Existence of Almost Periodic Solutions to Delay Differential Equations with Lipschitz Nonlinearities

WILLIAM LAYTON *

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Received February 18, 1982; revised February 17, 1983

Differential equations whose nonlinearities depend upon both \( x(t) \) and \( x(t - \tau) \) arise in many settings. In this paper equations of this form subject to periodic and almost periodic forcing are studied:

\[
x'(t) + g(x(t), x(t - \tau)) = e(t), \quad -\infty < t < \infty. \tag{E}
\]

A nonresonance type condition is found under which it is shown that (E) will have a unique Besicovitch almost periodic solution for any Besicovitch almost periodic forcing term \( e(t) \). These results are then generalized to systems of equations of the same form as (E). These results hold without any small parameter type restriction upon \( g \).

1. INTRODUCTION

Many predator–prey interactions can be described by equations of the general form

\[
x'(t) + g(x(t), x(t - \tau)) = e(t) \tag{1.1}
\]

(see, e.g., Wagnersky and Cunningham [8]). Here the response of equations of the form (1.1) to periodic and almost periodic forcing is considered. In particular, a nonresonance condition, motivated by the case when (1.1) reduces to a linear equation, is formulated. Under this condition upon \( g \) it is shown that (1.1) will have a unique Besicovitch almost periodic \( (B^2) \) solution for any Besicovitch almost periodic \( e(t) \). When \( e(t) \) is periodic and square integrable over one period the unique solution to (1.1) will be periodic (with the same period as \( e \)) and square integrable over one period. These results are extended in a natural way to systems of equations having the same form as (1.1).

* The research described herein was supported in part by grant MCS-8202025 from the National Science Foundation.
† Current address: Mathematisch Instituut, Katholieke Universiteit, Nijmegen, The Netherlands.
The existence of almost periodic solutions to linear equations of the form (1.1) has been studied previously by many people (see Corduneanu [3], Hale [4], and the references in [3, 5]). Hale [4] also considers the case where a Lipschitz nonlinear term multiplied by a small parameter is present. He proves existence of an almost periodic solution for the nonlinear equation, provided the linear equation has a unique almost periodic solution and the parameter is sufficiently small. For the analogous result for ordinary differential equations see Theorem 4.4, pp. 98–99 of Corduneanu [3]. Alexiades [1] has studied the question of the existence of solutions to equations with infinite delay with $B^2$ almost periodic forcing terms. In [1] he proved the existence of almost periodic solutions under conditions similar to the ones formulated here.

Here the equation (1.1) is studied in the Hilbert space of Besicovitch almost periodic functions instead of the Banach space of Bohr almost periodic functions. Using the extra structure, the small parameter type restriction of Hale [4] can be avoided.

The extra structure of $B^2$ gives rise to other types of complications however. Because of the size of $B^2$, finding a unique solution to (1.1) for any $e(t) \in B^2$ gives a very strong uniqueness result and an existence result for a very wide class of forcing functions. Conversely, existence of a $B^2$ solution is a correspondingly weaker statement. To illustrate the peculiarities of $B^2$ type solutions consider the simple ordinary differential equation

$$x'(t) + x(t) = e(t),$$

(1.2)

where $e(t) = \sum_{n=1}^{\infty} (1/n) \cos(t/n)$ is a $B^2$ forcing term. The $B^2$ solution to (1.2) is easily found to be

$$x(t) = \sum_{n=1}^{\infty} \left( \frac{n}{n^2 + 1} \cos \left( \frac{t}{n} \right) + \frac{1}{n^2 + 1} \sin \left( \frac{t}{n} \right) \right).$$

(1.3)

Like the series for $e(t)$, the series giving the solution, $x(t)$, to (1.2) converges for no value of $t$. Another peculiarity is that although (1.3) converges for no value of $t$, the series for $x'(t)$ converges uniformly and absolutely for every value of $t$. When the Fourier exponents of the solution have finite limit points this and other properties can be qualitatively different from those in the periodic case.

