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On densities of box products

Moti Gitik [a,*], Saharon Shelah [b,1]

[a] School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel
[b] Hebrew University of Jerusalem, Department of Mathematics, Givat Ram, Jerusalem, Israel

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Abstract

We construct two universes V_1, V_2 satisfying the following GCH below $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and the topological density of the space ${}^{\aleph_\omega}2$ with $\aleph_0$ box product topology $d_{<\aleph_1}(\aleph_\omega)$ is $\aleph_{\omega+1}$ in V_1 and $\aleph_{\omega+2}$ in V_2. Further related results are discussed as well. © 1998 Elsevier Science B.V. All rights reserved.

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W.W. Comfort asked us (see history below) the following question: Assume λ is a strong limit singular, $\kappa > \mathrm{cf}(\lambda)$. Is $d_{<\kappa}(\lambda) = 2^\lambda$? Is it always $> \lambda^+$ when $2^\lambda > \lambda^+$?

$d_{<\kappa}(\lambda)$ denotes the density of the topological space ${}^\lambda 2$ with topology generated by the following family of clopen sets:

$$\{[f] \mid f \in {}^a 2 \text{ for some } a \subseteq \lambda,\ |a| < \kappa\},$$

where $[f] = \{g \in {}^\lambda 2 \mid g \supseteq f\}$, i.e.,

$$d_{<\kappa}(\lambda) = \min\{|F| \mid F \subseteq {}^\lambda 2 \text{ and if } a \subseteq \lambda\ |a| < \kappa \text{ and } g \in {}^a 2 \text{ then there is } f \in F,\ g \subseteq f\}.$$

The aim of this paper will be to show that under $\neg$SCH $d_{<\aleph_1}(\lambda)$ may be λ^+ even if $2^\lambda > \lambda^+$. Surprisingly, it turned out that it is easier to get $d_{<\aleph_1}(\lambda) = \lambda^+$ than $d_{<\aleph_1}(\lambda) = 2^\lambda$ for a strong limit λ of cofinality $\aleph_0$ with $2^\lambda > \lambda^+$. We refer to the ZFC results using the cardinal arithmetic of Shelah [16, Section 5].

* Corresponding author.
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$d(2^{\aleph_0}) = \aleph_0$ by the classical Hewitt–Marczewski–Pondiczery theorem [8,10,11]. This has been generalized by Engelking and Karlowicz [5] and by Comfort and Negrepontis [2,3] to show, for example, that $d_{<\kappa}(2^\alpha, \alpha) = \alpha$ if and only if $\alpha = \alpha^{<\kappa}$ [2, Theorem 3.1]. Cater et al. [1] show that every nondegenerate space X satisfies $\mathrm{cf}(d_{<\kappa}(\lambda, X)) \geqslant \mathrm{cf}(\kappa)$ when $\kappa \leqslant \lambda^+$, and they note (in our notation) that "$d_{<\kappa}(\lambda)$ is usually (if not always) equal to the well-known upper bound $(\log\lambda)^{<\kappa}$". It is known (cf. [1,4]) that SCH $\Rightarrow d_{<\aleph_1}(\lambda) = (\log\lambda)^{\aleph_0}$, but it is not known whether $d_{<\aleph_1}(\lambda) = (\log\lambda)^{\aleph_0}$ is a theorem of ZFC.

The point in those theorems is the upper bound, as, of course, $d_{<\kappa}(\mu, \theta) > \chi$ if $\mu > 2^\chi$ und $\theta > 2$ [why? because if $F = \{f_i\colon i < \chi\}$ exemplify $d_{<\kappa}(\mu, \theta) \leqslant \chi$, the number of possible sequences $\langle \mathrm{Min}\{1, f_i(\zeta)\}\colon i < \chi\rangle$ (where $\zeta < \mu$) is $\leqslant 2^\chi$, so for some $\zeta \neq \xi$ they are equal and we get contradiction by g, $g(\zeta) = 0$, $g(\xi) = 1$, $\mathrm{Dom}\, g = \{\zeta, \xi\}$].

Also trivial is: for κ limit, $d_{<\kappa}(\lambda, \theta) = \kappa + \sup_{\sigma<\kappa} d_{<\sigma}(\lambda, \theta)$, so we only use κ regular; $d_{<\kappa}(\lambda, \theta) \geqslant \sigma^\theta$ for $\sigma < \kappa$.

Also if $\mathrm{cf}(\lambda) < \kappa$, λ strong limit then $d_{<\kappa}(\lambda) > \lambda$. The general case (say $2^{<\mu} < \lambda < 2^\mu$, $\mathrm{cf}(\mu) \leqslant \theta$) is similar; we ignore it in order to make the discussion simpler.

So the main problem is:

Problem. Assume λ is strong limit singular, $\lambda > \kappa > \mathrm{cf}(\lambda)$, what is $d_{<\kappa}(\lambda)$? Is it always 2^λ? Is it always $> \lambda^+$ when $2^\lambda > \lambda^+$?

In [13] this question was raised (later and independently) for model theoretic reasons. We thank Comfort for asking us about it in the Fall of 1990.

The paper is organized as follows. Section 1 is less involved and provides a model with a strong limit λ, $\mathrm{cf}(\lambda) = \aleph_0$, $2^\lambda > \lambda^+$ and $d_{<\aleph_1}(\lambda) = \lambda^+$. The main disadvantage is that λ is rather large and it is unclear how to move everything down to say $\aleph_\omega$. But as a bonus this construction gives a normal ultrafilter over λ generated by λ^+ sets and $2^\lambda > \lambda^+$. Originally such models were produced by T. Carlson and H. Woodin (both unpublished). In Section 2 it is fixed at the cost of using more involved techniques. Also initial assumptions reduced from huge to hypermeasurable.

Both sections can be read independently. Most of the construction in Section 1 is due to the second author. Only the final argument using a huge cardinal is of the first author. The construction in Section 2 is due to the first author.

1. Density of box products from huge cardinal

In this section, we prove the following:

Theorem 1.1. *Suppose that λ is a huge cardinal. Then there exists a generic extension satisfying the following*:

(a) *λ is a strong limit of cofinality ω.*

(b) $2^\lambda > \lambda^+$.

(c) *for every $\mu < \lambda$ $d_{<\mu}(\lambda) = \lambda^+$.*

Let us recall the definition of the Prikry forcing with a normal ultrafilter $\mathcal{U}$ over a measurable cardinal κ. The set of conditions $\mathcal{P}_{\mathcal{U}}$ consists of all pairs $\langle \nu, B\rangle$ such that $\nu \in [\kappa]^{<\omega}$, $B \in \mathcal{U}$ and $\max \nu < \min B$. A condition $\langle \eta, A\rangle \geqslant \langle \nu, B\rangle$ iff

(a) $A \subseteq B$,

(b) η extends ν,

(c) $\eta \setminus \nu \in [B]^{<\omega}$.

The main property of this forcing (called the Prikry property) says that for every condition $\langle \nu, B\rangle$ and any sentence σ of the forcing language there is $A \subseteq B$, $A \in U$ such that $\langle \nu, A\rangle \parallel \sigma$, i.e., $\langle \nu, A\rangle \Vdash \sigma$ or $\langle \nu, A\rangle \Vdash \neg\sigma$. This is crucial for showing that all the cardinals are preserved. We refer to Kanamori's book [9] for a detailed presentation.

Let κ be a measurable cardinal and D be a normal ultrafilter over κ.

Definition 1.2. Let Q_D be a forcing notion consisting of all triples $\langle f, \alpha, A\rangle$ so that

(a) $A \in D$;

(b) $\alpha < \kappa$;

(c) f is a function on $[A]^{<\omega}$ such that

  (c1) for every $\eta \in [A]^{<\omega}$, $f(\eta)$ is a partial function from $\min(A \setminus (\max \eta + 1))$ to 2,

  (c2) $\sup\{|\operatorname{dom} f(\eta)| \mid \eta \in [A]^{<\omega}\} < \kappa$.

Definition 1.3. Let $\langle f_1, \alpha_1, A_1\rangle, \langle f_2, \alpha_2, A_2\rangle \in Q_D$. We define $\langle f_2, \alpha_2, A_2\rangle \geqslant \langle f_1, \alpha_1, A_1\rangle$ iff

(a) $\alpha_1 \leqslant \alpha_2$,

(b) $A_1 \supseteq A_2$,

(c) $A_1 \cap \alpha_1 = A_2 \cap \alpha_1$,

(d) for every $\eta \in [A_2]^{<\omega}$ $f_1(\eta) \subseteq f_2(\eta)$,

(e) $f_1 \restriction [A_1 \cap \alpha_1]^{<\omega} = f_2 \restriction [A_2 \cap \alpha_1]^{\omega}$.

Intuitively, the forcing is intended to add a set $A \subseteq \kappa$ which is almost contained in every set of D and a function f on $[A]^{<\omega}$ which is a name of a function in a Prikry forcing for changing cofinality of κ to $\aleph_0$. This function will be eventually a member of a desired dense set of cardinality κ^+.

The idea will be to add λ new subsets to k ($\lambda = \kappa^{++}$ or any desired value for the final 2^κ) preserving supercompactness of κ together with iteration of the length κ^+ of forcings $Q_{\underset{\sim}{D}_i}$ $(i < \kappa^+)$, where $\underset{\sim}{D}_i$'s are picked to increase. Finally we will obtain $\underset{\sim}{D} = \bigcup \underset{\sim}{D}_i$ and force with the Prikry forcing for $\underset{\sim}{D}$. The interpretation of the generic functions f_i's $(i < \kappa^+)$ from each stage of the iteration will form the dense set of cardinality κ^+.

Let us start with a basic fact about names in the Prikry forcing.

