_1$ (*resp.,* $x_i = x_1$).

Proof. (1) The bijection comes from a simple flip of the edge (p, v) into the edge $(v-1, v)$.

(2) As in part (1), the first step is to perform a flip of the edge (p, v). This introduces an edge (v, y), where $y = v + 1$ unless both $v_2 = v + 1$ and $x_2 > v_2$ hold, in which case $y = x_2$. (See Fig. 7.)

In any case, we now have a pointed pseudo-triangulation T that belongs to $\mathcal{PPT}_{W\setminus\{v\}}$ and with the property that it contains the triangle $t = (x_1, v, y)$. This triangle decomposes T into three parts: a triangulation T_1 of the convex $(v - x_1 + 1)$-gon with vertices $\{x_1, \dots, v\}$, the triangle (x_1, v, y) itself, and a pointed pseudo-triangulation T_2 of the single chain with $l - (v - x_1)$ vertices $A \setminus \{x_1 + 1, \dots, v\}$.

We are going to rearrange these three pieces in order to obtain a different pointed pseudo-triangulation of A. We embed T_2 as a pointed pseudo-triangulation of the vertex set $A \setminus \{x_1, \dots, v-1\}$, add the triangle $(x_1 - 1, v - 1, v)$ to it, and then place the triangulation T_1 on the polygon $\{x_1 - 1, \dots, v - 1\}$; see Fig. 8. Essentially, in T_2 we are substituting vertex v for vertex x_1, and then we are changing the rest to be consistent with this replacement. Since everything in T_2 previously joined to x_1 is now joined to v, the new pointed pseudo-triangulation is indeed in $\mathcal{PPT}_{W\setminus\{v\},(x_1-1,x_2^*,\dots,x_k^*)}$.

This process can be reversed: Starting with a pointed pseudo-triangulation in $\mathcal{PPT}_{W\setminus\{v\},(y_1,y_2,\dots,y_k)}$, with $y_1 \in \{0, \dots, v-2\}$, the triangle $(y_1, v-1, v)$ decomposes it into three parts: a triangulation T_1' of the convex polygon with vertices $\{y_1, \dots, v-1\}$, the triangle itself, and a pointed pseudo-triangulation T_2' of $A \setminus \{y_1 + 1, \dots, v - 1\}$. We place T_2' on $A \setminus \{y_1 + 2, \dots, v\}$, T_1' on $\{y_1 + 1, \dots, v\}$ and insert the triangle $(y_1 + 1, v, v + 1)$. We now flip the edge $(v, v + 1)$, and get a pointed pseudo-triangulation in $\mathcal{PPT}_{W,(y_1+1,y_2^*,\dots,y_k^*)}$ where $y_i^* = y_i$ if $y_i \neq v$ and $y_i^* = y_1 + 1$ otherwise.

(3) The process is exactly the same as in part (2), except that since $x_1 = 0$ we have to use $l + 1$ in the role that was played by $x_1 - 1$. That is, T_1 will be a triangulation of the convex polygon $\{0, \dots, v\}$ before the rearrangement, and placed as a triangulation T_1' of the convex polygon $\{l + 1, 0, 1, \dots, v - 1\}$. The rest is unchanged. □

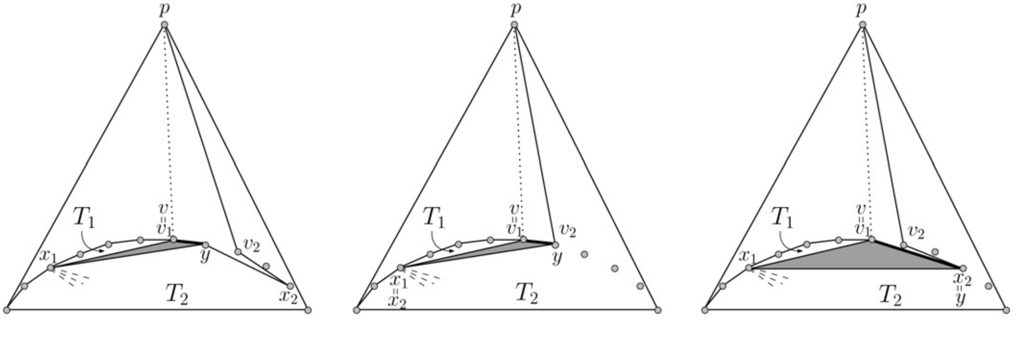


Fig. 7. Three examples of flipping edge (p, v) in part (2) of the proof of Lemma 13. In the left and middle cases $y = v+1$, in the right case $y = x_2$.

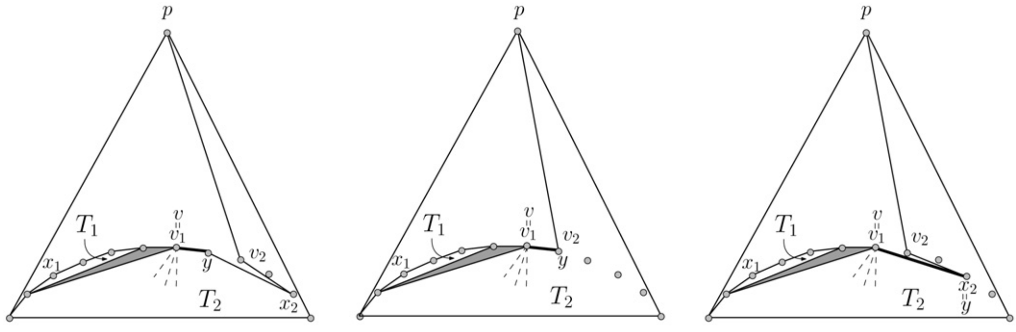


Fig. 8. Rearrangement for the three examples in Fig. 7.