In Section 2 the notation used and spaces of almost periodic functions required are defined. Section 3 contains an analysis of the linear equation associated with (1.1). In Section 4 conditions are found under which the nonlinear problem (1.1) has a unique solution. These conditions are sharp in the linear case. The precise sense in which $x(t)$ is a solution to (1.1) is discussed there. When $e(t)$ is periodic, a unique (classical) periodic solution is shown to exist in Section 5. The periodic case follows easily from the
analysis of Section 4. Finally, in Section 6 it is shown how the analysis of the previous sections on the scalar equation can be extended to systems of equations of the same form.

2. PRELIMINARIES

The space of Bohr almost periodic functions $f: \mathbb{R} \to \mathbb{R}^n$, denoted by $AP(\mathbb{R}; \mathbb{R}^n)$, is defined to be the closure of the trigonometric polynomials

$$
\sum_{s=1}^{n} (\alpha_s \cos \lambda_s t + \beta_s \sin \lambda_s t), \quad \alpha_s, \beta_s \in \mathbb{R}^n, \lambda_s \in \mathbb{R},
$$

(2.1)
in the maximum norm. The trigonometric polynomials (2.1) may be represented in a more compact complex notation as follows. Let $\lambda_0 = 0$, $\lambda_{-n} = -\lambda_n$, $\alpha_0 = \beta_0 = 0$,

$$
a_n = \frac{1}{2}(\alpha_n - i\beta_n), \quad a_{-n} = \overline{a_n}
$$

so that (2.1) becomes

$$
\sum_{s=-n}^{n} a_s e^{i\lambda_s t}, \quad a_s \in \mathbb{C}^n, a_s = \overline{a_{-s}}, \lambda_{-s} = -\lambda_s.
$$

(2.2)

Conversely, any sum (2.2) represents a real trigonometric polynomial (2.1).

The Besicovitch space of almost periodic functions, $B^2(\mathbb{R}; \mathbb{R}^n)$, is the closure of the trigonometric polynomials of the form (2.2) (or (2.1)) under the norm

$$
\|f\|^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt.
$$

$\| \cdot \|$ on $B^2(\mathbb{R}; \mathbb{R}^n)$ is induced by the inner product

$$
\langle f, g \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f^*(t) \overline{g(t)} \, dt.
$$

Alternately, $B^2$ could be defined as the set of all $f(t) = \sum_{-\infty}^{\infty} a_j e^{i\lambda_j t}$ with $\lambda_{-j} = -\lambda_j$, $a_{-j} = \overline{a_j}$, and

$$
\|f\|^2 = \sum_{-\infty}^{\infty} |a_j|^2 < \infty.
$$

Note that $B^2$ possesses a continuum of orthonormal basis functions $\{e^{i\lambda t} | \lambda \in \mathbb{R} \}$. If $L: B^2 \to B^2$ is a bounded linear operator, the operator norm of $L$ will be denoted by $\|L\|$. 


\[ |x| \] will denote the length of the vector \( x \), \(|x|^2 = x^*x \), and also the induced operator norm on the \( n \times n \) matrices: \(|A| = \sup\{|Ax|: |x| = 1\} \). For \( A \subset \mathbb{R} \) the closed subspace \( B^2_A \) of \( B^2 \) is defined as

\[
B^2_A = \left\{ f(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\lambda_j t} \mid \lambda_j \in A, \lambda_{-j} = -\lambda_j, \sum_{-\infty}^{\infty} |a_j|^2 < \infty \right\}.
\]

The space \( B^{2,1} \) is defined to be the closure of the trigonometric polynomials (2.2) in the norm

\[
\|f\|_2^2 = \sum_{j=-\infty}^{\infty} (1 + |\lambda_j|^2) |a_j|^2.
\]

For \( A \subset \mathbb{R} \), \( B^{2,1}_A \) is defined as \( B^2_A \cap B^{2,1} \).