Lemma 1.4. *Let D be a normal ultrafilter over κ, $\mathcal{P}_D$ the Prikry forcing with D, τ a $\mathcal{P}_D$-name of a partial function of cardinality $< \mu$ ($\mu < \kappa$) from κ to 2. Then there are A and f satisfying the conditions* (a), (c) *of Theorem* 1.1 *so that $\langle \phi, A\rangle \Vdash \underset{\sim}{\tau} = \bigcup_{n<\omega} f((\underset{\sim}{\kappa}_0, \underset{\sim}{\kappa}_1, \ldots, \underset{\sim}{\kappa}_n))$ where $\langle \underset{\sim}{\kappa}_n \mid n < \omega\rangle$ is the canonical name of the Prikry sequence. Also $|f(\eta)| < \mu$ for each $\eta \in [A]^{<\omega}$.*

Proof. Using normality and the Prikry property, we pick $A \in D$ and $\langle a_\eta \mid \eta \in [A]^{<\omega}\rangle$, $|a_\eta| < \mu$ $(\eta \in [A]^{<\omega})$ such that for every $\eta \in [A]^{<\omega}$, $\langle \eta, A \setminus \max \eta\rangle \Vdash \underset{\sim}{\tau} \cap$ ($\max \eta$, the first element of the Prikry sequence above η) $= a_\eta$. Thus, let $\langle \nu, B\rangle \in \mathcal{P}_D$. Shrinking if necessary we can assume that B consists of inaccessible cardinals. Let $\xi \in B$. Applying the Prikry condition, we can find $B_\xi \subseteq B$ in D such that $\langle \nu^\frown\langle\xi\rangle, B_\xi\rangle$ decides $\underset{\sim}{\tau} \cap (\max \nu, \xi)$. Take B_ν to be the diagonal intersection of B_ξ's. Then $B_\nu \in D$ and $\langle \nu, B_\nu\rangle$ decides $\underset{\sim}{\tau} \cap$ ($\max \nu$, the first element of the Prikry sequence above ν). With the above in mind it is easy to start from the weakest condition $\langle \emptyset, \kappa\rangle$ and climb up to define A.

Define $f(\eta) = a_\eta$ for $\eta \in [A]^{<\omega}$. Then, clearly

$$\langle \phi, A\rangle \Vdash \underset{\sim}{\tau} = \bigcup_{n<\omega} f\big(\underset{\sim}{\kappa}_0, \ldots, \underset{\sim}{\kappa}_n\big). \qquad \square$$

Let $G \subseteq Q_D$. We define $A_D = \bigcap\{A \in D \mid \text{for some } \alpha,\ f,\ \langle f, \alpha, A\rangle \in G\}$ and $f_{D,\mu}$ will be a function with domain $[A_D]^{<\omega}$ so that for every $\eta \in [A_D]^{<\omega}$, $f_D(\eta) = \bigcup\{f(\eta) \mid \text{for some } \alpha,\ A,\ \langle f, \alpha, A\rangle \in G\}$.

Let $\underset{\sim}{A}_D, \underset{\sim}{f}_D$, be a canonical name of A_D, f_D. Let $\mathcal{P}_D$ denote the Prikry forcing with D. The following lemma is crucial.

Lemma 1.5. *Suppose that D is a normal ultrafilter over κ and $\underset{\sim}{\tau}$ is a $\mathcal{P}_D$-name of a partial function of cardinality $< \mu$ (for some $\mu < \kappa$) from κ to 2.*

Suppose that $\langle \phi, 0, \kappa\rangle \Vdash_{Q_D}$ "there is a normal ultrafilter $\underset{\sim}{D}_1$, over κ with $\underset{\sim}{A}_D \in \underset{\sim}{D}_1$". Then there is a generic $G \subseteq Q_D$, so that if D_1 is a normal ultrafilter in $V[G]$ with $A_G \in D_1$, then, in $V[G]$

$$\langle \phi, A_G\rangle \underset{\mathcal{P}_{D_1}}{\Vdash} \underset{\sim}{\tau} \subseteq \bigcup_{n<\omega} f_G\big(\langle \kappa_0, \ldots, \kappa_n\rangle\big).$$

Proof. Applying Lemma 1.4 to D, τ in V we pick A, f as in the conclusion of the lemma. Now let $G \subseteq Q_D$ be generic with $\langle f, 0, A\rangle \in G$. Then $A_G \subseteq A$ and for every $\eta \in [A_G]^{<\omega}$, $f(\eta) \subseteq f_G(\eta)$, by Definition 1.2. But since

$$\langle \phi, A\rangle \underset{\mathcal{P}_D}{\Vdash} \underset{\sim}{\tau} = \bigcup_{n<\omega} f\big(\langle \underset{\sim}{\kappa}_0, \ldots, \underset{\sim}{\kappa}_n\rangle\big)$$

and $D \subseteq D_1$ we are done. $\square$

Now the plan will be as follows: We will blow up the power of κ to some cardinal of cofinality κ^+ using $<\kappa$-support iteration of forcings of the type $Q_{\underset{\sim}{D}}$. Using hugeness, a sequence

$$D_0 \subseteq \underset{\sim}{D}_1 \subseteq \underset{\sim}{D}_2 \subseteq \cdots \subseteq \underset{\sim}{D}_\alpha \subseteq \cdots \quad (\alpha < \kappa^+)$$

will be generated and $Q_{\underset{\sim}{D}_\alpha}$'s will be used cofinally. The final step will be to use the Prikry forcing with $\bigcup_{\alpha<\kappa^+} \underset{\sim}{D}_\alpha$.

Let us observe first that the forcing Q_D is quite nice.

Lemma 1.6. *Q_D is $< \kappa$-directed complete.*

Proof. It is obvious from the definition. □

Definition 1.7 (Shelah [12]). Let P be a forcing notion. P satisfies a "stationary" κ^+-c.c. iff for every $\langle p_i \mid i < \kappa^+\rangle$ in $\mathcal{P}$ there is a closed unbounded set $C \subseteq \kappa^+$ and a regressive function $f:\kappa^+ \to \kappa^+$ such that for $\alpha, \beta \in C$ if $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \kappa$ and $f(\alpha) = f(\beta)$ then p_α and p_β are compatible.

Lemma 1.8. *Q_D satisfies the "stationary" κ^+-c.c.*

Proof. Let $p_i = \langle f_i, \alpha_i, A_i\rangle$ $(i < \kappa^+)$ be conditions in Q_D.

For every $\sigma, \alpha < \kappa$, $a \subseteq \alpha$, g a function on $[a]^{<\omega}$ we set

$$A_{\sigma,\alpha,a,g} = \{ i < \kappa^+ \mid \sigma = \sup \{ \mathrm{dom}\, f_i(\eta) \mid \eta \in [A_i]^{<\omega} \}, \\ \alpha = \alpha_i,\ A_i \cap \alpha_i = a \text{ and } f_i \restriction [a]^{<\omega} = g \}.$$

Then κ^+ is a disjoint union of these κ sets.

It is enough to prove the following claim:

Claim. *For every σ, α, a, g as above among any $(2^{|\sigma|+|\alpha|})^+$ members of $A_{\sigma,\alpha,a,g}$ at least two are compatible.*

Let us first complete the proof of the lemma using the claim and then we prove the claim.

Denote $A_{\sigma,\alpha,a,g}$ by A. Assume that $\{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \kappa\} \cap A$ is stationary. Clearly, there are σ, α, a, g for which this is true. Let $\delta \in A$, $\mathrm{cf}(\delta) = \kappa$. We define by induction on ε an increasing sequence of ordinals $\alpha_{\delta,\varepsilon} < \delta$ in A such that $p_{\alpha_{\delta,\varepsilon}}$ is incompatible with p_δ and with $p_{\alpha_{\delta,\rho}}$ for $\rho < \varepsilon$. At stage ε just pick $\alpha < \delta$, $\alpha \in A$ such that p_α is incompatible with p_δ and every $p_{\alpha_{\delta,\rho}} (\rho < \varepsilon)$ if there is such an α. Otherwise we stop. Let $\langle \alpha_{\delta,\varepsilon} \mid \varepsilon < \tau_\delta\rangle$ be such a sequence. Then, by the claim, $\tau_\delta < (2^{|\alpha|+|\sigma|})^+ < \kappa$. Hence, if we take a regressive function $g(\delta) =$ a code of $\langle \alpha_{\delta,\varepsilon} \mid \varepsilon < \tau_\delta\rangle$, then whenever $g(\delta_1) = g(\delta_2)$, p_{δ_1}, p_{δ_2} will be compatible. So, we obtain a "stationary" κ^+-c.c. □

Proof of Claim. Let $\langle i_\xi \mid \xi < (2^{|\sigma|+|\alpha|})^+\rangle$ be a sequence from A. Set

$$B_0 = \bigcap_{\xi < 2^{|\sigma|+|\alpha|}} A_{i_\xi}.$$

Then $B_0 \in D$. There is $B_1 \subseteq B_0$, $B_1 \in D$, such that the isomorphism types of structures

$$\langle \alpha, \rho, a, \langle \mathrm{dom}\, f_{i_\xi}(\nu^\frown \rho \restriction \ell) \mid \xi < (2^{|\sigma|+|\alpha|})^+,\ \nu \in [a]^{<\omega},\ \ell \leqslant \text{length}(\rho)\rangle, \leqslant \rangle$$

depends only on the length of ρ for $\rho \in [B_1]^{<\omega}$. Choose $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n < \cdots$ $(n < \omega)$ an ω-sequence of elements of B_1. Now using the Erdös–Rado theorem it is easy to find $\xi_0 < \xi_1 < (2^{|\sigma|+|\alpha|})^+$ such that for every $\rho \in [a \cup \{\varepsilon_\ell \mid \ell < \omega\}]^{<\omega}$, $f_{i_{\xi_0}}(\rho)$ and $f_{i_{\xi_1}}(\rho)$ are compatible. But then $f_{i_{\xi_0}}(\rho)$ and $f_{i_{\xi_1}}(\rho)$ will be compatible for every $\rho \in [B_1]^{<\omega}$. Which implies a compatibility of $p_{i_{\xi_0}}$ and $p_{i_{\xi_1}}$. □

Let now κ be an almost huge cardinal with a measurable target point, i.e., there is $j : V \to M$, critical$(j) = \kappa$, $j(\kappa) = \lambda$, ${}^{\lambda>}M \subseteq M$ and λ is a measurable cardinal in V. Fix such an embedding $j : V \to M$, $j(\kappa) = \lambda$ and a normal measure $\mathcal{U}_\lambda$ over λ.