Proof of Proposition 12. We now show how Lemma 13 can be used to finish the proof of Proposition 12, hence that of Proposition 10 (and therefore the one of Theorem 6). What we need to show is that the sets in the right-hand sides of Lemma 13 cover the set $\mathcal{PPT}_{W\setminus\{v\}}(A)$ without repetitions if we exclude from the left-hand side the ones with $x_1 = l+1$, which are the elements in $\mathcal{PPT}_W(A) \setminus \mathcal{PPT}_W(A)^*$. To this end, we consider an end-point vector $Y = (y_1, \dots, y_k)$ of an element in $\mathcal{PPT}_{W\setminus\{v\}}(A)$, and show that it comes from a unique end-point vector $X = (x_1, \dots, x_k)$ of an element of $\mathcal{PPT}_W(A)^*$ via the bijections in Lemma 13. First, observe that the end-points in the right-hand sides of parts (1), (2) and (3) are distinguished by the properties $v < y_1 < l+1$, $y_1 < v-1$ and $y_1 = l+1$, respectively. Also, a valid Y cannot have y_1 equal to v or $v-1$, by the definition of end-point vector in $\mathcal{PPT}_{W\setminus\{v\}}(A)$. It only remains to show how to recover the vector X from Y:

(a) If $v < y_1 < l+1$, then just let $X = Y$.
(b) If $y_1 < v-1$, then let $x_1 = y_1 + 1$ and for $i > 1$ let x_i equal x_1 or x_i depending on whether $y_i = v$ or $y_i \neq v$.
(c) If $y_1 = l+1$, then let $x_1 = 0$ and let x_i equal x_1 or x_i depending on whether $y_i = v$ or $y_i \neq v$.

Note that, in all cases, $x_1 \neq l+1$, trivially for (a) and (c) and because $v - 1 \leqslant l - 1$ in (b). □

5. Additional bounds and properties for the single chain

Corollary 8 gives the approximations $|\mathcal{PPT}(A)| \simeq 2^l C_l$ and $|\mathcal{PT}(A)| \simeq 3^l C_l$ within a factor of four. In this section we show that $|\mathcal{PPT}(A)| \simeq 2^{l+1} C_l$ and $|\mathcal{PT}(A)| \simeq 3^{l+1} C_l/2$ are much better approximations, with errors of 12.5% and 4% respectively when l goes to infinity. This is in contrast with the fact that we do not know such good and simple approximations for the individual summands $|\mathcal{PPT}_W(A)|$. Our first step is to compute the sum of all the $\mathcal{PPT}_W(A)$'s for each cardinality of W, via the following recursive formulae.

Theorem 14. *Let* $a(l,i) := \sum_{|W|=i} |\mathcal{PPT}_W(A)|$. *Then*:

(1) $a(l,0) = C_l$, *and* $a(l,1) = (l+1)C_l$.
(2) *For every* $i \geqslant 2$,

$$a(l,i) = \binom{l+1}{i} C_l - a(l-1, i-2).$$

As a preparation for the proof of Theorem 14, observe that the number $\binom{l+1}{i}C_l$ that appears in the statement equals the number of ways of specifying a triangulation of the $(l+2)$-gon together with i of the $l+1$ boundary edges of the $(l+2)$-gon visible from the tip. We say that a pointed pseudo-triangulation T of A is *compatible* with this specification if T restricted to the interior of the $(l+2)$-gon gives that triangulation and when restricted to the boundary of the $(l+2)$-gon the i edges chosen above are precisely the ones not appearing. Note that this notion of compatibility is usable in both directions, i.e., "pointed pseudo-triangulations compatible with a choice" and "choices compatible with a pointed pseudo-triangulation." Some pointed pseudo-triangulations of A may not produce a triangulation of the $(l+2)$-gon, and hence they are not compatible with any choice. Reciprocally, some choices are not compatible with any pointed pseudo-triangulation, but the next statement describes them:

Lemma 15. *Let a choice of a triangulation of the* $(l+2)$*-gon and a choice of a subset of boundary edges of the* $(l+2)$*-gon visible from the tip be given. Then*:

(1) *The choice is compatible with a pointed pseudo-triangulation of A if and only if no ear of the triangulation is incident to two missing boundary edges.*
(2) *A compatible choice determines uniquely a pointed pseudo-triangulation. This pseudo-triangulation uses i interior edges incident to the tip vertex, where i equals the number of missing boundary edges.*

Proof. Observe that compatible means that from the given choice of triangulation and subset of boundary edges we can get a pointed pseudo-triangulation, by adding to the chosen triangulation some edges incident to the tip and removing the chosen boundary edges. We call an ear of the triangulation incident to two missing boundary edges of the $(l+2)$-gon a *bad ear* (see Fig. 9).

Clearly, if a bad ear appears at p_i the choice cannot be compatible with a pointed pseudo-triangulation, because a vertex cannot have degree 1 in a pointed pseudo-triangulation. Hence, assume that we have a choice with no bad ears and let us prove that it is compatible with one and only one pointed pseudo-triangulation. The way to obtain the pointed pseudo-triangulation is as follows: let $p_i p_{i+1}$ be a missing edge in the choice. Let p_k be the vertex of the triangulation of the $(l+2)$-gon joined to it. We add the edge (p, p_i) or (p, p_{i+1}) depending on whether $k < i$ or $k > i+1$. The assumption of no bad ears implies that we add as many edges as missing edges were in the choice. In particular, the set of edges obtained in this way has cardinality $2l+3$, the same as an element of $\mathcal{PPT}_\emptyset(A)$. Since every pointed graph with $2l+3$ edges of a vertex set of size $l+3$ is a pointed pseudo-triangulation, we have shown existence. For uniqueness, just observe that every compatible pointed pseudo-triangulation must have at least the edges we have

