Continua whose homeomorphism groups are generated by arbitrarily small neighborhoods of the identity

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Abstract

The first author has shown that the pseudo-arc has the property that every self-homeomorphism is a composition of ε-homeomorphisms. We present results and examples concerning other continua and self-homeomorphisms which have this property.

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As part of an effort to determine the topological structure of the space of homeomorphisms of the pseudo-arc, the first author has shown [19] that every self-homeomorphism of the pseudo-arc is a composition of homeomorphisms arbitrarily close to the identity. Despite being able to go from one self-homeomorphism of the pseudo-arc to another with arbitrarily small steps, he has shown [15] that the space of homeomorphisms of the pseudo-arc contains no nondegenerate compact connected subsets. It remains unknown whether the

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space of homeomorphisms of the pseudo-arc contains any nondegenerate (noncompact) connected subsets.

In this paper we investigate other continua which have the property that every self-homeomorphism is a composition of homeomorphisms arbitrarily close to the identity.

**Definition 1.** A self-homeomorphism \( h: X \to X \) of the continuum \( X \) is said to be **locally generated** if, for every \( \varepsilon > 0 \), there exist self-homeomorphisms \( h_1, h_2, \ldots, h_n \) of \( X \), each moving no point of \( X \) a distance greater than \( \varepsilon \), such that \( h = h_n \circ h_{n-1} \circ \cdots \circ h_2 \circ h_1 \). Equivalently, \( h \) is a composition of homeomorphisms in any neighborhood of the identity \( 1_X \).

**Definition 2.** For a continuum \( X \), the group of self-homeomorphisms \( H(X) \) is **locally generated** if every self-homeomorphism of \( X \) is locally generated, i.e., \( H(X) \) is generated by arbitrarily small neighborhoods of the identity \( 1_X \).

We are interested in the following question.

**Question 3.** For which continua \( X \) is the homeomorphism group \( H(X) \) locally generated?

For some continua, the group of self-homeomorphisms is trivially locally generated. There exist examples of nondegenerate continua constructed by Anderson and Choquet [2], Cook [8], Maćkowiak [22] and others which are rigid in the sense of admitting no self-homeomorphisms other than the identity. Cook [8] has examples of continua with the even more restrictive property of having no continuous functions between any subcontinua other than constant maps or the identities on the domain subcontinua.

The following is an easy observation for any continuum.

**Theorem 4.** For any continuum \( X \) the collection of self-homeomorphisms of \( X \) which are locally generated is a closed subgroup of the collection of all self-homeomorphisms of \( X \).

Neither the interval \([0, 1]\) nor the circle \( S^1 \) has a homeomorphism group which is locally generated, though each has an abundance of homeomorphisms which are locally generated. For each, the group of self-homeomorphisms can be divided into two classes, those which preserve orientation and are locally generated and those which reverse orientation and can be obtained as the composition of any orientation reversing homeomorphism with a locally generated homeomorphism.

Clearly, if \( H(X) \) is connected for a given continuum \( X \) then it is locally generated. Must the converse be true? It is suspected, but no proof is currently available, that the pseudo-arc provides a negative answer to this question.

**Question 5.** If \( H(X) \) is locally generated for the continuum \( X \), must \( H(X) \) be connected?

**Theorem 6.** Every self-homeomorphism of the Hilbert cube \( \mathbb{Q} \) is stable and isotopic to the identity. Thus, \( H(\mathbb{Q}) \) is locally generated.
It is not in general the case that having locally generated homeomorphism groups is preserved by products.

**Theorem 7** [4]. Every self-homeomorphism \( h : P \times P \to P \times P \) of the product, \( P \times P \), of two pseudo-arcs is a product of homeomorphisms on the individual factor spaces composed either with the identity or with a permutation of coordinates.

**Corollary 8.** The product, \( P \times P \), of two pseudo-arcs does not have locally generated homeomorphism group. However, the collection of locally generated homeomorphisms of \( P \times P \) does act transitively on \( P \times P \), i.e., given \((p_1, p_2)\) and \((q_1, q_2)\) in \( P \times P \) there is a locally generated homeomorphism \( h : P \times P \to P \times P \) such that \( h((p_1, p_2)) = (q_1, q_2) \).

**Proof.** Any homeomorphism of \( P \times P \) which interchanges factor spaces must move some point at least half the diameter of a factor space. Thus, such a homeomorphism cannot be locally generated.