We define an iteration $\langle P_\alpha, Q_{0\alpha} * Q_{1\alpha} \mid \alpha < \kappa \rangle$ as follows: if α is not measurable in V^{P_α} then $Q_{0\alpha} * Q_{1\alpha} = \emptyset$; if α is a measurable cardinal in V^{P_α}, then $Q_{0\alpha}$ will be an atomic forcing picking an ordinal $F(\alpha) < \kappa$ and $Q_{1\alpha}$ will be a $<\alpha$-support iteration of maximal possible length $\leqslant F(\alpha)$ of forcings of the form $Q_{\underset{\sim}{D}}$ over all normal ultrafilters $\underset{\sim}{D}$ over α. I.e., first over α we force with a $<\alpha$-supported product of forcings Q_D where $\underset{\sim}{D}$ runs over all normal ultrafilters over α. If α remains measurable after this forcing, then again we force with Q_D's for each normal ultrafilter D of this extension and so on as far as possible up to $F(\alpha)$. Easton support is used at limit stages of the iteration. By Shelah [12] and Lemmas 1.8 and 1.6, $Q_{1\alpha}$ satisfies α^+-c.c. and is α-directed closed over $V^{\mathcal{P}_\alpha * Q_{0\alpha}}$, for $\alpha < \kappa$.

The role of the trivial forcing $Q_{0\alpha}$ is to bound the length of the iteration of $Q_{1\alpha}$. It is needed, since, for example, if α is a supercompact and α-directed closed indestructible, then forcings Q_D will preserve its supercompactness and hence also the measurability. So new ultrafilters will appear over α forever.

Let us work now over κ. Let $G_\kappa \subseteq P_\kappa$ be generic. We consider in M $j(P_\kappa) = P_{j(\kappa)}$ and $P_{j(\kappa)}/G_\kappa$ in $M[G_\kappa]$. Let us split $P_{j(\kappa)}/G_\kappa$ into $Q_{0\kappa} * Q_{1\kappa}$ and $P_{>\kappa}$. The generic object for $Q_{0\kappa}$ is just any ordinal $F(\kappa) < j(\kappa) = \lambda$. We want that κ should remain measurable after forcing with $P_\kappa * Q_{0,\kappa} * Q$ where Q is an initial segment of the desired $Q_{1,\kappa}$. As is standard the method to show this is to show that some elementary embedding k can be extended to a k^* in $V[G_\kappa * \{\delta\} * G]$. By standard techniques this will be the case so long as the generic extends to one (over $k[V]$) for $k(P_\kappa * Q_{0,\kappa} * Q)$. The key here is that the $k(\kappa)$th term of $P_{k(\kappa)}$ should be an extension of $k''(Q_{0,\kappa} * Q)$. This, of course, is why we defined P_κ as we did. By standard arguments on backwards Easton forcing (see, for example, A. Kanamori [9]), for every $F(\kappa) \leqslant \lambda$ the length of $Q_{1\kappa}$ will be $F(\kappa)$. For a while set $F(\kappa) = \lambda$, i.e., we like to deal with iteration $Q_{1\kappa}$ of the length λ. We consider an enumeration $\langle \underset{\sim}{A}_\tau \mid \tau < \lambda \rangle$ of $Q_{1\kappa}$-names of subsets of κ in $M[G_\kappa][\{\lambda\}]$, such that $\tau_1 < \tau_2 < \lambda$ implies that $\underset{\sim}{A}_{\tau_1}$ depends on the part of $Q_{1\kappa}$ of the length at most those of $\underset{\sim}{A}_{\tau_2}$. Since λ is measurable and $Q_{1\kappa}$ has $<\kappa$-support there will be $C \in \mathcal{U}_\lambda$ consisting of inaccessibles such that for every $\delta \in C$, $\langle \underset{\sim}{A}_\tau \mid \tau < \delta \rangle$ enumerates the names of all subsets of κ appearing before the stage δ, i.e., $Q_{1\kappa} \restriction \delta$-names. Equivalently, all the subsets for the $Q_{1\kappa}$ with $F(\kappa) = \delta$. Now let δ be in C. For every $\underset{\sim}{D}$ appearing in $Q_{1\kappa} \restriction \delta$ let $r_{\underset{\sim}{D}} \in Q_{j(\underset{\sim}{D})}$ be defined as follows:

$$r_{\underset{\sim}{D}} = \langle \underset{\sim}{f}, \kappa, \underset{\sim}{A} \rangle,$$

where

$$\underset{\sim}{A} = A_{\underset{\sim}{D}} \cup \Big(\bigcap \{ \underset{\sim}{j}(\underset{\sim}{B}) \mid \underset{\sim}{B} \in \underset{\sim}{D} \} \Big),$$

$\underset{\sim}{f} \restriction \kappa = f_{\underset{\sim}{D}}$ and above κ we take

$$\underset{\sim}{f}(\eta) = \bigcup \big\{ j(\underset{\sim}{f})(\eta) \mid \underset{\sim}{f} \text{ appear in a condition in } G_{\underset{\sim}{D}} \big\}.$$

Let $q_\delta \in Q_{1j(\kappa)}$ consists of these $r_{\underset{\sim}{D}}$'s sitting in the right place.

Clearly, if $\rho > \delta$ is also in C, then $q_\rho \restriction \delta = q_\delta$.

Let ρ be in C. Pick a master condition $p_\rho \in P_{>\kappa} * (j(\rho) * Q_{1j(\kappa)})$ deciding all the statements "$\kappa \in j(\underset{\sim}{A}_\tau)$" for $\tau < \rho$ and stronger than q_ρ, i.e., p_ρ satisfies the following: for every $\tau < \rho$ there is $s \in G_\kappa * \{\rho\} * Q_{1\kappa} \restriction \rho$ so that $\langle s, p_\rho\rangle \parallel \kappa \in j(\underset{\sim}{A}_\tau)$.

Shrink the set C to a set $C^* \in \mathcal{U}_\lambda$ so that for any two $\rho_1 \leqslant \rho_2 \in C^*$ decisions are the same, i.e., for every $s, \tau < \rho_1$ as above

$$\langle s, p_{\rho_1}\rangle \parallel \kappa \in j(\underset{\sim}{A}_\tau)$$

iff

$$\langle s, p_{\rho_2}\rangle \parallel \kappa \in j(\underset{\sim}{A}_\tau),$$

and

$$\langle s, p_{\rho_1}\rangle \Vdash \kappa \in j(\underset{\sim}{A}_\tau)$$

iff

$$\langle s, p_{\rho_2}\rangle \Vdash \kappa \in j(\underset{\sim}{A}_\tau).$$

For every $\rho \in C^*$ we define in $V[G_\kappa * \{\rho\} * G(Q_{1\kappa})]$ a normal ultrafilter $D(\rho)$ over κ, where $G(Q_{1\kappa}) \subseteq Q_{1\kappa}$ is generic and $Q_{1\kappa}$ has length ρ. Let us set $A \in D(\rho)$ iff for some $s \in G_\kappa * \{\rho\} * G(Q_{1\kappa})$, $\langle s, p_\rho\rangle \Vdash \kappa \in j(\underset{\sim}{A}_\tau)$ where the interpretation of $\underset{\sim}{A}_\tau$ is A for $\tau < \rho$.

Suppose now that $\rho < \rho_1$ are two elements of C^*. Work in $V[G_\kappa * \{\rho_1\} * G(Q_{1\kappa})]$. Then, clearly, $G(Q_{1\kappa}) \restriction \rho$ will be $V[G_\kappa * \{\rho\}]$ generic for $Q_{1\kappa}$ (or $Q_{1\kappa} \restriction \rho$ in the sense of the iteration to ρ_1). So $D(\rho) \in V[G_\kappa * \{\rho_1\} * G(Q_{1\kappa})]$.

Claim. $D(\rho) \subseteq D(\rho_1)$.

Proof. Let $A \in D(\rho)$. Pick $\tau < \rho$, $\underset{\sim}{A}_\tau$ and s to be as in the definition of $D(\rho)$. By the choice of C^*, then $\langle s, p_{\rho_1}\rangle \Vdash \kappa \in j(\underset{\sim}{A}_\tau)$. So, $i_{G(Q_{1\kappa})}(\underset{\sim}{A}_\tau) = i_{G(Q_{1\kappa}\restriction\rho)}(\underset{\sim}{A}_\tau) = A$ is in $D(\rho_1)$ as well as in $D(\rho)$, where i_G is the function interpreting names. □

Now we are about to complete the construction. Thus, let δ be a limit of an increasing sequence $\langle \rho_i \mid i < \kappa^+\rangle$ of elements of C^*. We consider $V[G_\kappa * \{\delta\}]$, i.e., the iteration $Q_{1\kappa}$ will be of the length δ. By the claim,

$$D(\rho_0) \subseteq D(\rho_1) \subseteq \cdots \subseteq D(\rho_i) \subseteq \cdots \quad (i < \kappa^+).$$

For every $i < \kappa^+$, $D(\rho_i)$ is a normal ultrafilter over κ in $V[G_\kappa * \{\delta\} * G(Q_{1\kappa}) \restriction \rho_i]$. Hence, the forcing $Q_{D(\rho_i)}$ was used at the stage $\rho_i + 1$. Finally set $D = \bigcup_{i<\kappa^+} D(\rho_i)$.

Lemma 1.9. *In $V[G_\kappa * \{\delta\} * G(Q_{1\kappa})]$ D is a normal ultrafilter over κ generated by κ^+ sets and $2^\kappa = \delta > \kappa^+$.*

Proof. $2^\kappa = \delta$ since at each stage of the iteration $Q_{1\kappa}$ a new subset of κ is produced and δ is a limit of inaccessibles of cofinality κ^+.

Notice that D is a normal ultrafilter over κ since it is an increasing union of κ^+ normal ultrafilters $D(\rho_i)$ ($D(\rho_i)$ is such in $V[G_\kappa * \{\delta\} * Q_{1\kappa} \restriction \rho_i]$) and $Q_{1\kappa}$ satisfies κ^+-c.c. It is κ^+-generated since for every $i < \kappa^+$ a set $A_{D(\rho_i)}$ generating $D(\rho_i)$ is added at stage $\rho_i + 1$. $\square$

Let $\langle f_{D(\rho_i)} \mid i < \kappa^+\rangle$ be the generic functions added by $Q_{D(\rho_i)}$'s. Use the Prikry forcing with D. Let $\langle \kappa_n \mid n < \omega\rangle$ be the Prikry sequence. Then by Lemma 1.5 we obtain the following:

Theorem 1.10. *The following holds in the model* $V[G_\kappa * \{\delta\} * G(Q_{1\kappa}) * \langle \kappa_n \mid n < \omega\rangle]$

(a) *κ is a strong limit cardinal of cofinality ω;*

(b) $2^\kappa = \delta > \kappa^+$;

(c) *the functions $\langle f_{D(\rho_i)} \mid i < \kappa^+\rangle$ are witnessing $d_{<\aleph_1}(\kappa) = \kappa^+$.*

Remarks.