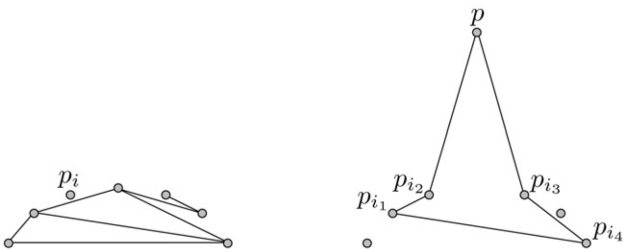


Fig. 9. Left: bad ear at p_i. Right: bad pseudo-triangle in the proof of Theorem 14.

added: an interior vertex not joined to the tip must be joined to vertices both to its right and to its left. □

Proof of Theorem 14. The equation $a(l,0) = |\mathcal{PPT}_\emptyset(A)| = C_l$ is obvious. For $a(l,1)$, observe that every pointed pseudo-triangulation with a single interior edge joined to the tip uniquely gives rise, by a flip of that edge, to another one with no edges joined to the tip. Conversely, every pointed pseudo-triangulation with no edges to the tip gives rise to $l+1$ pointed pseudo-triangulations with a single edge to the tip, by the $l+1$ possible flips of the boundary edges of the $(l+2)$-gon.

For the proof of part (2), let us call $b(l,i) := \binom{l+1}{i} C_l$ and let $c(l,i)$ be the number of pointed pseudo-triangulations of A that are compatible with a choice of triangulation and boundary. We will abuse notation and use $a(l,i)$, $b(l,i)$ and $c(l,i)$ to represent not only the numbers but also the sets of objects counted by them; $\bigcup_{|W|=i} \mathcal{PPT}_W(A)$ for $a(l,i)$, choices of a triangulation and a subset of edges as above for $b(l,i)$, and the double meaning of "pointed pseudo-triangulations compatible with a choice" and "choices compatible with a pointed pseudo-triangulation" for $c(l,i)$.

Clearly, every element of $b(l,i)$ with k "bad ears" can be considered a member of $c(l-k, i-2k)$: just delete the k interior points where the bad ears occur. Reciprocally, each member of $c(l-k, i-2k)$ can give a member of $b(l,i)$ in $\binom{l-k-(i-2k)+1}{k}$ ways: we choose k of the $l-k-(i-2k)+1$ used boundary edges and place a new vertex (a bad ear) beyond each of those k edges. Hence:

$$b(l,i) = \sum_{k\geqslant 0} \binom{l-i+1+k}{k} c(l-k, i-2k).$$

Now, what can make a pointed pseudo-triangulation not compatible with a choice of triangulation plus boundary edges is the existence of a *bad pseudo-triangle* $[p, p_{i_2}, p_{i_1}, p_{i_4}, p_{i_3}]$ with $i_1 < i_2 < i_3 < i_4$, see Fig. 9 (note that $p_{i_3} = p_{i_2} + 1$ in order to be a pseudo-triangle). In this case the restriction to the $(l+2)$-gon has a quadrangle, let us call it a *bad quadrangle*, instead of being a triangulation. More that one bad quadrangle can occur, but the two edges (p, p_{i_2}) and (p, p_{i_3}) that join one bad quadrangle to the tip cannot join any other bad quadrangle to the tip.

In particular, if an element of $a(l,i)$ produces k bad quadrangles, contracting the edge (p_{i_2}, p_{i_3}) of each quadrangle and removing the $2k$ corresponding edges incident to the tip we get an element of $c(l-k, i-2k)$, because of Lemma 15. To get back an element of $a(l,i)$ from one of $c(l-k, i-2k)$ one must choose k of the $l-i+k$ interior vertices not incident to the tip and split them into two vertices, joining both to the tip. Clearly, there are $\binom{l-i+k}{k}$ ways to do that. Hence:

$$a(l,i) = \sum_{k\geqslant 0} \binom{l-i+k}{k} c(l-k, i-2k).$$

Then:

$$a(l-1, i-2) = \sum_{k\geqslant 1} \binom{l-i+k}{k-1} c(l-k, i-2k),$$

where the index k has been shifted by one after evaluating with the previous formula. To get the statement, add the two last equalities and compare them to the one for $b(l,i)$. □

Table 4
Values of $a(l,i)$ and $|\mathcal{PPT}(A)|$ for $l, i \leqslant 5$

$l \backslash i$	0	1	2	3	4	5	$\lvert\mathcal{PPT}(A)\rvert = \sum a(l,i)$
0	1						1
1	1	2					3
2	2	6	5				13
3	5	20	28	14			67
4	14	70	135	120	42		381
5	42	252	616	770	495	132	2307

Theorem 14 allows us to compute all the values of $a(l,i)$ recursively, starting from those stated in part (1). The first few values of $a(l,i)$ are shown in Table 4. The recursion also tells us that the array $a(l,i)$ equals the sequence A062991 in Sloane's Encyclopedia [26]. The row sums, that is, the numbers $|\mathcal{PT}_{A_I}(A)| = |\mathcal{PPT}(A)|$ of all pointed pseudo-triangulations, form the sequence A062992 and satisfy:

$$\left|\mathcal{PPT}(A)\right| = \left|\mathcal{PT}_{A_I}(A)\right| = \sum_{i=0}^{l} a(l,i) = 2\sum_{j=0}^{l}(-1)^{l-j}C_j 2^j - (-1)^l.$$

We can obtain them by adding over all values of i in the formula of Theorem 14.