Any homeomorphism which preserves factor spaces is locally generated, being a product of locally generated homeomorphisms. The collection of such homeomorphisms acts transitively on \( P \times P \). \( \Box \)

**Theorem 9.** The hyperspace \( C(P) \) of all nonempty subcontinua of the pseudo-arc \( P \) has locally generated homeomorphism group.

**Proof.** There is a continuous unique association of homeomorphisms of the pseudo-arc \( P \) with homeomorphisms of the hyperspace \( C(P) \) and vice versa. \( \Box \)

We can make a number of observations about classes of continua having locally generated homeomorphism groups.

**Theorem 10** [28, Theorem 5.1.1]. If a compact ANR \( X \) has locally generated homeomorphism group \( H(X) \), then every self-homeomorphism of \( X \) induces the identity homomorphism on the (co)homology groups \( H_*(X) \).

**Proof.** For any open cover \( U \) of a compact ANR \( X \), any homeomorphism of \( X \) which is sufficiently close to the identity on \( X \) is \( U \)-homotopic to the identity. Thus, if the homeomorphism group of \( X \) is locally generated then every self-homeomorphism of \( X \) is homotopic to the identity \( 1_X \) and induces the identity homomorphism on \( H_*(X) \). \( \Box \)

**Theorem 11.** No nondegenerate compact finite-dimensional Abelian group \( G \) has locally generated homeomorphism group \( H(G) \).

**Proof.** If \( G \) is not connected then it can be separated into two closed sets a positive distance apart. No locally generated homeomorphism can map a point in one of these two closed sets to a point in the other. However, the translations on \( G \) act transitively, taking any point to any other point.
Thus, suppose $G$ is a connected compact metrizable Abelian group with $\dim(G) = n$. $G$ can be represented [23] as an inverse limit $G = \lim \leftarrow \{ T_i, P^i_j \}$ where each factor space $T_i$ is an $n$-torus and each bonding map $P^i_j: T_j \to T_i$ is a covering map and group homomorphism.

In this case a self-homeomorphism of $G$ generated by a reversal of orientation on each factor space is not locally generated. $\square$

The examples presented so far of continua with locally generated homeomorphism groups have either been continua lacking an abundance of self-homeomorphisms or have been acyclic.

However, this need not be the case. Theorem 9 places restrictions on compact ANRs which have locally generated homeomorphism groups, but does not require that such have trivial (co)homology.

**Theorem 12.** The real projective plane is a non-cell-like 2-dimensional manifold which has locally generated homeomorphism group.

**Proof.** Every self-homeomorphism of the real projective plane is isotopic to the identity and hence locally generated. $\square$

Many of the examples which we have presented of continua having locally generated homeomorphism groups have been ones having a sense of orientation or possessing distinguished subsets preserved or interchanged by any self-homeomorphism.

The Warsaw circle possesses a single distinguished set, the limit bar, but because of how the arc from the limit ray connects to this bar any homeomorphism must be orientation preserving on the limit bar. This removes the possibility of an orientation reversing homeomorphism and might at first appear to allow for all self-homeomorphisms to be locally generated. However, Bellamy [5] has observed that a homeomorphism of the Warsaw circle which is the identity on the limit bar and shifts each oscillation of the limit ray one oscillation closer to the limit bar cannot be locally generated.

**Theorem 13.** While every self-homeomorphism of the Warsaw circle must be orientation preserving on the limit bar (and hence locally generated when restricted to the limit bar) there exist homeomorphisms of the Warsaw circle which are not locally generated.

**Proof.** See the description immediately before the theorem. $\square$

Another way to have cyclicity and to have an abundance of homeomorphisms but to require all self-homeomorphisms to preserve a sense of directionality or orientation occurs in homogeneous continua which are not bihomogeneous. Recall that a continuum is said to be **homogeneous** if, for each $x_1, x_2 \in X$, there is a self-homeomorphism $h: X \to X$ with $h(x_1) = x_2$. A continuum $X$ is said to be **bihomogeneous** if, for each $x_1, x_2 \in X$, there is a self-homeomorphism $h: X \to X$ with both $h(x_1) = x_2$ and $h(x_2) = x_1$. Examples of continua which are homogeneous but not bihomogeneous have been constructed by
K. Kuperberg [12]. These continua have cyclicity, while also having a sense of direction-ality preserved by all self-homeomorphisms which prevents bihomogeneity.

**Question 14.** Does there exist a (locally connected) homogeneous continuum $X$ which is not bihomogeneous such that $X$ has a locally generated homeomorphism group?

Both the pseudo-arc [5] and the Hilbert Cube are bihomogeneous.