(1) If one likes to have $2^\kappa = \kappa^{+7}$ then just collapse δ to κ^{+6} using the Levy collapse. No new subset of κ will be added. So $d_{<\aleph_1}$ will still be κ^+.

(2) κ^+ as the density can be replaced by κ^{++}, κ^{+7}, etc. Just pick a longer sequence of ρ_i's and argue that no smaller family is dense. It requires simple arguments about names in the Prikry forcing.

(3) $\aleph_1$ can be replaced by any regular $\theta < \kappa$.

2. The basic construction

In this section we will show how to apply [6] in order to produce a model with a strong limit κ, $\mathrm{cf}(\kappa) = \aleph_0$, $2^\kappa = \kappa^{++}$ and $d_{<\aleph_1}(\kappa) = \kappa^+$.

The idea will be that we can reflect the situation above κ in the ground model below κ once changing its cofinality.

Theorem 2.1. *Suppose that $V \Vdash$ GCH and there exists an elementary embedding $j: V \to M$ with a critical point κ such that*

(a) $M \supseteq V_{\kappa+2}$;

(b) $M = \{j(f)(\delta_1, \ldots, \delta_n) \mid n < \omega,\ \delta_1 < \cdots < \delta_n < \kappa^{++}$ *and* $f: [\kappa]^n \to V\}$;

(c) ${}^\kappa M \subseteq M$.

Then there is a cardinal preserving extension $V[G]$ of V so that

(1) *for every $\alpha < \kappa$ or $\alpha > \kappa$ $2^\alpha = \alpha^+$;*

(2) $2^\kappa = \kappa^{++}$;

(3) $\mathrm{cf}(\kappa) = \aleph_0$;

(4) $d_{<\aleph_1}(\kappa) = \kappa^+$.

Remark 2.2.

(1) The assumption used in Theorem 2.1 is actually the $\mathcal{P}^2(\kappa)$-hypermeasurability of κ or in the Mitchell order $o(\kappa) = \kappa^{++} + 1$.

(2) Let us determine the cardinality of $j(\kappa)$ in order to get a feeling of the matter. First, $j(\kappa) > \kappa^{++}$, since, in M, $j(\kappa)$ is inaccessible and by (a) $(\kappa^{++})^M = \kappa^{++}$. Second, $j(\kappa) = j(c_\kappa)(0) = c_{j(\kappa)}(0)$, where $c_\kappa : \kappa \to \kappa + 1$ is the constant function with the value κ. Then every ordinal below $j(\kappa)$ is represented by $j(f)(\delta_1, \ldots, \delta_n)$ for some $f : [\kappa]^n \to \kappa$ and some $\delta_1 < \cdots < \delta_n < \kappa^{++}$. But V satisfies GCH, so the number of possibilities is κ^{++}. Hence, $\kappa^{++} < j(\kappa) < \kappa^{+++}$. By (c) it follows that $\mathrm{cf}(j(\kappa)) \geqslant \kappa^+$. Using further considerations it is not hard to see that $\mathrm{cf}(j(\kappa))$ is κ^+.

Proof of Theorem 2.1. Let $\mathcal{U}_0 = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Then $\mathcal{U}_0$ is a normal ultrafilter over κ. Let $i : V \to N \simeq \mathrm{Ult}(V, \mathcal{U}_0)$ be the corresponding elementary embedding. Then the following diagram is commutative.

$$\begin{tikzcd} & M \\ V \arrow[ur, "j"] \arrow[dr, "i"'] & \\ & N \arrow[uu, "k"'] \end{tikzcd}$$

where $k(i(f)(\kappa)) = j(f)(\kappa)$.

Since, if $a \in V$ then $i(a) = i(c_a)(\kappa)$, $k(i(a)) = k(i(c_a)(\kappa)) = j(c_a)(\kappa) = a$, where c_a is the constant function with the value a.

The critical point of k is $(\kappa^{++})^N$. Thus, $k(\kappa) = k(i(\mathrm{id})(\kappa)) = j(\mathrm{id})(\kappa) = \mathrm{id}(\kappa) = \kappa$, ${}^\kappa N \subseteq N$ and $2^\kappa = \kappa^+$ imply $k(\kappa^+) = \kappa^+$. But $(\kappa^{++})^N < \kappa^{++} = (\kappa^{++})^M$, since $2^{\kappa^+} = \kappa^{++}$ and $\mathcal{U}_0 \notin N$.

Lemma 2.3. *There is a sequence $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ so that*

(i) $j(\kappa) = \bigcup_{\alpha < \kappa^+} A_\alpha$ *and for every* $\alpha < \kappa^+$,

(ii) $A_\alpha \in M$,

(iii) $|A_\alpha| \leqslant \kappa^{++}$, *and*

(iv) $A_\alpha \in \mathrm{rng}\, k$.

Proof. For every $\tau < j(\kappa)$ there are $\delta_1, \ldots, \delta_n < \kappa^{++}$ and $f : [\kappa]^n \to \kappa$ such that $j(f)(\delta_1, \ldots, \delta_n) = \tau$. Consider a function $f' : \kappa \to [\kappa]^{<\kappa}$ defined as follows:

$$f'(\nu) = \{f(\nu_1, \ldots, \nu_n) \mid \nu_1, \ldots, \nu_n < \nu^{++}\}.$$

Then, in M, $|j(f')(\kappa)| \leqslant \kappa^{++}$ and $\tau \in j(f')(\kappa)$. Clearly, $k(i(f'))(\kappa) = j(f')(\kappa)$. Hence $j(f')(\kappa) \in \mathrm{rng}\, k$.

So,

$$j(\kappa) = \bigcup \{j(f')(\kappa) \mid f' : \kappa \to [\kappa]^{<\kappa} \text{ and for every } \nu < \kappa,\ |f'(\nu)| \leqslant \nu^{++}\}.$$

Since the number of such f' is κ^+, we are done. $\square$

Lemma 2.4. *There exists a dense set F of cardinality κ^+ in the topological space ${}^{j(\kappa)}\kappa$ with the topology generated by $< \kappa^+$ products such that every element of F belongs to* $\mathrm{rng}\, k$ *and in particular also to M.*

Proof. Let a sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ be given by Lemma 2.3. Assume also that it is increasing. For every $\alpha < \kappa^+$ there is $A^*_\alpha \in N$, such that $k(A^*_\alpha) = A_\alpha$ and $N \models |A^*_\alpha| = \kappa^{++}$. Working in N and using GCH, we pick a dense subset F^*_α, with $|F^*_\alpha| = \kappa^+$ of the topological space ${}^{A^*_\alpha}\kappa$ with the topology generated by countable products. Then let $F_\alpha = k(F^*_\alpha)$ and $F = \bigcup_{\alpha<\kappa^+} F_\alpha$. Notice, that $|F_\alpha| = \kappa^+$ in both M and V, since $\mathrm{crit}(k) > \kappa^+$. Clearly, F is as required. □

The family F of Lemma 2.4 will be used to generate a dense set in the space ${}^\kappa 2$ with countable product topology once the cofinality of κ is changed to ω and its power is blown up to κ^{++}. Thus, if $\langle \kappa_n \mid n < \omega\rangle$ is the Prikry sequence for the normal measure of the extender, i.e., for $\mathcal{U}_0$ and $f = k(f^*) \in F$, then let f^{**} be a function such that $(i)(f^{**})(\kappa) = f^*$. The dense set will consist of functions $\bigcup_{n<\omega} f^{**}(\kappa_n) \restriction (\kappa_{n+1} \setminus \kappa_n)$.

Now, in order to show that this works, we need to deal with names of clopen sets in ${}^\kappa 2$ in the forcing of [6]. Finite iterations described below are needed for a nice representation of such names.

The model M is the ultrapower of V by the extender $E = \langle E_a \mid a \in [\kappa^{++}]^{<\omega}\rangle$, where

$$X \in E_a \quad \text{iff} \quad a \in j(X).$$

Going the other direction, an embedding π can be generated from the extender E and a model M which is to be the domain of π:

$$\pi : M \to \mathrm{Ult}(M, E) = \bigl\{[a, f]\colon\ a \in [\kappa^{++}]^{<\omega} \text{ and } f \in M \text{ and } f : {}^a\kappa \to M\bigr\},$$

where $[a, f] = [a', f']$ if and only if $\{\sigma \in {}^{a\cup a'}\kappa\colon\ f(\sigma \restriction a) = f'(\sigma \restriction a')\} \in E_{a\cup a'}$.

This will define an embedding on M provided that E_a is an ultrafilter on at least the subsets of $[\kappa]^{|a|}$ which are in M.

Now, $j(E) =_{\mathrm{df}} E_1 \in M =_{\mathrm{df}} M_1$ and it is an extender over $j(\kappa^{++})$. Using E_1 we obtain $j'_1 : M_1 \to M_2 \simeq \mathrm{Ult}(M_1, E_1)$ with a critical point $j(\kappa) =_{\mathrm{df}} \kappa_1$. Let $j'_0 = j$, $V = M_0$ and $\kappa = \kappa_0$. In the same fashion we can use $j'_1(E_1) =_{\mathrm{df}} E_2$ over M_2 and form $j'_2 : M_2 \to M_3 \simeq \mathrm{Ult}(M_2, E_2)$ with a critical point $j'_1(\kappa_1) =_{\mathrm{df}} \kappa_2$, and so on. Thus, for $n < \omega$, we will have $j'_n : M_n \to M_{n+1} \simeq \mathrm{Ult}(M_n, E_n)$, $\mathrm{crit}(j_n) = \kappa_n$. Let $j_n : V \to M_n$, $\mathrm{crit}(j_n) = \kappa$ be the composition of $j'_0, j'_1, \ldots, j'_{n-1}$. Also set $j_0 = \mathrm{id}$, i.e., the identity map. Another way to obtain M_n's is using finite products of E and their ultrapower. Thus we consider $E^2 = \langle E^2_a \mid a \in [\kappa^{++}]^{<\omega}\rangle$ where for $a \in [\kappa]^m$ $(m < \omega)$ and $X \subseteq [\kappa]^m \times [\kappa]^m$, $X \in E^2_a$ iff

$$\Bigl\{(\alpha_1, \ldots, \alpha_m) \bigm| \bigl\{(\beta_1, \ldots, \beta_m) \bigm| (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) \in X\bigr\} \in E_a\Bigr\} \in E_a.$$

It is not hard to see that $M_2 \simeq \mathrm{ult}(V, E^2)$ and the corresponding embedding is the same as j_{02}. In the same fashion for every n, $0 < n < \omega$, we can reach M_n using only one ultrapower. Thus if $E^n = \langle E^n_a \mid a \in [\kappa^{++}]^{<\omega}\rangle$, then $M_n \simeq \mathrm{Ult}(V, E^n)$. Instead of dealing with finite a's we can replace them just by ordinals using a reasonable coding.