Corollary 16. *The number $a_l = |\mathcal{PPT}(A)|$ of pointed pseudo-triangulations of the single chain satisfies*:

$$a_l = 2^{l+1}C_l - a_{l-1}.$$

Hence,

$$\left(1 - \sum_{i=\lfloor 2+\frac{(-1)^l}{2}\rfloor}^{l} (-1)^{l-i}\prod_{j=i}^{l}\frac{j+1}{4(2j-1)}\right)\cdot 2^{l+1}C_l \leqslant \left|\mathcal{PPT}(A)\right| \leqslant 2^{l+1}C_l.$$

Observe that the parenthesis in the left-hand side tends to $\frac{8}{9}$ when l goes to infinity.

Proof. The first statement follows from $|\mathcal{PPT}(A)| = \sum_{i=0}^{l} a(l,i)$ and Theorem 14, using that $a(l,l) = C_{l+1}$ (Theorem 6, with $W = \{1,\dots,l\}$). For an example, $381 = 2^5 C_4 - 67 = 32\cdot 14 - 67$. For the second part, the upper bound is straightforward and for the lower bound the first part gives

$$a_l = (-1)^l + \sum_{i=1}^{l}(-1)^{l+i}2^{i+1}C_i$$

and then one can use the fact that $C_l = C_{l-1}(4l-2)/(l+1)$. $\square$

We now turn our attention to the total number of pseudo-triangulations $\mathcal{PT}(A)$. Lemma 5 implies that:

$$\left|\mathcal{PT}(A)\right| = \sum_{W'\subseteq A_I} 2^{|A_I\setminus W'|}\left|\mathcal{PPT}_{W'}(A)\right| = \sum_{i=0}^{l} 2^{l-i}a(l,i).$$

Corollary 17. *The number $b_l = |\mathcal{PT}(A)|$ of pseudo-triangulations of the single chain satisfies:*

$$2b_l = 3^{l+1} C_l - b_{l-1}.$$

Hence,

$$\left(1 - \sum_{i=\lfloor 2+\frac{(-1)^l}{2}\rfloor}^{l} (-1)^{l-i} \prod_{j=i}^{l} \frac{j+1}{12(2j-1)}\right) \cdot \frac{3^{l+1}}{2} C_l \leqslant \left|\mathcal{PPT}(A)\right| \leqslant \frac{3^{l+1}}{2} C_l.$$

Observe that the parenthesis in the left-hand side tends to $\frac{24}{25}$ when l goes to infinity.

Proof. Similar to the proof of Corollary 16. For the second part we use $|\mathcal{PT}(A)| = \sum_{i=0}^{l} 2^{l-i} a(l,i)$ and the first part to get

$$b_l = \frac{1}{(-2)^l} + \sum_{i=1}^{l} (-1)^{l+i} \frac{3^{i+1}}{2^{l-i+1}} C_i. \qquad \square$$

6. The double chain

For any two numbers $l, m \geqslant 0$, we call *double chain* with parameters (l, m) the point set consisting of a convex 4-gon with l and m points, respectively, placed forming concave chains next to opposite edges of the 4-gon in a way that they do not cross the two diagonals of the convex 4-gon (see Fig. 10). The double chain decomposes into a convex $(l+2)$-gon, a convex $(m+2)$-gon, and a non-convex $(l+m+4)$-gon, the latter with $\binom{l+m+2}{l+1}$ triangulations [13]. Hence, the double chain has exactly

$$C_l C_m \binom{l+m+2}{l+1}$$

triangulations. In the extremal case $l = m = (n-4)/2$ this gives $\Theta(8^n n^{-7/2})$. The double chain has been, until very recently (see [5]), the example of a point set in the plane with asymptotically the biggest number of triangulations known.

Throughout this section, let A be a double chain with l and m interior points in the two chains, respectively (so A has $l+m+4$ points in total). We call the $l+2$ and $m+2$ vertices in the two chains the "top" and "bottom" parts.

In order to count the number of pseudo-triangulations of A, let us call B and C single chains with l and m interior points each. B can be considered the subset of A consisting of the top part plus a bottom vertex, and analogously for C. Every pseudo-triangulation T_A of A induces on the one hand a pseudo-triangulation T_B of B by contracting all bottom vertices to a single one, and on the other hand a pseudo-triangulation T_C of C by doing the same with all top vertices (see

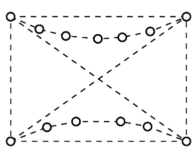
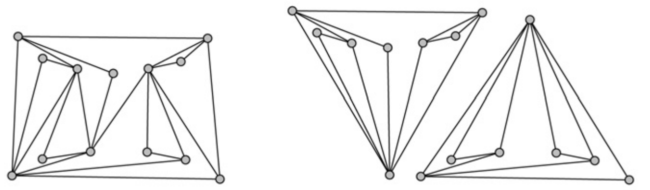


Fig. 10. A double chain: $l = 5$ and $m = 4$.



(a) T_A. (b) T_B and T_C.

Fig. 11. Decomposing a pseudo-triangulation of a double chain.

Fig. 11). Since no pseudo-triangle of T_A contains both more than one top vertex and more than one bottom vertex, every pseudo-triangle survives either in T_B or in T_C but not in both.

Conversely, given a pair of pseudo-triangulations T_B and T_C of B and C, if i (resp. j) denotes the number of interior edges incident to the bottom point in T_B (respectively to the top point in T_C), there are exactly $\binom{i+j+2}{i+1}$ ways to recover a pseudo-triangulation of A from that data, by shuffling the $i+1$ pseudo-triangles of T_B incident to the bottom and the $j+1$ of T_C incident to the top.