### 3. The Linear Problem

For \( e(t) \in B^2(\mathbb{R}; \mathbb{R}^n) \),

\[
e(t) = \sum_{j=-\infty}^{\infty} e_j e^{i\lambda_j t},
\]

consider the problem of finding the solution \( x(t) \in B^2 \) to

\[
L_{a,b} x \equiv x'(t) + ax(t) + bx(t - \tau) = e(t).
\]

Let \( A \) be the Fourier exponents of \( e \). Then, formally, the solution \( x \in B^2_A \) to (3.1) is given by

\[
x(t) = \sum_{n=-\infty}^{\infty} x_n e^{i\lambda_n t},
\]

\[
x_n = \left[ i\lambda_n + a + be^{-i\lambda_n \tau} \right]^{-1} e_n, \quad n \in \mathbb{Z}.
\]

Provided this series converges to an element of \( B^2 \), this is the unique solution to (3.1) in \( B^2 \). Thus, (3.2) represents the unique solution to (3.1) provided \( i\lambda_n + a + be^{-i\lambda_n \tau} \) is bounded away from zero. \( i\lambda_n + a + be^{-i\lambda_n \tau} \) can equal zero only when \( a + b \cos(\lambda_n \tau) = 0, \lambda_n - b \sin(\lambda_n \tau) = 0 \), for some \( n \). For \( L_{a,b}^{-1} \) to exist for general \( e \in B^2 \) and \( \tau \), then, necessarily, \( a + b \cos(\lambda_n \tau) \) must be bounded away from zero. Otherwise \( \lambda_n - b \sin(\lambda_n \tau) \) can become arbitrarily close to zero infinitely often for any \( b \) and (3.2) will not converge. Alternately, \((a, b)\) must not belong to the region \( R \) in Fig. 1.
LEMMA 3.1. If \((a, b) \notin R\), then \(L_{a,b}^{-1}\) exists on \(B^2\). The solution, \(x(t)\), to (3.1) exists, is an element of \(B^2\), is unique (up to a function with \(B^2\)-norm zero), and is given by (3.2).

Next \(\|L_{a,b}^{-1}\|\) will be estimated, where

\[
L_{a,b}^{-1}\left(\sum_{n=-\infty}^{\infty} e_n e^{i\lambda_n t}\right) = \sum_{n=-\infty}^{\infty} (i\lambda_n + a + be^{-i\lambda_n t})^{-1} e_n e^{i\lambda_n t}.
\]

LEMMA 3.2. If \((a, b) \notin R\), then

\[
\|L_{a,b}^{-1}e\| \leq d^{-1} \|e\|, \quad d = \min \{|a + b|, |a - b|\},
\]

and hence, \(\|L_{a,b}^{-1}\| \leq d^{-1}\).

Proof. From (3.2),

\[
\|L_{a,b}^{-1}e\|^2 = \sum_{n=-\infty}^{\infty} |(i\lambda_n + a + be^{-i\lambda_n t})^{-1} e_n|^2.
\]

But,

\[
|i\lambda_n + a + be^{-i\lambda_n t}|^2 = (a + b \cos(\lambda_n \tau))^2 + (\lambda_n - b \sin(\lambda_n \tau))^2
\]

\[
\geq (a + b \cos(\lambda_n \tau))^2
\]

\[
\geq \inf\{|a + sb|; -1 \leq s \leq +1\}^2 = d^2,
\]

since \((a, b) \notin R\).

LEMMA 3.3. Assume \((a, b) \notin R\), then \(L_{a,b}^{-1}\) maps \(B^2\) into \(B^{2,1}\) continuously.
Proof. From (3.2),

\[ \| L_{a,b} e \|_1^2 = \sum (1 + |\lambda_j|^2) |i\lambda_j + a + be^{-i\lambda_j \tau}|^{-2} |e_j|^2 \]

\[ = \sum (1 + |\lambda_j|^2)[(a + b \cos(\lambda_j \tau))^2 + (\lambda_j - b \sin(\lambda_j \tau))^2]^{-1} |e_j|^2 \]

\[ \leq \sum (1 + |\lambda_j|^2)[d^2 + (\lambda_j - b \sin(\lambda_j \tau))^2]^{-1} |e_j|^2 \]

\[ \leq C \sum |e_j|^2 = C \| e \|^2. \]

When \((a, b) \in R\) the solution to the linear equation can be calculated formally by manipulating the series for \(e\) and \(x\). Next, it is shown that this formal solution is precisely the \(B^2\)-weak solution to (3.1):

\[ \langle x(t), -p'(t) + ap(t) + bp(t + \tau) \rangle = \langle e, p \rangle, \quad \forall p \in \mathcal{F}, \]

where \(\mathcal{F}\) denotes the set of all trigonometric polynomials of the form (2.1) or (2.2).