The following lemma is folklore.

Lemma 2.5. *For every $\alpha < j_n(\kappa) = \kappa_n$ there are $f_\alpha : [\kappa]^n \to \kappa$ and $\delta < \kappa^{++}$ such that $\alpha = j_n(f_\alpha)(\delta, j_1(\delta), j_2(\delta), \ldots, j_{n-1}(\delta))$.*

Let us sketch the proof. We deal only with the case $n = 2$. So let $\alpha < j_2(\kappa) = \kappa_2$. Then $\alpha < j_1'(\kappa_1)$ and it is an element of $M_2 \simeq \mathrm{Ult}(M_1, E_1)$. Hence, in M_1, for some $\delta < \kappa_1^{++}$ and $f : \kappa_1 \to \kappa_1$, $j_1'(f)(\delta) = \alpha$. Also, $M_1 = M \simeq \mathrm{Ult}(V, E)$ and $E_1 = j(E)$. First we like to show that it is possible to choose δ of the form $j(\tilde{\delta})$ for some $\tilde{\delta} < \kappa^{++}$. Thus, there are $g : \kappa \to \kappa^{++}$ and $\xi < \kappa^{++}$ so that $j(g)(\xi) = \delta$. Let $\tilde{\delta} = \sup(\mathrm{rng}\, g)$. Denote $E_{\langle\gamma\rangle}$ by $\mathcal{U}_\gamma$ for $\gamma < \kappa^{++}$. Then $\mathcal{U}_{\tilde{\delta}}$ is above $\mathcal{U}_\gamma$ in the Rudin–Keisler ordering for every $\gamma < \tilde{\delta}$, i.e., $\mathcal{U}_{\tilde{\delta}}$ projects onto $\mathcal{U}_\gamma$. It is easy to see this considering the following commutative diagram:

$$\begin{tikzcd}
 & M \\
V \arrow[ur, "j"] \arrow[dr, "i_{\tilde{\delta}}"'] & \\
 & N_{\tilde{\delta}} \simeq \mathrm{Ult}(V, U_{\tilde{\delta}}) \arrow[uu, "k_{\tilde{\delta}}"']
\end{tikzcd}$$

where $k_{\tilde{\delta}}(i_{\tilde{\delta}}(t)([\mathrm{id}]_{\mathcal{U}_{\tilde{\delta}}})) = j(t)(\tilde{\delta})$. As in the beginning of the proof of Theorem 2.1, the critical point of $k_{\tilde{\delta}}$ is $(\kappa^{++})^{N_{\tilde{\delta}}}$. But $k_{\tilde{\delta}}([\mathrm{id}]_{\mathcal{U}_{\tilde{\delta}}}) = \tilde{\delta} < \kappa^{++}$. So, $[\mathrm{id}]_{\mathcal{U}_{\tilde{\delta}}} = \tilde{\delta}$. Then for every $\gamma < \tilde{\delta}$, function representing γ in $\mathrm{Ult}(V, \mathcal{U}_{\tilde{\delta}})$ will project $\mathcal{U}_{\tilde{\delta}}$ onto $\mathcal{U}_\gamma$.

In particular, for every $\gamma \in \mathrm{rng}\, g$, $\mathcal{U}_\gamma \leqslant_{RK} \mathcal{U}_{\tilde{\delta}}$ (i.e., $\leqslant$ in the Rudin–Keisler order). Now in M_1, $\mathcal{U}_{j(\tilde{\delta})} \geqslant_{RK} \mathcal{U}_\delta$ since $\delta \in \mathrm{rng}\, j(g)$. Hence for some $\widetilde{f} : \kappa_1 \to \kappa_1$ in M_1, $j_1'(\widetilde{f})(j(\tilde{\delta})) = \alpha$. Find some $h : \kappa \to \kappa_\kappa$ and ρ, $\tilde{\delta} \leqslant \rho < \kappa^{++}$ such that $\widetilde{f} = j(h)(\rho)$. Then $(j_1'(j(h)(\rho)))(j(\tilde{\delta})) = \alpha$. Replacing h by a two-place function $g : [\kappa]^2 \to \kappa$ we obtain $j_2(g)(\rho, j(\tilde{\delta})) = \alpha$. Since $\rho \geqslant \tilde{\delta}$, $\mathcal{U}_\rho \geqslant_{RK} U_{\tilde{\delta}}$. Hence $\mathcal{U}_\rho \times \mathcal{U}_\rho \geqslant_{RK} \mathcal{U}_\rho \times \mathcal{U}_{\tilde{\delta}}$. It means that for some $g' : \kappa \times \kappa \to \kappa$, $j_2(g')(\rho, j(\rho)) = \alpha$.

Now fix n, $1 < n < \omega$. We would like to describe one more way of constructing M_n. Thus, we consider E^{n-1} and M_1. E^{n-2} and even E is not in M_1 but we still can measure subsets of κ of M_1 from the outside. So we can form $\mathrm{Ult}(M_1, E^{n-1})$. Since $V_{\kappa+2} \subseteq M_1$ and ${}^\kappa M_1 \subseteq M_1$, it is routine to check that $\mathrm{Ult}(M_1, E^{n-1}) \simeq M_n$. Let ℓ be the corresponding embedding. Then $\ell(\kappa) = \kappa_{n-1}$, $\ell(\kappa_1) = \kappa_n$.

Lemma 2.6. *For every $\alpha < \kappa_n$ there are $g_\alpha : [\kappa]^{n-1} \to \kappa_1$ and $\delta < \kappa^{++}$ such that*

$$\alpha = \ell(g_\alpha)\big(\delta, j_1(\delta), \ldots, j_{n-2}(\delta)\big).$$

Proof. Let g_α be a function representing α in the ultrapower by E^{n-1}, i.e., for some $\delta < \kappa^{++}$, $j_{n-1}(g_\alpha)(\delta, j_1(\delta), \ldots, j_{n-2}(\delta)) = \alpha$. Then $g_\alpha : [\kappa]^{n-1} \to \kappa_1$, since $\alpha < \kappa_n$ and $j_{n-1}(\kappa_1) = \kappa_n$. But then also $\ell(g_\alpha)(\delta, j_1(\delta), \ldots, j_{n-2}(\delta)) = \alpha$, since ${}^\kappa M_1 = {}^\kappa V$. $\square$

Further let us add to such ℓ the subscript n.

Let F be the family given by Lemma 2.4. We define $F_n = \ell_n''(F)$ for every n, $0 < n < \omega$. Let $\widetilde{F}_{nk} = \{f \restriction [\kappa_{k-1}, \kappa_k) \mid f \in F_n\}$ for every k, $0 < k \leqslant n$. For n, $0 < n < \omega$

and $t \in \prod_{k=1}^{n} \widetilde{F}_{nk}$ $\bigcup \operatorname{rng} t$ is a partial function from κ_n to 2 and it belongs to M_n as a finite union of its elements. Set

$$F_n^* = \left\{ \bigcup t \mid \text{for some } m,\ 0 < m \leqslant n,\ t \in \prod_{k=1}^{m} \widetilde{F}_{mk} \right\}.$$

Lemma 2.7. *For every n, $1 < n < \omega$, F_n^* is dense in the topological space ${}^{\kappa_n}2$ with countable product topology.*

Proof. Let $\langle \alpha_m \mid m < \omega \rangle$ be an ω-sequence of ordinals below κ_n, for some n, $1 < n < \omega$. Let $\varphi \in {}^{\{\alpha_m \mid m<\omega\}}2$. By the definition of F_n^* it is enough to prove the lemma in the situation when all α_m's are in some fixed interval $[\kappa_{k-1}, \kappa_k)$ for $0 < k \leqslant n$. Also by Lemma 2.4, we can assume that $k > 1$. Since nothing happens between κ_k and κ_n, we can assume that $k = n$. For every $m < \omega$, by Lemma 2.6 there are $g_m : [\kappa]^{n-1} \to \kappa_1$ and $\delta_m < \kappa^{++}$ such that $\alpha_m = \ell_n(g_m)(\delta_m, j_1(\delta_m), \ldots, j_{n-2}(\delta_m))$. We can code the sequence $\langle \delta_m \mid m < \omega \rangle$ into one $\delta < \kappa^{++}$, since $(\kappa^{++})^{\aleph_0} = \kappa^{++}$ and M is ω-closed. Hence, for every $m < \omega$

$$\alpha_m = \ell_n(g_m)\big(\delta, j_1(\delta), \ldots, j_{n-2}(\delta)\big).$$

Since α_m's are all different, there will be $A \in \mathcal{U}_\delta^{n-1}$ such that for every $m \neq \ell < \omega$ and $\vec{s} \in A$ $g_m(s) \neq g_\ell(s)$, where $\mathcal{U}_\delta$ denotes $E_{\langle\delta\rangle}$ and $\mathcal{U}_\delta^{n-1} = E_{\langle\delta\rangle}^{n-1}$. Let us also show that the ranges of g_m's can be made disjoint. Let us do this for two g_0 and g_1. Using the completeness of $\mathcal{U}_\delta$ it is easy then to get the full result.

Claim 2.8. *There is $B \subseteq A$ in $\mathcal{U}_\delta^{n-1}$ such that* $\operatorname{rng} g_0 \restriction B \cap \operatorname{rng} g_1 \restriction B = \emptyset$.

Remark. It may not be true iff either α_0, α_1 are in different intervals $[\kappa_k, \kappa_{k+1})$ or if a same measure appears in the extender several times.