Theorem 18. *Let V and W be subsets of the top and bottom interior points. For each $i \leqslant v \leqslant l$ and $j \leqslant w \leqslant m$ let $t_{i,j}^{v,w} := \binom{l-v+i+m-w+j+2}{l-v+i+1}$. Then*:

(1) $$\left|\mathcal{PT}_{V\cup W}(A)\right| = \sum_{\substack{V'\subseteq V\\ W'\subseteq W}} t_{|V'|,|W'|}^{|V|,|W|}\left|\mathcal{PPT}_{V'}(B)\right|\left|\mathcal{PPT}_{W'}(C)\right|.$$

(2) *In particular, if $v = |V|$ and $w = |W|$, then*

$$\left|\mathcal{PT}_{V\cup W}(A)\right| \in \Theta\left(C_l C_m \sum_{i=0}^{v}\sum_{j=0}^{w}\binom{v}{i}\binom{w}{j}t_{i,j}^{v,w}\right).$$

Proof. The first observation is that the "shuffling" described above preserves pointedness. Then, part (1) follows from the fact that in the expression

$$\left|\mathcal{PT}_V(B)\right| = \sum_{V'\subseteq V}\left|\mathcal{PPT}_{V'}(B)\right|$$

of Lemma 5, each element of $\mathcal{PPT}_{V'}(B)$ corresponds to an element of $\mathcal{PT}_V(B)$ with exactly $l - |V\setminus V'| = l - |V| + |V'|$ interior edges incident to the bottom point (same for C).

Part (2) follows from part (1) using that $|\mathcal{PPT}_{V'}(B)| \in \Theta(C_l)$, $|\mathcal{PPT}_{W'}(C)| \in \Theta(C_m)$, and in the sum of part (1) there are exactly $\binom{v}{i}\binom{w}{j}$ summands with $|V'| = i$ and $|W'| = j$. $\square$

Corollary 19. *The double chain satisfies Conjecture* 1.

Proof. When we add a point p to V, Theorem 18 gives that $|\mathcal{PT}_{V\cup\{p\}\cup W}(A)|$ equals

$$\sum_{\substack{V'\subseteq V\\ W'\subseteq W}}\left(t_{|V'|,|W'|}^{|V\cup\{p\}|,|W|}\left|\mathcal{PPT}_{V'}(B)\right| + t_{|V'\cup\{p\}|,|W'|}^{|V\cup\{p\}|,|W|}\left|\mathcal{PPT}_{V'\cup\{p\}}(B)\right|\right)\left|\mathcal{PPT}_{W'}(C)\right|.$$

We neglect the first summand, and use monotonicity of $|\mathcal{PPT}_{V'}|$ (part (1) of Corollary 7) in the second summand, which is then greater than

$$\sum_{\substack{V'\subseteq V\\ W'\subseteq W}} t^{|V\cup\{p\}|,|W|}_{|V'\cup\{p\}|,|W'|}\big|\mathcal{PPT}_{V'}(B)\big|\big|\mathcal{PPT}_{W'}(C)\big|.$$

But this equals $|\mathcal{PT}_{V\cup W}(A)|$ by Theorem 18 since, clearly,

$$t^{v,w}_{i,j}=t^{v+1,w}_{i+1,j}. \qquad \square$$

Part (2) of Theorem 18 implies that, to understand the asymptotics of pseudo-triangulations of the double chain, we need to understand the expressions

$$\sum_{i=0}^{v}\sum_{j=0}^{w}\binom{v}{i}\binom{w}{j}t^{v,w}_{i,j}=\sum_{i=0}^{v}\sum_{j=0}^{w}\binom{v}{i}\binom{w}{j}\binom{l-i+m-j+2}{l-i+1}. \tag{10}$$

The second form is obtained from the first by the substitutions $i\to v-i$ and $j\to w-j$, since $t^{v,w}_{v-i,w-j}=\binom{l-i+m-j+2}{l-i+1}$.

For the special case $l=v$ and $m=w$, this expression has a very nice combinatorial interpretation and has appeared in the literature (see below). In particular, we can give the exact asymptotics of the number of pointed pseudo-triangulations of a double chain with $l=m$ (Theorem 22). For general values of l, m, $|V|$, $|W|$, or for the total number of pseudo-triangulations, we can only offer the upper and lower bounds in the following two statements:

Lemma 20.

(1) *For every $V\subseteq\{1,\dots,l\}$ and $W\subseteq\{1,\dots,m\}$,*
$$\big|\mathcal{PT}_{V\cup W}(A)\big|\in O\big(2^{l+m}(3/2)^{v+w}C_lC_m\big),$$
where $v=|V|$ and $w=|W|$.

(2) *In particular,*
$$\big|\mathcal{PPT}(A)\big|\in O\big(3^{l+m}C_lC_m\big)=O\big(12^{l+m}(lm)^{-3/2}\big).$$

(3) $$\big|\mathcal{PT}(A)\big|\in O\big(5^{l+m}C_lC_m\big)=O\big(20^{l+m}(lm)^{-3/2}\big).$$

Proof. Starting with the equality in Theorem 18, we bound $|\mathcal{PPT}_{V'}(B)|$ by C_{l+1} and $|\mathcal{PPT}_{W'}(C)|$ by C_{m+1}, respectively, using part (2) of Corollary 7. We also bound

$$t^{v,w}_{|V'|,|W'|}=\binom{l-v+|V'|+m-w+|W'|+2}{l-v+|V'|+1}\leqslant 2^{l+m-v-w+|V'|+|W'|+2}.$$

Thus,

$$\frac{|\mathcal{PT}_{V\cup W}(A)|}{C_{l+1}C_{m+1}}\leqslant 2^{l+m-v-w+2}\sum_{\substack{V'\subseteq V\\ W'\subseteq W}}2^{|V'|}2^{|W'|}=2^{l+m-v-w+2}3^v3^w. \tag{11}$$