If \(p(t), q(t) \in \mathcal{F}\), then the formula

\[ \langle p', q \rangle = -\langle p, q' \rangle \]

holds using integration by parts.

Define the norm on \(\mathcal{F}\)

\[ \| p \|^2 = \| p \|^2 + \| p' \|^2, \]

where \(\cdot \) is the normal \(B^2\) norm. An easy calculation reveals that \(\| \cdot \|\) coincides with the \(B^{2,1}\) norm \(\| \cdot \|\) and thus \(B^{2,1}\) consists of the closure of \(\mathcal{F}\) in the norm \(\| \cdot \|\). Let \(R^{2,-1}\) denote \((R^{2,1})^*\) then \(B^{2,-1}\) can be characterized as a set of formal Fourier series so that the pairing between \(B^{2,1}\) and \(B^{2,-1}\) is the usual \(B^2\) inner product. Specifically, \(f = \sum \lambda_j e^{i\lambda_j t} \in B^{2,-1}\) if

\[ \| f \|_{2,-1}^2 = \sum \left(1 + |\lambda_j|^2\right)^{-1} |f_j|^2 < \infty. \]

Proposition 3.1. Let \(x \in B^2\) be given, then there is a unique \(Dx \in B^{2,-1}\) such that

\[ \langle Dx, p \rangle = -\langle x, p' \rangle, \quad \forall p \in \mathcal{F}. \]

Further, \(Dx\) is the formal derivative of the Fourier series giving \(x\).

Proof. Define the linear functional on \(\mathcal{F}\), \(\phi_x: p \rightarrow -\langle x, p' \rangle\). \(\phi_x\) is bounded on \(\mathcal{F}\) in the \(B^{2,1}\) norm since

\[ | - \langle x, p' \rangle | \leq \| x \| \| p' \| \leq \| x \| \| p \|_1. \]
Thus, the Riesz-representation theorem gives a unique function \( Dx \in B^{2,-1} \) such that
\[
\langle Dx, p \rangle = -\langle x, p' \rangle, \quad \forall p \in \mathcal{F}.
\]

To see that \( Dx \) agrees with the formal derivative of the Fourier series for \( x \), let \( x = \sum x_j e^{i\lambda_j t} \). Then,
\[
\begin{align*}
i\lambda_j x_j &= i\lambda_j \langle x, e^{i\lambda_j t} \rangle \\
&= -\langle x, i\lambda_j e^{i\lambda_j t} \rangle = -\langle x, (e^{i\lambda_j t})' \rangle,
\end{align*}
\]
so that \( Dx = \sum x_j i\lambda_j e^{i\lambda_j t} \).

Note also that \( B^{2,1} \) becomes, \( B^{2,1} = \{ x \in B^2 \mid Dx \in B^2 \} \).

**COROLLARY 3.1.** Let \( x \) be the solution to (3.1) given by (3.2) (where \((a, b) \in \mathbb{R}\)). Then, \( x \) is the unique weak solution to (3.1):
\[
\langle x, -p'(t) + ap(t) + bp(t + \tau) \rangle = \langle e, p \rangle, \quad \forall p \in \mathcal{F}.
\]

4. THE NONLINEAR EQUATION

In this section the scalar equation (1.1) is considered. Here \( g \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \( e(t) \in B^2(\mathbb{R}, \mathbb{R}) \) is a known forcing function and \( x: \mathbb{R} \rightarrow \mathbb{R} \). The equation (1.1) is rewritten as
\[
x'(t) + ax(t) + bx(t - \tau) = ax(t) + bx(t - \tau) - g(x(t), x(t - \tau)) + e(t),
\]
where \( e \in B^2 \) and \((a, b) \in \mathbb{R}\). Define the nonlinear operator
\[
N_{a,b}x = ax(t) + bx(t - \tau) - g(x(t), x(t - \tau)).
\]
Equation (1.1) will be studied as the fixed point problem in \( B^2 \):
\[
x = T_{a,b}x = L_{a,b}^{-1}(N_{a,b}x + e). \tag{4.1}
\]

First, some properties of \( N_{a,b} \) will be collected. Alexiades [1] has shown that if the function \( h(x) \) is Lipschitz then the map \( x(t) \rightarrow h(x(t)) \) maps \( B^2 \) into \( B^2 \) continuously. That fact is now extended to the present situation.