Proof. In order to simplify the notation, let us assume that $n = 3$. So $\kappa_2 \leqslant \alpha_0 < \alpha_1 < \kappa_3$. Recall that $\ell_3(\kappa) = \kappa_2$ and $\ell_3(\kappa_1) = \kappa_3$. So, for almost all $(\operatorname{mod} \mathcal{U}_\delta^2)(\beta, \gamma) \in [\kappa]^2$, $\kappa \leqslant g_0(\beta, \gamma) < g_1(\beta, \gamma) < \kappa_1$. Consider $\rho_i = \inf_{C \in \mathcal{U}_\delta^2}(\sup \operatorname{rng}(g_i \restriction C))$ for $i < 2$. If $\rho_0 \neq \rho_1$, then everything is trivial. Suppose that $\rho_0 = \rho_1 =_{\mathrm{df}} \rho$. Then $\operatorname{cf}(\rho) = \kappa$ by κ-completeness of $\mathcal{U}_\delta$. Notice also that g_0 or g_1 cannot be constant $(\operatorname{mod} \mathcal{U}_\delta^2)$ since then this constant will be ρ. Consider sets $X_0 = (g_0''[\kappa]^2) \cap \rho$ and $X_1 = (g_1''[\kappa]^2) \cap \rho$. We define a κ-complete ultrafilters $\mathcal{W}_0$ and $\mathcal{W}_1$ over X_0 and X_1 as follows:

$$\mathcal{S} \in \mathcal{W}_k \quad \text{iff} \quad g_k^{-1''}\mathcal{S} \in \mathcal{U}_\delta^2, \text{ where } k < 2.$$

Then $\mathcal{W}_0, \mathcal{W}_1 \leqslant_{RK} \mathcal{U}_\delta^2$ (less in the Rudin–Keisler ordering) and g_0, g_1 are the corresponding projection functions. Now, $g_0 \neq g_1 \operatorname{mod} \mathcal{U}_\delta^2$ and the extender E has the length κ^{++}. So, $\mathcal{W}_0 \neq \mathcal{W}_1$ (just use the argument of Lemma 2.5 (or see Mitchell [14]) for similar arguments). Now we pick $B_0 \in \mathcal{W}_0 \setminus \mathcal{W}_1$ and set $B_1 = X_1 \setminus B_0$. The set

$$B = (g_0^{-1''}B_0) \cap (g_1^{-1''}B_1)$$

is as desired. □

So let $B \in \mathcal{U}_\delta^2$ be so that $g''_m B \cap g''_k B = \emptyset$ for every $m \neq k < \omega$. Denote $g''_m B$ by B_m $(m < \omega)$. Consider now the clopen set in ${}^{\kappa_1}2$ with κ-products generated by $\psi \in {}^{\bigcup_{m<\omega} B_m} 2$ where $\psi \restriction B_m$ takes the constant value $\varphi(\alpha_m)$. Now pick $f \in F_n$ $f \supseteq \psi$. Then $\ell_n(f) \supseteq \varphi$, since for every $m < \omega$,

$$\{(\beta, \gamma) \in [\kappa]^2 \mid f(g_m(\beta, \gamma)) = \varphi(\alpha_m)\} \supseteq g_m^{-1''}(B_m) \in \mathcal{U}_\delta^2.$$

Suppose now that the extender E has the length κ^{+7} instead of κ^{++}. We would like to apply previous arguments in order to produce a dense set of cardinality κ^{+6}. We change the assumptions (a) and (b) of Theorem 2.1 as follows:

(a′) $M \supseteq V_{\kappa+7}$;

(b′) $M = \{j(f)(\delta_1, \ldots, \delta_n) \mid n < \omega,\ \delta_1 < \cdots < \delta_n < \kappa^{+7}$ and $f : [\kappa]^n \to V\}$.

In Lemmas 2.3–2.7 and Claim 2.8 we replace κ^{++} by κ^{+7} and the proofs do not change.

The only obstacle is that Lemma 2.7 breaks down if we use the family F_n^* defined there. The problem is that once the length of the extender is $\geqslant \kappa^{+++}$ the same measures are starting to appear in it at different places. It was crucial for Claim 2.8 that this does not happen. The solution is going to be to take a larger family and use the fact that for any two measures in the extenders there is a measure with index $< \kappa^{+6}$ which provides a difference between them inside the extender.

First let us define the new F_n. Let $F'_n = \{t \mid t : [\kappa]^n \to F\}$. Clearly, $|F'_n| = |F|^\kappa = (\kappa^{+6})^\kappa = \kappa^{+6}$.

Also, every t in F'_n is in M since ${}^\kappa M \subseteq M$. Now for every $\delta < \kappa^{+6}$ and $t \in F'_n$ we consider $\ell_n(t)(\delta, j_1(\delta), \ldots, j_{n-2}(\delta))$. It is an element of M_n. Set

$$F_n = \{\ell_n(t)(\delta, j_1(\delta), \ldots, j_{n-2}(\delta)) \mid \delta < \kappa^{+6} \text{ and } t \in F'_n\}.$$

Now, we define F_n^* as in case κ^{++} using this new F_n. We need to show that the analog of Lemma 2.7 is true with our new F_n. The argument of Lemma 2.7 and Claim 2.8 are running smooth until the point where it is claimed that $\mathcal{W}_0 \neq \mathcal{W}_1$.

Suppose now that $\mathcal{W}_0 = \mathcal{W}_1$. Let us assume in order to simplify the presentation that

$$\kappa < \kappa_1 < \alpha_0, \alpha_1 < \kappa_2 = \kappa_n.$$

Thus g_0, g_1 are now one-place functions.

The ultrafilters $\mathcal{W}_0$, $\mathcal{W}_1$ are then isomorphic to some measures $\mathcal{U}_{\tau_0}$, $\mathcal{U}_{\tau_1}$ of extender, where $\tau_0, \tau_1 < \kappa^{+7}$ and for $\tau < \kappa^{+7}$, $\mathcal{U}_\tau = \{S \subseteq \kappa \mid \tau \in j(S)\}$. Just take the bijections ρ_0, ρ_1 between κ and X_0, X_1. The general case is slightly more complicated since we need to deal with E^2, E^3 etc. and instead of $\mathcal{U}_\tau$ it will be $\mathcal{U}_\tau^2$, $\mathcal{U}_\tau^3$, etc.

Let $\tau_0 < \tau_1 < \kappa^{+7}$. The opposite case is identical.

Claim. *There is $\tau < \kappa^{+6}$ such that $E_{\{\tau,\tau_0\}} \neq E_{\{\tau,\tau_1\}}$, where for $a \in [\kappa]^{<\omega}$*

$$E_a = \{s \subseteq \kappa^{|a|} \mid a \in j(s)\}.$$

Proof. Let us assume that $\mathcal{U}_{\tau_0} = E_{\{\tau_0\}} = E_{\{\tau_1\}} = \mathcal{U}_{\tau_1}$, since otherwise everything is trivial. Suppose also that $\tau_0 > \kappa^{+6}$. It will be clear from the argument below that this only needed for simplification of the notation.

Consider the following commutative diagram

$$\begin{tikzcd} & M \\ V \arrow[ur, "j"] \arrow[dr, "i_{\tau_1}"'] & \\ & N_{\tau_1} \simeq \mathrm{Ult}(V,\mathcal{U}_{\tau_1}) \arrow[uu, "k_{\tau_1}"'] \end{tikzcd}$$

where $k_{\tau_1}([f]_{\mathcal{U}_{\tau_1}}) = j(f)(\tau_1)$ for every $f:\kappa \to V$. Let $\widetilde{\tau}_1 = [\mathrm{id}]_{\mathcal{U}_{\tau_1}}$. By elementarity, $(\kappa^{+6})^{N_{\tau_1}} < \widetilde{\tau}_1 < (\kappa^{+7})^{N_{\tau_1}}$. Let $\widetilde{f}:(\kappa^{+6})^{N_{\tau_1}} \leftrightarrow \widetilde{\tau}_1$ be a function in N_{τ_1}. Pick $h:\kappa \to V$ representing $\widetilde{f}$ in N_{τ_1}, i.e., $i_{\tau_1}(h)(\widetilde{\tau}_1) = \widetilde{f}$. Now consider $f = k_{\tau_1}(\widetilde{f})$. Then by elementarity of k_{τ_1}, $f:\kappa^{+6} \leftrightarrow \tau_1$. We pick $\tau < \kappa^{+6}$ to be $f^{-1}(\tau_0)$. Let us show that $E_{\{\tau,\tau_0\}} \neq E_{\{\tau,\tau_0\}}$. Suppose otherwise. Let $\mathcal{U} = E_{\{\tau,\tau_0\}} = E_{\{\tau,\tau_1\}}$. We consider the following commutative diagram:

$$\begin{tikzcd} & M & \\ V \arrow[ur, "j"] \arrow[rr, "i_{\tau\tau_1}"] \arrow[dr, "i_{\tau_1}"'] & & N_{\tau\tau_1} \simeq \mathrm{Ult}(V,\mathcal{U}) \arrow[ul, "k_{\tau\tau_1}"'] \\ & N_{\tau_1} \arrow[uu, "k_{\tau_1}"'] \arrow[ur, "\ell"'] & \end{tikzcd},$$