That finishes part (1), since $C_{l+1}\in\Theta(C_l)=\Theta(4^l l^{-3/2})$. For the upper bound in part (2), we simply specialize $v=l$ and $w=m$. For the upper bound in part (3) we add over all values of V and W the inequality (11) obtained above, since

$$\big|\mathcal{PT}(A)\big|=\sum_{\substack{V\subseteq\{1,\dots,l\}\\ W\subseteq\{1,\dots,m\}}}\big|\mathcal{PT}_{V\cup W}(A)\big|.$$

Hence:

$$\frac{|\mathcal{PT}(A)|}{C_{l+1}C_{m+1}} \leqslant \sum_{v=0}^{l}\sum_{w=0}^{m}\binom{l}{v}\binom{m}{w}2^{l+m-v-w+2}3^{v+w}$$
$$= 2^{l+m+2}\sum_{v=0}^{l}\sum_{w=0}^{m}\binom{l}{v}\binom{m}{w}\left(\frac{3}{2}\right)^{v+w} = 2^{l+m+2}\left(\frac{5}{2}\right)^{l+m} = 4\cdot 5^{l+m}. \qquad \square$$

We now look at lower bounds. We obtain the following ones by simply taking the greatest summand in the expressions derived from Theorem 18. Observe that in the case $l=m$ they differ from the upper bounds only by a polynomial factor of $l^{-3/2}$ and $l^{-5/2}$, respectively.

Theorem 21.

$$\left|\mathcal{PPT}(A)\right| \in \Omega\left(3^{l+m}C_lC_m\frac{(l+m)^{1/2}}{lm}\left(\frac{1}{2}\right)^{2|l-m|/3}\right). \tag{1}$$

$$\left|\mathcal{PT}(A)\right| \in \Omega\left(5^{l+m}C_lC_m\frac{(l+m)^{1/2}}{(lm)^{3/2}}\left(\frac{1}{2}\right)^{4|l-m|/5}\right). \tag{2}$$

(3) *In particular, if $l=m=(n-4)/2$ (where n is the total number of vertices), we have*

$$\left|\mathcal{PPT}(A)\right| \in \Theta^*\left(12^n\right), \quad \textit{and} \quad \left|\mathcal{PT}(A)\right| \in \Theta^*\left(20^n\right).$$

Proof. For part (1) we start with

$$\left|\mathcal{PPT}(A)\right| = \sum_{\substack{V'\subseteq\{1,\dots,l\}\\ W'\subseteq\{1,\dots,m\}}} t^{l,m}_{|V'|,|W'|}\left|\mathcal{PPT}_{V'}(B)\right|\left|\mathcal{PPT}_{W'}(C)\right|$$
$$\geqslant \sum_{\substack{V'\subseteq\{1,\dots,l\}\\ W'\subseteq\{1,\dots,m\}}} t^{l,m}_{|V'|,|W'|}C_lC_m = C_lC_m\sum_{i=0}^{l}\sum_{j=0}^{m}\binom{l}{i}\binom{m}{j}\binom{i+j+2}{i+1}.$$

In this expression we substitute the sum by the summand with $i=2l/3$ and $j=2m/3$. That is:

$$\frac{|\mathcal{PPT}(A)|}{C_lC_m}\binom{l}{2l/3}\binom{m}{2m/3}\binom{\frac{2(l+m)}{3}+2}{2l/3} \sim \binom{l}{2l/3}\binom{m}{2m/3}\binom{2(l+m)/3}{2l/3}.$$

Next we approximate the binomial coefficients using Stirling approximation, which gives:

$$\binom{l}{2l/3} \in \Theta\left(\frac{3^l}{2^{2l/3}}l^{-1/2}\right)$$

and

$$\binom{2(l+m)/3}{2l/3} \in \Theta\left(\left(\frac{l+m}{l}\right)^{2l/3}\left(\frac{l+m}{m}\right)^{2m/3}\left(\frac{l+m}{lm}\right)^{1/2}\right).$$

Putting things together we get

$$\frac{|\mathcal{PPT}(A)|}{C_l C_m} \in \Omega\left(3^{l+m}\left(\frac{l+m}{2l}\right)^{2l/3}\left(\frac{l+m}{2m}\right)^{2m/3}\frac{(l+m)^{1/2}}{lm}\right)$$
$$= \Omega\left(3^{l+m}\left(\frac{(l+m)^2}{4lm}\right)^{\frac{2\min(l,m)}{3}}\left(\frac{l+m}{2\max(l,m)}\right)^{\frac{2|l-m|}{3}}\frac{(l+m)^{1/2}}{lm}\right).$$

This gives part (1), since $\frac{(l+m)^2}{4lm} \geqslant 1$ and $\frac{l+m}{2\max(l,m)} \geqslant \frac{1}{2}$.

For part (2) we use the same ideas. We start with

$$|\mathcal{PT}(A)| \geqslant \sum_{v=0}^{l}\sum_{w=0}^{m}\binom{l}{v}\binom{m}{w}\sum_{i=0}^{v}\sum_{j=0}^{w}\binom{v}{i}\binom{w}{j}\binom{l-v+i+m-w+j+2}{l-v+i+1}C_l C_m.$$

Here, we substitute the sum with the summand $i = 2l/5$, $v = 3l/5$, $j = 2m/5$, and $w = 3m/5$. This gives:

$$\frac{|\mathcal{PT}(A)|}{C_l C_m} \geqslant \binom{l}{3l/5}\binom{m}{3m/5}\binom{3l/5}{2l/5}\binom{3m/5}{2m/5}\binom{4(l+m)/5}{4l/5}.$$