**Lemma 4.1.** If \( g(x, y) \) is Lipschitz then \( g(\cdot, \cdot): B^2 \times B^2 \rightarrow B^2 \) continuously.

**Proof.** Every \( x, y \in B^2 \) is the \( B^2 \)-limit of a sequence \( \{p_n\}, \{q_n\} \) of trigonometric polynomials. Each \( p_n, q_n \in \text{AP} \) (i.e., almost periodic in the
sense of Bohr). Since \( g(\cdot, \cdot) \) is uniformly continuous, Theorem 1.7, p. 13 of Corduneanu [3] implies that \( g(p_n, q_n) \in \text{AP} \). Hence \( g(p_n, q_n) \in B^2 \).

Also, since \( g \) is Lipschitz,

\[ |g(p_n, q_n) - g(x, y)| \leq C(|p_n - x| + |q_n - y|), \]

thus,

\[
\frac{1}{2T} \int_{-T}^{T} |g(p_n, q_n) - g(x, y)|^2 \, dt \leq \frac{1}{2T} \int_{-T}^{T} (|p_n - x| + |q_n - y|)^2 \, dt
\]

\[
\leq 2C \frac{1}{2T} \int_{-T}^{T} |p_n - x|^2 + |q_n - y|^2 \, dt.
\]

Therefore,

\[ \|g(p_n, q_n) - g(x, y)\| \leq 2C(|p_n - x|^2 + |q_n - y|^2), \]

and \( g(x(t), y(t)) \in B^2 \).

Let \( H(x, y) = (\partial g/\partial x)(x, y) \), \( K(x, y) = (\partial g/\partial y)(x, y) \). The mean value theorem and a density argument give the following estimate on \( N_{a,b} \) for any \( x, y \in B^2 \):

\[ \|N_{a,b}x - N_{a,b}y\| \leq \sup \{|a - H(u, v)| + |b - K(u', v')|\} \|x - y\|. \quad (4.2) \]

For the linearized equation associated with (1.1) to be invertible \((H, K)\) must be bounded away from the region \( R \). The next assumption is a precise statement of this condition.

**Assumption A1.** Assume \( g \in C^1 \), and that there are numbers \( p, q, r, s \) such that \( p < q < 0 < r < s \) and either

\[ p \leq K(u', v') - H(u, v) \leq q, \quad r \leq H(u, v) + K(u', v') \leq s, \]

for all \( u, v, u', v' \in \mathbb{R} \), or

\[ p \leq H(u, v) + K(u', v') \leq q, \quad r \leq K(u', v') - H(u, v) \leq s, \]

for all \( u, v, u', v' \in \mathbb{R} \).

In the first instance A1 states that \((H, K)\) is bounded away from \( R \) in the region to the right of \( R \). The second case is when \((H, K)\) is located to the left of \( R \). Note that A1 also implies that \( g \) is Lipschitz.

**Lemma 4.2.** Under assumption A1,

\[ \|N_{a,b}x - N_{a,b}y\| \leq \max \{|a + b - s|, |a + b - r|, |a - b + q|, |a - b + p|\} \|x - y\|. \]
Proof. For definiteness, consider the case when \((H, K)\) lies to the right of \(R\). In this case the first alternative in A1 gives that
\[
-s + a + b \leq (a - H) + (b - K) \leq -r + a + b,
\]
\[
p + a - b \leq (a - H) - (b - K) \leq q + a - b.
\]
Hence,
\[
|a - H| + |b - K| \leq \max\{|-s + a + b|, |-r + a + b|\},
\]
\[
|(a - H) - (b - K)| \leq \max\{|q + a - b|, |p + a - b|\}.
\]
Considering the cases when \((a - H) \leq 0, (b - K) \geq 0\) separately gives
\[
|a - H| + |b - K| \leq \max\{|a + b - s|, |a + b - r|, |a - b + q|, |a - b + p|\}.
\]
The lemma now follows from (4.2) in this case. When \((H, K)\) lies to the left of \(R\) an analogous argument yields the result. 