where $k_{\tau,\tau_1}([f]_{\mathcal{U}}) = j(f)(\tau,\tau_1)$ and $\ell([g]_{\mathcal{U}_{\tau_1}}) = [g']_{\mathcal{U}_{\tau\tau_1}}$ with $g'(\alpha,\beta) = g(\beta)$ for all $\alpha < \beta < \kappa$. Let $\tau^* = [\mathrm{id}_1]_{\mathcal{U}}$, $\tau_1^* = [\mathrm{id}_2]_{\mathcal{U}}$ and $f^* = \ell(\widetilde{f})$, where $\mathrm{id}_1(\alpha,\beta) = \alpha$ and $\mathrm{id}_2(\alpha,\beta) = \beta$ for every $\alpha < \beta < \kappa$. Using commutativity and elementarity, then $\ell(\widetilde{\tau}_1) = \tau_1^*$, $k_{\tau,\tau_1}((\tau^*,\tau_1^*)) = (\tau,\tau_1)$ and $f^*:(\kappa^{+6})^{N_{\tau,\tau_1}} \leftrightarrow \tau_1^*$. Let $\tau_0^* = f^*(\tau^*)$. Clearly, $k_{\tau,\tau_1}(\tau_0^*) = \tau_0$. Also, $X \in \mathcal{U}$ iff $(\tau^*,\tau_1^*) \in i_{\tau\tau_1}(X)$. But since $\mathcal{U} = E_{\{\tau,\tau_0\}}$, $X \in \mathcal{U}$ iff $(\tau,\tau_0) \in j(X)$ iff $(\tau^*,\tau_0^*) \in i_{\tau\tau_1}(X)$. Hence, $X \in \mathcal{U}$ iff $(\tau^*,\tau_1^*) \in i_{\tau\tau_1}(X)$ iff $(\tau^*,\tau_0^*) \in i_{\tau\tau_1}(X)$. Now $i_{\tau\tau_1}(h)(\tau_1^*) = f^*$. There is $X \in \mathcal{U}$ so that for every $(\alpha,\beta) \in X$ $h(\beta):|\beta| \leftrightarrow \beta$ and $\alpha < |\beta|$. Define a projection π on X: $\pi(\alpha,\beta) = (\alpha, h(\beta)(\alpha))$. Then π projects $\mathcal{U}$ on itself, since whenever $Y \subseteq X$ is in $\mathcal{U}$ $(\tau^*,\tau_1^*) \in i_{\tau\tau_1}(Y)$. Then $(\tau^*,\tau_0^*) \in i_{\tau\tau_1}(\pi''(Y))$ which implies $\pi''(Y) \in \mathcal{U}$. Now we are ready to get a contradiction. Just consider the following sets in $\mathcal{U}$: $X = X_0, \pi^{-1''}(X) = X_1, \pi^{-1''}(\pi^{-1''}(X)) = X_2, \ldots, X_{n+1} = \pi^{-1''}(X_n)\cdots$. Let $X^* = \bigcap_{n<\omega} X_n$ and $(\alpha,\beta) \in X^*$. Then $(\alpha,\beta) \in \pi^{-1''}(X) \cap X$, so $(\alpha, h_\beta(\alpha))$ is well defined and it is in X. In particular, $\beta > h(\beta)(\alpha)$. But $(\alpha,\beta) \in \pi^{-1''}(\pi^{-1''}(X)) \cap X$, hence $(\alpha, h(\beta)(\alpha)) \in \pi^{-1''}(X)$ and hence $(\alpha, h(h(\beta)(\alpha))(\alpha))$ is well defined, is in X and so $\beta > h(\beta)(\alpha) > h(h(\beta)(\alpha))(\alpha)$. We continue in the same fashion and obtain a decreasing ω-sequence of ordinals. Contradiction. $\square$

Next, we replace δ by some $\delta^* < \kappa^{+7}$ coding $\{\tau, \tau_0, \tau_1, \delta\}$. Or in other words, we find U_{δ^*} in the extender E which is Rudin–Keisler above $\mathcal{U}_\delta$, $E_{\{\tau,\tau_0\}}$, $E_{\{\tau,\tau_1\}}$. Let π_δ be the corresponding projection of $\mathcal{U}_{\delta^*}$ onto U_δ. Define $g_i^* : \kappa \to \kappa_1$ $(i < 2)$ as follows

$$g_i^*(\beta) = g_i(\pi_\delta(\beta)).$$

Then, $\alpha_i = \ell_2(g_i^*)(\delta^*)$. Hence g_i^* projects $\mathcal{U}_{\delta^*}$ onto W_i. Consider an ultrafilter E_i over $\kappa \times X_i$ defined as follows:

$$\begin{aligned} &\mathcal{S} \in E_i \quad \text{iff} \quad \text{for some } \mathcal{S}' \in E_{\{\tau,\tau_i\}}, \\ &\mathcal{S} = \{(\beta, \rho_i(\gamma)) \mid (\beta, \gamma) \in \mathcal{S}'\}. \end{aligned}$$

I.e., we are using the bijection ρ_i to transfer $E_{\{\tau_i\}}$ back to W_i. Pick projections π_i and π of $\mathcal{U}_{\delta^*}$ to E_i and $\mathcal{U}_\tau$ such that $\pi_i(\xi) = (\pi(\xi), g_i^*(\xi))$ for almost all $\xi \bmod \mathcal{U}_{\delta^*}$.

Now we find disjoint $B_0' \in E_0$ and $B_1' \in E_1$. There is $B \in \mathcal{U}_{\delta^*}$ such that $\pi_0''(B) \subseteq B_0'$ and $\pi_1''(B) \subseteq B_1'$. Let $C = \pi''(B)$, $B_0 = \pi_0''(B)$ and $B_1 = \pi_1''(B)$. Then $C \in \mathcal{U}_\tau$, $B_0 \in E_0$ and $B_1 \in E_1$. The following is important:

(∗) for every $\beta \in C$ and $\gamma < \kappa_1$ it is impossible to have both $(\beta, \gamma) \in B_0$ and $(\beta, \gamma) \in B_1$.

For $\beta \in C$ we consider the set $C_\beta = \{\gamma \in X_0 \cup X_1 \mid (\beta, \gamma) \in B_0 \cup B_1\}$. For every $\beta \in C$ let $\psi_\beta : C_\beta \to 2$ be defined as follows:

$$\psi_\beta(\gamma) = \begin{cases} \varphi(\alpha_0) & \text{if } (\beta, \gamma) \in B_0, \\ \varphi(\alpha_1) & \text{if } (\beta, \gamma) \in B_1. \end{cases}$$

Notice that by (∗) such defined ψ_β is a function. Since $|C_\beta| \leqslant \kappa$, $C_\beta \in M$ and $C_\beta \subseteq \kappa_1$, there is $f_\beta \in F$, $f_\beta \supseteq \psi_\beta$. Let $t : \kappa \to F$ be defined by $t(\beta) = f_\beta$ for $\beta \in C$ and arbitrarily (but in F) otherwise. Then, $\ell_2(t)(\tau) \in F_2$ and let us show that $\ell_2(t)(\tau) \supseteq \varphi \restriction \{\alpha_0, \alpha_1\}$. It is enough to show that the set

$$\{\xi < \kappa \mid \pi(\xi) \in C,\ g_0^*(\xi), g_1^*(\xi) \in C_{\pi(\xi)} \text{ and } f_{\pi(\xi)}(g_i^*(\xi)) = \varphi(\alpha_i) \text{ for } i < 2\}$$

is in $\mathcal{U}_{\delta^*}$. We claim that it contains B. Thus let $\xi \in B$. Then, $\pi(\xi) \in C$, $(\pi(\xi), g_0^*(\xi)) \in B_0$ and $(\pi(\xi), g_1^*(\xi)) \in B_1$. Hence, $g_0^*(\xi), g_1^*(\xi) \in C_{\pi(\xi)}$ and $f_{\pi(\xi)}$ was chosen so that $f_{\pi(\xi)}(g_i(\xi)) = \varphi(\alpha_i)$ where $i < 2$.

This show the density for α_0, α_1. In order to deal with $\langle \alpha_m \mid m < \omega \rangle$ instead of only two α_0, α_1, just produce disjoint $\langle B_m \mid m < \omega \rangle$ using ω_1-completeness of the ultrafilters involved.

Now we are ready to complete the proof of Theorem 2.1. For every n, $0 < n < \omega$ let F_n^* be a set given by Lemma 2.7. Then for every $f \in F_n^*$ $(1 < n < \omega)$ there will be $\overline{f} : [\kappa]^n \to {}^{\kappa \geqslant}\kappa$ representing f in the ultrapower by $\mathcal{U}_0^n$, i.e.,

$$j_{n-1}(\overline{f})(\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) = f.$$

Set $\overline{F}_n = \{\overline{f} \mid \overline{f} : [\kappa]^n \to {}^{\kappa \geqslant}\kappa$ and $j_{n-1}(\overline{f})(\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) \in F_n^*\}$, where $0 < n < \omega$. Let $\overline{F}_0 = \{\overline{f} \mid \operatorname{dom} \overline{f} = \{0\}$ and $\overline{f}(0) : \kappa \to \kappa\}$. Define $\overline{F} = \prod_{n<\omega} \overline{F}_n$. Clearly,

$$|\overline{F}| = \prod_{n<\omega} |\overline{F}_n| = (\kappa^+)^{\aleph_0} = \kappa^+.$$

Suppose now that we have forced with the forcing of [6], then basically, a Prikry sequence was added for every measure of the extender E and no new bounded subset of κ was added. So, GCH holds below κ, $\operatorname{cf}\kappa = \aleph_0$ and $2^\kappa = \kappa^{++}$. Let $\langle \nu_n \mid n < \omega \rangle$ be the Prikry sequence for $\mathcal{U}_0$, i.e., for the normal measure. We are going to use it in order to define a dense set D in the topological space ${}^\kappa 2$ with topology generated by countable products. The idea is to transfer F_n^*'s to the space ${}^\kappa 2$. We are going to take functions representing elements of F_n^*'s, i.e., the members of $\overline{F}_n$ and apply them to the $\langle \nu_1, \ldots, \nu_n \rangle$. Then, in order to show density we will notice that a name of a basic clopen set can be transferred back to κ_n's using the same process but in the opposite direction. Over κ_n we find an element of F_n^* inside such clopen set and pull it back to ${}^\kappa 2$.

Now let us do this formally. For every $t \in \overline{F}$ we define a partial function t^* from κ to 2 as follows. Let $\alpha < \kappa$. We find $n_\alpha < \omega$ such that $\nu_{n_\alpha} \leqslant \alpha < \nu_{n_\alpha + 1}$, where ν_0 denotes 0. If $n_\alpha = 0$ and $t(0)(\alpha) < \nu_1$, then set $\alpha \in \operatorname{dom} t^*$ and $t^*(\alpha) = t(0)(\alpha)$. Suppose now that $n_\alpha > 0$. If $\alpha \in \operatorname{dom} t(n_\alpha)(\nu_1, \ldots, \nu_{n_\alpha})$ and $\nu_{n_\alpha} \leqslant t(n_\alpha)(\nu_1, \ldots, \nu_{n_\alpha})(\alpha) < \nu_{n_\alpha + 1}$ then set $\alpha \in \operatorname{dom} t^*$ and $t^*(\alpha) = t(n_\alpha)(\nu_1, \ldots, \nu_{n_\alpha})(\alpha)$. Otherwise $t^*(\alpha)$ is undefined or if one likes to have it total just set then $t^*(\alpha) = 0$. Set $D = \{t^* \mid t \in \overline{F}\}$. Obviously, $|D| \leqslant |\overline{F}| = \kappa^+$.

Lemma 2.9. *D is dense in the topological space ${}^\kappa 2$ with the topology generated by countable products.*

Proof. Suppose $\varphi \in {}^{\{\tau_m \mid m < \omega\}}2$. We need to find some $f \in D$ $f \supseteq \varphi$. Let us work in V with names instead of working in the generic extension. So, let $\underset{\sim}{\tau}_m$ be a name of an ordinal τ_m $(m < \omega)$ and $\underset{\sim}{\varphi}$ a name for φ.