As before, Stirling's approximation gives:

$$\binom{l}{3l/5} \in \Theta\left(\frac{5^l}{3^{3l/5}2^{2l/5}}l^{-1/2}\right), \qquad \binom{3l/5}{2l/5} \in \Theta\left(\frac{3^{3l/5}}{2^{2l/5}}l^{-1/2}\right)$$
$$\binom{4(l+m)/5}{4l/5} \in \Theta\left(\left(\frac{l+m}{l}\right)^{4l/5}\left(\frac{l+m}{m}\right)^{4m/5}\left(\frac{l+m}{lm}\right)^{1/2}\right).$$

That is,

$$\frac{|\mathcal{PT}(A)|}{C_l C_m} \in \Omega\left(5^{l+m}\left(\frac{l+m}{2l}\right)^{4l/5}\left(\frac{l+m}{2m}\right)^{4m/5}\frac{(l+m)^{1/2}}{(lm)^{3/2}}\right)$$
$$= \Omega\left(5^{l+m}\left(\frac{(l+m)^2}{4lm}\right)^{\frac{4\min(l,m)}{5}}\left(\frac{l+m}{2\max(l,m)}\right)^{\frac{4|l-m|}{5}}\frac{(l+m)^{1/2}}{(lm)^{3/2}}\right).$$

Part (3) is straightforward from parts (1) and (2), by Lemma 20. □

Let us now restrict our attention to the case of pointed pseudo-triangulations. Applying Theorem 18 with $v = l$ and $w = m$ we get the following, which has been used in the proofs of the last two results:

$$\frac{|\mathcal{PPT}(A)|}{C_l C_m} \in \Theta\left(\sum_{i=0}^{l}\sum_{j=0}^{m}\binom{l}{i}\binom{m}{j}\binom{i+j+2}{i+1}\right).$$

Let us call $E^{l,m}$ the expression inside the $\Theta(-)$. It turns out that $E^{l,m}$ has the following nice interpretation: it equals the number of lattice paths from $(0,0)$ to $(l+1, m+1)$ when horizontal and vertical steps of arbitrary positive length are allowed. In other words, it equals the number of monotone rook paths from $(0,0)$ to $(l+1, m+1)$ (a path is specified not only by the squares traversed, but also by the positions where the rook stops. The rook is allowed to do several consecutive horizontal or vertical moves). Indeed, for a particular path, $i+1$ and $j+1$ represent the numbers of horizontal and vertical moves taken by the rook. The coefficient $\binom{l}{i}$ (respectively $\binom{m}{j}$) accounts for the possibilities of columns (respectively rows) where the rook makes at least one stop, and the coefficient $\binom{i+j+2}{i+1}$ accounts for the relative ordering of the $i+1$ horizontal and $j+1$ vertical moves.

The sequence $E^{l,m}$ appears (with a shift in the indices) as A035002 in [26] and has been studied in Section 7 of [9]. It satisfies, among others, the following formulas:

$$E^{l+1,m+1} = 2E^{l,m+1} + 2E^{l+1,m} - 3E^{l,m} \quad \forall l, m > 1; \qquad E^{l,0} = E^{0,l} = (l+4)2^{l-1},$$

or

$$\sum_{l+m=n} E^{l,m} = 2\big(3^{n+1} - 2^{n+1}\big).$$

In particular, the generating function of its diagonal sequence $E^{m,m}$ (sequence A051708) is known, and from it we can derive the asymptotics very precisely:

Theorem 22. *Let A be a double chain with n vertices in total and with $l = m = (n-4)/2$. Then, $|\mathcal{PPT}(A)| \in \Theta(12^n n^{-7/2})$.*

Proof. The generating function of $E^{m,m}$ is

$$f(t) = \frac{9t - 1 + \sqrt{9t^2 - 10t + 1}}{2(9t-1)} = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1-t}{1-9t}}.$$

(This is Theorem 7.1(c) of [9], except there a negative sign is wrongfully taken before the square root. The correct sign is positive since otherwise $f(t)$ is negative near zero, which does not make sense.)

The dominant (i.e., smallest in absolute value) singularity of $f(t)$ is at $t = 1/9$, and near the singularity one has

$$f(t) \sim \sqrt{\frac{2}{9}}(1-9t)^{-1/2}.$$

Then, the singularity analysis of [11] (see also [12]) implies that

$$E^{m,m} \sim \sqrt{\frac{2}{9}} \cdot \frac{9^m}{\sqrt{\pi m}}.$$

Hence, $|\mathcal{PPT}(A)| \in \Theta(\frac{9^m {C_m}^2}{\sqrt{m}}) = \Theta(12^n n^{-7/2})$. $\square$

A similar analysis could be undertaken for $\mathcal{PT}(A)$. By Theorem 18 and the equality (10) we have $|\mathcal{PT}(A)|/C_l C_m \in \Theta(F^{m,n})$, where

$$F^{m,n} = \sum_{\substack{0 \leqslant i \leqslant v \leqslant l \\ 0 \leqslant j \leqslant w \leqslant m}} \binom{l}{v}\binom{m}{w}\binom{v}{i}\binom{w}{j}\binom{l+m-i-j+2}{l-i+1}.$$

$F^{m,n}$ can still be interpreted (although less directly) in terms of rook paths, and satisfies formulas such as

$$F^{l+1,m+1} = 3F^{l,m+1} + 3F^{l+1,m} - 5F^{l,m} \quad \forall l, m > 1, \qquad F^{l,0} = F^{0,l} = (l+6)3^{l-1},$$

or

$$\sum_{l+m=n} F^{l,m} = 5^{n-1} - 3^{n-1}.$$

The latter, since the biggest summand is obtained with $l = m = n/2$, implies that $F^{n,n}$ is between $\Omega(5^n n^{-1})$ and $O(5^n)$, in agreement with—but also refining—the result in part (3) of Theorem 21. We believe that $F^{n/2,n/2} \in \Theta(5^n n^{-1/2})$ and, hence, that the total number of pseudo-triangulations of a double chain with the same number of points on both sides is in $\Theta(20^n n^{-7/2})$.