**Theorem 4.1.** Under assumption A1, the scalar equation (1.1) has an almost periodic solution \(x \in B^2\). That solution is unique in \(B^2\) and belongs to \(B^2_1\).

**Proof.** Consider the fixed point problem (4.1). Lemmas 3.2 and 4.2 imply
\[
\|T_{a,b}x - T_{a,b}y\| \leq a(a, b) \|x - y\|,
\]
where \(a(a, b) = d^{-1} \max\{|a + b - s|, |a + b - r|, |a - b + p|, |a - b + q|\}\). Next it will be shown that \((a, b) \notin R\) can be chosen to ensure that \(a(a, b) < 1\), or, equivalently,
\[
\max\{|a + b - s|, |a + b - r|, |a - b + p|, |a - b + q|\} < \min\{|a + b|, |a - b|\}.
\]
(4.3)

For definiteness, assume that \((H, K)\) lies to the right of \(R\).

The first part of A1 bounds \((H, K)\) inside a rectangle to the right of \(R\) whose sides are given by lines with slope +1 and \(b\) intercepts \(p\) and \(q\) and lines with slope −1, and \(b\) intercepts \(r\) and \(s\). Choose \((a, b)\) to be the center of the rectangle containing \((H, K)\):
\[
b = \frac{p + q + r + s}{4}, \quad a = \frac{r + s - p - q}{4}.
\]
(4.4)

Note that by increasing \(|p|\) or \(|s|\) the rectangle containing \((H, K)\) can be enlarged to ensure that \(|p - q| = |r - s|\) (i.e., that the rectangle is a square). Without loss, assume \(|p - q| = |r - s|\).
Using the definition of $a, b$ (4.3) becomes

$$\max \left\{ \frac{s-r}{2}, \frac{q-p}{2} \right\} < \min \left\{ \frac{r+s}{2}, \frac{p+q}{2} \right\}. \tag{4.5}$$

Consider $|(r+s)/2| = (r+s)/2$ and $|(s-r)/2| = (s-r)/2$; clearly from Fig. 2, $(s-r)/2 < ((r+s)/2)$. For a similar reason $(q-p)/2 < ((p+q)/2)$. Since $|p-q| = |r-s|$ (4.5) follows. Thus, $a(a, b) < 1$ and $T_{a,b}$ is a contradiction on $B^2$. The fact that $x \in B^{2,1}$ follows since $x \in B^2$ implies that $z = N_{a,b} x \in B^2$. Thus, $x = L^{-1} z \in B^{2,1}$. \[\square\]

Theorem 4.1 ensures that under A1, (1.1) will have a unique $B^2$ solution. Since the term by term derivative of $x$ is precisely the weak $B^2$ derivative of $x$ this solution is the weak $B^2$ solution of (1.1). Naturally, if $x$ is sufficiently smooth the weak derivative agrees with the classical derivative and $x$ is then a classical solution of (1.1).

**COROLLARY 4.1.** Assume A1 holds $e \in B^2$ and $x$ is the solution to (1.1) given in Theorem 4.1. Then, $x$ is the unique weak solution to (1.1). That is,

$$-\langle x, p' \rangle + \langle g(x(t), x(t-\tau)), p(t) \rangle = \langle e, p \rangle, \quad \forall p \in \mathcal{F}. \quad \square$$

This can be formulated in the following manner.

**THEOREM 4.2.** Suppose A1 holds, and let $x(t) = \sum x_j e^{i\lambda_j t}$ be the unique solution to (4.3) given by Theorem 4.1. Let $s_n(t)$ denote the trigonometric polynomials

$$s_n(t) = \sum_{j=-n}^{n} x_j e^{i\lambda_j t}.$$ 

Then,

$$s_n'(t) + g(s_n(t), s_n(t-\tau)) \to e(t)$$

strongly in $B^2$.