Our basic tool will be Lemma 2.11 of [6] or actually the condition $p^* = p \cup \{\langle \beta, \emptyset, S^* \rangle\}$ produced in this lemma if instead of $\underset{\sim}{g}$ we deal with $\langle \underset{\sim}{\tau}_m \mid m < \omega \rangle$ and $\underset{\sim}{\varphi}$ there. In order to make the presentation as self-contained as possible, let us state here the main properties of p^*. Thus S^* is a subtree of $[\kappa]^{<\omega}$ such that for every $s \in S^*$, $\operatorname{Suc}_{S^*}(s) \in \mathcal{U}_\beta$. For every $m < \omega$ there is a level $n_m < \omega$ in S^* such that for every $s_1, s_2 \in S^*$ from this level, i.e., $|s_1| = |s_2| = n_m$ there are γ_1, γ_2 and $i < 2$ such that the following holds for $k = 1, 2$.

$$\big(s_k(n_m)\big)^0 \leqslant \gamma_k < \big(\min \operatorname{Suc}_{S^*}(s_k)\big)^0, \tag{a}$$

$$\big(p \cup \{\langle \beta, \emptyset, S^* \rangle\}\big)_{s_k} \Vdash \big(\underset{\sim}{\alpha}_m = \gamma_k \text{ and } \underset{\sim}{\varphi}(\underset{\sim}{\alpha}_m) = i\big), \tag{b}$$

where ^{0}-denotes the projection function to the normal measure $\mathcal{U}_0$ and $(p \cup \{\langle \beta, \emptyset, S^* \rangle\})_{s_k}$ is the condition obtained from $p \cup \{\langle \beta, \emptyset, S^* \rangle\}$ by adding s_k to be the initial segment of the Prikry sequence for β (or $\mathcal{U}_\beta$) and then shrinking S^* to the tree above s_k and projecting s_k to the appropriate coordinates in p.

Now consider the following set

$$A = \{n < \omega \mid \exists m < \omega,\ n = n_m\}.$$

Let $n \in A$. Denote $\{m < \omega \mid n_m = n\}$ by A_n. We define a function g_n on $\mathrm{Lev}_n(S^*)$. Let $s \in \mathrm{Lev}_n(S^*)$. By (a), (b), for every $m \in A_n$ there are $\gamma_{m,s}$ and $i_m < 2$ such that

$$\big(s(n)\big)^0 \leqslant \gamma_m < \big(\min \mathrm{Suc}_{S^*}(s)\big)^0, \tag{1}$$

$$\big(p \cup \{\langle \beta, \emptyset, S^*\rangle\}\big)_s \Vdash \big(\underset{\sim}{\alpha}_m = \gamma_{m,s} \text{ and } \underset{\sim}{\varphi}(\underset{\sim}{\alpha}_m) = i_m\big). \tag{2}$$

Set $g_n(s) = \{\langle \gamma_{m,s}, i_m\rangle \mid m \in A_n\}$. Hence, $g_n(s) \in {}^{\{\gamma_{m,s} \mid m \in A_n\}}2$. Then, g_n represents a basic clopen set in ${}^{\kappa_n}2$ in M_n. Namely, $j_{n-1}(g_n)(\beta, j_1(\beta), \dots, j_{n-1}(\beta))$. Using the density of F_n^*, we find $f_n \in F_n^*$, $f_n \supseteq j_{n-1}(g_n)(\beta, j_1(\beta), \dots, j_{n-1}(\beta))$. Pick $\overline{f}_n \in \overline{F}_n$ such that $j_{n-1}(\overline{f}_n)(\kappa_0, \kappa_1, \dots, \kappa_{n-1}) = f_n$. Then for almost all

$$(\mathrm{mod}\, \mathcal{U}_\beta^n)_s \in \mathrm{Lev}_n(S^*)\overline{f}_n\big((s)^0\big) \supseteq g_n(s).$$

Now let us do it for every $n \in A$ we will get a sequence $\langle \overline{f}_n \mid n \in A\rangle$. Let $t \in \overline{F}$ be such that for every $n \in A$, $t(n) = \overline{f}_n$. Then the corresponding t^* or here its name $\underset{\sim}{t}^*$ will be as desired, i.e., $p \cup \{\langle \beta, \emptyset, S^*\rangle\} \Vdash \underset{\sim}{t}^* \supseteq \underset{\sim}{\varphi}$. This completes the proof of the lemma and hence of the theorem. □

3. Some generalizations

Under the same lines we obtain the following theorem:

Theorem 3.1. *Suppose that $o(\kappa) = \lambda^+ + 1$ (i.e., extender of the length λ^+) and $\mathrm{cf}\,\lambda > \kappa$. Then the following holds in a generic extension $V[G]$:*

(1) *for every $\alpha < \kappa$ or $\alpha \geqslant \lambda$ $2^\alpha = \alpha^+$;*
(2) $2^\kappa = \lambda^+$;
(3) $\mathrm{cf}\,\kappa = \aleph_0$;
(4) $d_{<\aleph_1}(\kappa) = \lambda$.

Proof. Apply the construction of Section 2 with extender E of the length λ^+ instead of κ^{++}. An additional property that we need to show in the present situation is that $d_{<\aleph_1}(\kappa)$ cannot be below λ. But this follows by [16, 5.3, 5.4] and the *pcf* structure of the models of [6] or just directly using the correspondence established in Lemma 2.9 between basic clopen sets of ${}^\kappa 2$ of $V[G]$ and ${}^{\kappa_n}2$ of M_n. Since already ${}^{\kappa_1}2$ cannot have a dense set of cardinality less than λ because ${}^{\lambda^+}2$ embeds it and GCH holds. □

The following two results are straightforward applications of the techniques for pushing everything down to $\aleph_\omega$ [6, Section 2] or changing cofinality to $\aleph_1$ Segal [15,7] and pushing down to $\aleph_{\omega_1}$.

Theorem 3.2. *Suppose $o(\kappa) = \kappa^{++} + 1$. Then the following holds in a generic extension:*

(1) *for every $\alpha < \omega$ or $\alpha > \omega$ $2^{\aleph_\alpha} = \aleph_{\alpha+1}$;*
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$;
(3) $d_{<\aleph_1}(\aleph_\omega) = \aleph_{\omega+1}$.

Theorem 3.3. *Suppose $o(\kappa) = \kappa^{++} + \omega_1$. Then the following holds in a generic extension*:

(1) *for every* α $2^{\aleph_{\alpha+1}} = \aleph_{\alpha+2}$;
(2) GCH *above* $\aleph_{\omega_1+1}$;
(3) $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$;
(4) $d_{<\aleph_2}(\aleph_{\omega_1}) = \aleph_{\omega_1+1}$.

For Theorem 3.3 we need also to replace $\aleph_0$-box products by $\aleph_1$-base products. Notice that all the considerations of Section 2 are going smoothly if we replace $\aleph_0$-box product by θ-box product for any $\theta < \kappa$. Also instead of the space ${}^{\kappa}2$ we can work with ${}^{\kappa}\chi$ for any fixed $\chi < \kappa$. So the following holds:

Theorem 3.4. *Suppose $o(\kappa) = \kappa^{++}+1$, $\theta, \chi < \kappa$. Then the following holds in a generic cardinal preserving extension*:

(1) *for every $\alpha < \kappa$ or $\alpha > \kappa$* $2^{\alpha} = \alpha^{+}$;
(2) $\operatorname{cf}\kappa = \aleph_0$;
(3) $2^{\kappa} = \kappa^{++}$;
(4) *the density of the topological space ${}^{\kappa}\chi$ with the topology generated by θ-products is κ^{+}.*

The analogs of Theorems 3.2 and 3.3 hold as well.

4. Reaching the maximal density and wider gaps

In previous sections, we constructed models with density less that the maximal possible value 2^{κ}. Let us show now how to construct a model with the density 2^{κ} assuming singularity of κ and $2^{\kappa} > \kappa^{+}$.

Theorem 4.1. *Suppose $o(\kappa) = \kappa^{+3}+1$, then there is a generic extension $V[G]$ satisfying the following*:

(1) *for every $\alpha < \kappa$ or $\alpha > \kappa$* $2^{\alpha} = \alpha^{+}$;
(2) $\operatorname{cf}\kappa = \aleph_0$;
(3) $2^{\kappa} = \kappa^{++}$;
(4) $d_{<\aleph_1}(\kappa) = 2^{\kappa}$.

Proof. Let V_1 be a model of Theorem 3.1 with $\lambda = \kappa^{++}$. Collapse κ^{+++} to κ^{++} using the Levy collapse. Let V_2 be such generic extension. Then, in V_2, $2^{\kappa} = \kappa^{++}$. However, it is still true that $d_{<\aleph_1}(\kappa) = \kappa^{++}$. Thus, no new subsets of κ are added. Hence $({}^{\kappa}2)^{V_1} = ({}^{\kappa}2)^{V_2}$. But also no new subsets of cardinality κ^{+} are added to sets of V_1. So there is no dense set in ${}^{\kappa}2$ of cardinality $\leqslant \kappa^{+}$. $\square$

As in Section 3 it is possible to push this result down to $\aleph_{\omega}$ and $\aleph_{\omega_1}$.

Suppose now that one likes to have 2^{κ} big but still keep the density κ^{+}. A slight modification of the construction of Section 2 will give the following:

Theorem 4.2. *Suppose that $\lambda > \kappa$ is a regular cardinal $o(\kappa) = \lambda + 1$. Then there is a generic cardinal preserving extension satisfying the following*:

(1) *κ is a strong limit*;

(2) $\mathrm{cf}\,\kappa = \aleph_0$;

(3) $2^\kappa = \lambda$;

(4) $d_{<\aleph_0}(\kappa) = \kappa^+$.

Proof. Let $V \models$ GCH. E an extender of the length λ, $j : V \to M \simeq \mathrm{Ult}(V, E)$. Using Backward Easton forcing we blow up 2^{κ^+} to λ. By standard arguments E extends to an extender E^* in such generic extension $V[G]$ as well as $j \subseteq j^* : V[G] \to M[G^*]$. Now we proceed with $V[G]$, $M[G^*]$ and j^* as in Section 1. λ generic functions from κ^+ to κ^+ are used also to show that the analog of Claim 2.8 is valid. □

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