References

[1] P.K. Agarwal, J. Basch, L.J. Guibas, J. Hershberger, L. Zhang, Deformable free space tilings for kinetic collision detection, Int. J. Robot. Res. 21 (2002) 179–197.

[2] O. Aichholzer, F. Aurenhammer, H. Krasser, B. Speckmann, Convexity minimizes pseudo-triangulations, Comput. Geom. 28 (2004) 3–10.

[3] O. Aichholzer, F. Aurenhammer, P. Brass, H. Krasser, Pseudo-triangulations from surfaces and a novel type of edge Flip, SIAM J. Comput. 32 (2003) 1621–1653.

[4] O. Aichholzer, F. Aurenhammer, H. Krasser, Adapting (pseudo)-triangulations with a near-linear number of edge flips, in: Proc. 8th International Workshop on Algorithms and Data Structures, in: Lecture Notes in Comput. Sci., vol. 2748, 2003, pp. 12–24.

[5] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, B. Vogtenhuber, On the number of plane graphs, in: Proc. 17th ACM–SIAM Symposium on Discrete Algorithms, 2006, pp. 504–513.

[6] O. Aichholzer, F. Hurtado, M. Noy, A lower bound on the number of triangulations of planar point sets, Comput. Geom. 29 (2) (2004) 135–145.

[7] O. Aichholzer, H. Krasser, The point-set order-type database: A collection of applications and results, in: Proc. 13th Canadian Conference on Computational Geometry, 2001, pp. 17–20.

[8] B. Chazelle, H. Edelsbrunner, M. Grigni, L.J. Guibas, J. Hershberger, M. Sharir, J. Snoeyink, Ray shooting in polygons using geodesic triangulations, Algorithmica 12 (1994) 54–68.

[9] C. Coker, Enumerating a class of lattice paths, Discrete Math. 271 (1) (2003) 13–28.

[10] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (3) (1977) 291–301.

[11] P. Flajolet, A. Odlyzko, Singularity analysis of generating functions, SIAM J. Discrete Math. 3 (2) (1990) 216–240.

[12] P. Flajolet, R. Sedgewick, Analytic Combinatorics, book in preparation; online version of January 2007 available at http://algo.inria.fr/flajolet/Publications/books.html.

[13] A. García, M. Noy, J. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N, Comput. Geom. 16 (2000) 211–221.

[14] M. Goodrich, R. Tamassia, Dynamic ray shooting and shortest paths in planar subdivision via balanced geodesic triangulations, J. Algorithms 23 (1997) 51–73.

[15] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, W. Whiteley, Planar minimally rigid graphs and pseudo-triangulations, Comput. Geom. 31 (1–2) (2005) 31–61.

[16] F. Hurtado, M. Noy, Counting triangulations of almost-convex polygons, Ars Combin. 45 (1997) 169–179.

[17] D. Kirkpatrick, J. Snoeyink, B. Speckmann, Kinetic collision detection for simple polygons, Internat. J. Comput. Geom. Appl. 12 (2002) 3–27.

[18] D. Orden, F. Santos, The polytope of non-crossing graphs on a planar point set, Discrete Comput. Geom. 33 (2) (2005) 275–305.

[19] D. Orden, F. Santos, B. Servatius, H. Servatius, Combinatorial pseudo-triangulations, Discrete Math. 307 (2007) 554–566.

[20] M. Pocchiola, G. Vegter, Minimal tangent visibility graphs, Comput. Geom. 6 (1996) 303–314.

[21] M. Pocchiola, G. Vegter, Topologically sweeping visibility complexes via pseudo-triangulations, Discrete Comput. Geom. 16 (1996) 419–453.

[22] D. Randall, G. Rote, F. Santos, J. Snoeyink, Counting triangulations and pseudo-triangulations of wheels, in: Proc. 13th Canadian Conference on Computational Geometry, 2001, pp. 149–152.

[23] G. Rote, F. Santos, I. Streinu, Expansive motions and the polytope of pointed pseudo-triangulations, in: B. Aronov, S. Basu, J. Pach, M. Sharir (Eds.), Discrete and Computational Geometry—The Goodman–Pollack Festschrift, in: Algorithms Combin., Springer-Verlag, Berlin, 2003, pp. 699–736.

[24] G. Rote, F. Santos, I. Streinu, Pseudo-triangulations: A survey, preprint, arXiv: math/0612672v1, December 2006, in: J.E. Goodman, J. Pach, R. Pollack (Eds.), Proceedings of the Joint Summer Research Conference on Discrete and Computational Geometry, Snowbird, UT, June 18–22, 2006, in: Contemp. Math., American Mathematical Society, Providence, RI, in press.

[25] F. Santos, R. Seidel, A better upper bound on the number of triangulations of planar point sets, J. Combin. Theory Ser. A 102 (2003) 186–193.
[26] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://www.research.att.com/~njas/sequences.
[27] B. Speckmann, C.D. Tóth, Allocating vertex π-guards in simple polygons via pseudo-triangulations, Discrete Comput. Geom. 33 (2) (2005) 345–364.
[28] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge Univ. Press, Cambridge, 1999.
[29] I. Streinu, A combinatorial approach to planar non-colliding robot arm motion planning, in: Proc. 41st Symposium on Foundations of Computer Science, 2000, pp. 443–453.
[30] I. Streinu, Pseudo-triangulations, rigidity and motion planning, Discrete Comput. Geom. 34 (4) (2005) 587–635.