**Proof.** Let $r_n$ denote the residual $r_n = e - [s_n'(t) + g(s_n(t), s_n(t-\tau))]$ and consider $r_n - r_m$. For $(a, b) \in R$, $r_n - r_m$ satisfies

$$r_n - r_m = L_{a,b}(s_m - s_n) - [N_{a,b}s_n - N_{a,b}s_m].$$
Lemma 4.1 implies that \( N_{a,b} : B_{2}^{2,1} \to B^{2} \) continuously. Since \( L_{a,b}^{-1} : B^{2} \to B_{2}^{2,1} \) continuously it follows that \( L_{a,b}^{-1}[N_{a,b}s_{n} - N_{a,b}s_{m}] \to 0 \) in \( B_{2}^{2,1} \) as \( m, n \to \infty \). Thus,

\[
L_{a,b}^{-1}(r_{n} - r_{m}) \to 0 \quad \text{in} \quad B_{2}^{2,1} \quad \text{as} \quad m, n \to \infty.
\]

Since \( L_{a,b} : B_{2}^{2,1} \to B^{2} \) continuously, this gives that \( r_{n} - r_{m} \to 0 \) in \( B^{2} \) as \( m, n \to \infty \).

Thus, there is an \( r \in B^{2} \) such that \( r_{n} \to r \) as \( n \to \infty \). To see that \( r \) must be zero, consider for \( p \in \mathcal{F} \).

\[
\langle r_{n}, p \rangle = \langle e, p \rangle - \left[ -\langle s_{n}, p' \rangle + \langle g(s_{n}(t), s_{n}(t - \tau)), p \rangle \right].
\]

Since strong convergence implies weak convergence the above becomes as \( n \to \infty \)

\[
\langle r, p \rangle = \langle e, p \rangle - \left[ -\langle x, p' \rangle + \langle g(x(t), x(t - \tau)), p \rangle \right].
\]

By the previous theorem, \( r = 0 \), in \( B^{2} \).

5. THE PERIODIC CASE

In this section the results of the previous sections are specialized to the case when \( e(t) \) is \( \omega \)-periodic in \( t \). In this case a unique \( \omega \)-period solution to the scalar equation (1.1) will exist when A1 holds. Thus A1 gives a condition upon \( g \), independent of \( \omega \) or \( \tau \), under which periodic forcing gives rise to periodic response having the same period. The following proposition follows easily by the proof of Theorem 4.1. It will be specialized to the periodic case later.

**Proposition 5.1.** Suppose \( g(\cdot, \cdot) : B_{2} \times B_{2} \to B_{2}^{2} \), and A1 holds. Then, whenever \( e \in B_{2}^{2} \), a unique solution to (1.1) exists. That solution belongs to \( B_{2}^{2} \cap B_{2}^{2,1} \).

**Proof:** Under these hypotheses \( N_{a,b}(B_{2}^{2}) \subset B_{2}^{2} \). Since \( L_{a,b}^{-1} \) maps \( B_{2}^{2} \) into \( B_{2}^{2} \), \( T \) must map \( B_{2}^{2} \) into itself. Thus, the fixed point problem \( x = Tx \) can be posed on \( B_{2}^{2} \) so that the fixed point \( x \in B_{2}^{2} \).

In general the assumption that \( g(\cdot, \cdot) : B_{2} \times B_{2} \to B_{2}^{2} \) is very hard to verify. However, in the case when \( A = (j\omega(2\pi)^{-1} \mid j \in \mathbb{Z}) \) this assumption holds true. In this case \( B_{2}^{2} \) consists precisely of \( \omega \)-periodic functions square integrable over one period. The assumption then simply reduces to the statement that \( g(x(t), y(t)) \) is \( \omega \)-periodic whenever \( x(t) \) and \( y(t) \) are \( \omega \)-periodic. Also, with this choice of \( A \), \( B_{2}^{2,1} \) becomes the Sobolev space \( H_{t}^{1} \):

\[
B_{2}^{2,1} = H_{t}^{1} = \{f | f \in L^{2}[0, \omega], f' \in \text{A.C.}, f(0) = f(\omega), \text{and} f' \in L^{2}[0, \omega] \},
\]
where A.C. stands for the set of absolutely continuous functions. This then gives the result in the periodic case.

**Theorem 5.1.** Suppose A1 holds and \( e(t) \in L^2[0, \omega] \) is \( \omega \)-periodic in \( t \). Then, a unique \( \omega \)-periodic solution \( x(t) \in H^1_\omega \) to (1.1) exists. That solution is absolutely continuous and satisfies (1.1) pointwise.

### 6. Systems of Equations

In the previous sections the scalar equation (1.1) is analyzed. In this section it will be shown how the previous work can be extended to a system of delay equations. Here \( x: \mathbb{R} \to \mathbb{R}^n, \, e \in B^2(\mathbb{R}, \mathbb{R}^n) \) is a known forcing term, and \( g \) will denote a function in \( C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), so that the equation under consideration can be written in the form of the scalar equation:

\[
x'(t) + g(x(t), x(t - \tau)) = e(t); \quad (6.1)
\]

\( g \) is assumed to be such that for fixed \( y \in \mathbb{R}^n \) the equations

\[
x'(t) + g(x(t), y) = e(t), \quad x'(t) + g(y, x(t)) = e(t)
\]

are conservative. There are many (equivalent) ways of stating this condition, for example.

**Assumption A2.** Assume \( g \in C^1 \). Let \( H(x, y), K(x, y) \) denote the Jacobi matrices of \( g \) with respect to its first and second variables, respectively. Assume \( H \) and \( K \) are symmetric.

The simplest case when this hypothesis holds is when there are two functions \( G_1, G_2 \in C^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}) \) such that

\[
g(x, y) = \text{grad } G_1(x) + \text{grad } G_2(y).
\]

In this case \( H \) and \( K \) are the Hessians of \( G_1 \) and \( G_2 \), respectively, hence symmetric.

As in the previous section (6.1) is studied as the fixed point problem

\[
x = T_{a,b}x = L_{a,b}^{-1}(N_{a,b}x + e)
\]

with \( L_{a,b}, N_{a,b} \) as before. Assumption A2 allows the assumption A1 to be extended to the case of systems in the following way.

**Assumption A3.** Let \( \lambda_j, \mu_k \) denote the eigenvalues of \( H \) and \( K \), respec-
tively. Assume that there are numbers $p, q, r, s$ with $p < q < 0 < r < s$ such that either

$$ p \leq \mu_j(u', v') - \lambda_k(u, v) \leq q, \quad r \leq \lambda_j(u, v) + \mu_k(u', v') \leq s, $$

holds for all $j, k = 1, \ldots, n$ and $u, v, u', v' \in \mathbb{R}^n$, or

$$ p \leq \lambda_j(u, v) + \mu_k(u', v') \leq q, \quad r \leq \mu_j(u', v') - \lambda_k(u, v) \leq s, $$

holds for all $j, k = 1, \ldots, n$ and $u, v, u', v' \in \mathbb{R}^n$.

**Theorem 6.1.** Assume $A2$ and $A3$ hold. Then (6.1) has a unique solution in $B_{21}^*$. 

**Proof.** For $(a, b) \in \mathbb{R}$, there follows $\| T_{a,b}x - T_{c,d}y \| \leq d^{-1} \max \{|aI - H| + |bI - K|\} \| x - y \|$, where the max is taken over $u, v \in \mathbb{R}^n$. $A3$ implies

$$ |(\lambda_j - a) + (\mu_k - b)| \leq \max \{|a + b - s|, |a + b - r|\}, $$

$$ |(\lambda_j - a) - (\mu_k - b)| \leq \max \{|a - b + q|, |a - b + p|\}. $$

Since $H$ and $K$ are symmetric, this gives that

$$ \max \{|aI - H(u, v)| + |hI - K(u', v')|: u, v, u', v'\} \leq \max \{|a + b - s|, |a + b - r|, |a - b + q|, |a - b + p|\}. $$

Proceeding as in Section 4, the choice of $a, b$ given by (4.4) ensures that

$$ \| T_{a,b}x - T_{a,b}y \| \leq a(a, b) \| x - y \|, $$

where $a(a, b) < 1$. Thus, the result follows. 

Analogous results to those in Section 5 can naturally be shown in the case of systems of equations.

**Acknowledgments**

I would like to thank Lance Drager for helpful conversations concerning Proposition 3.1 and Theorem 4.2. I would also like to thank the referee for suggesting (1.2) and (1.3).

**References**