It follows from a result of Effros [9,10] that every homogeneous continuum $X$ is micro-homogeneous in the sense that, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1, x_2 \in X$ are points with $\text{dist}(x_1, x_2) < \delta$ then there exists a homeomorphism $h : X \to X$ with $h(x_1) = x_2$ and $\text{dist}(h(x), x) < \varepsilon$ for every $x \in X$.

Thus, for any homogeneous continuum $X$, any points $x_1$ and $x_2$ in $X$ and any $\varepsilon > 0$, there exists a homeomorphism $h : X \to X$ with $h(x_1) = x_2$ such that $h$ is a composition of homeomorphisms each of which moves no point a distance as much as $\varepsilon$ [17].

However, this does not imply that for any homogeneous continuum the collection of locally generated homeomorphisms acts transitively on the continuum. In fact, there may be no nonidentity locally generated self-homeomorphisms of the continuum.

The Menger universal curve $M$ has very rich homogeneity properties [1] and very rich self-homeomorphism structure. However, the only locally generated self-homeomorphism of the Menger curve is the identity.

**Theorem 15** [7]. If $f$ and $g$ are distinct self-homeomorphisms of the Menger universal curve $M$, then there exists a positive number $\varepsilon > 0$ and a separation of the space of self-homeomorphisms of $M$ into closed sets $F$ and $G$ with $f \in F$ and $g \in G$ such that, in terms of the sup metric on $H(M)$, $\text{dist}(F, G) > \varepsilon$.

The argument for the above result applies equally well to any 1-dimensional continuum (planar or not) locally homeomorphic to the Sierpiński universal plane curve. Every one-dimensional continuum locally homeomorphic to the Menger universal curve is homeomorphic to the Menger universal curve [1].

**Corollary 16.** The only locally generated self-homeomorphism of the Menger universal curve or of any continuum locally homeomorphic to the Sierpiński universal plane curve is the identity.

**Question 17.** Do results analogous to those in the above theorem and corollary hold for higher-dimensional Menger manifolds?

It was considerations such as the above which initially motivated the first author to investigate the topological structure of the homeomorphism group of the pseudo-arc.

While it remains unknown whether the homeomorphism group of the pseudo-arc contains any nondegenerate connected subsets, the considerations motivating the present investigation show that it does not admit the type of separations which the homeomorphism group of the Menger curve admits.
Many interesting results on homogeneous continua involve continuous decompositions and the approximate homeomorphism lifting property.

**Definition 18.** Let \( p : \tilde{X} \to X \) be a mapping from continuum \( \tilde{X} \) to continuum \( X \), \( h : X \to X \) a self-homeomorphism of \( X \) and \( \varepsilon > 0 \). If there exists a self-homeomorphism \( \tilde{h} : \tilde{X} \to \tilde{X} \) such that, for each \( z \in \tilde{X} \), \( \text{dist}(p \circ \tilde{h}(z), h \circ p(z)) < \varepsilon \) then \( \tilde{h} \) is called an \( \varepsilon \)-commutative lift of \( h \) (with respect to \( p \)). If for every self-homeomorphism \( h \) of \( X \) and every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-commutative lift of \( h \) (with respect to \( p \)) then the mapping \( p : \tilde{X} \to X \) is said to have the approximate homeomorphism lifting property.

If for every self-homeomorphism \( h : X \to X \) there exists a self-homeomorphism \( \tilde{h} : \tilde{X} \to \tilde{X} \) such that \( p \circ \tilde{h} = h \circ p \) then the mapping \( p : \tilde{X} \to X \) has the homeomorphism lifting property.

**Theorem 19.** Let \( p : \tilde{X} \to X \) have the approximate homeomorphism lifting property. If the continuum \( \tilde{X} \) has locally generated homeomorphism group and every self-homeomorphism of \( \tilde{X} \) respects the collection \( \{p^{-1} (x) : x \in X \} \) (i.e., the image under any self-homeomorphism of \( X \) of any element of this collection is an element of this collection), then the continuum \( X \) also has locally generated homeomorphism group.

**Proof.** Since self-homeomorphisms of \( \tilde{X} \) respect the fibers of \( p \), every self-homeomorphism \( h \) of \( X \) projects to a self-homeomorphism \( \tilde{h} \) of \( \tilde{X} \). By uniform continuity of \( p \), for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that any two self-homeomorphisms of \( \tilde{X} \) within \( \delta \) of each other project to self-homeomorphisms of \( X \) within \( \varepsilon \) of each other.

If \( h \) is a self-homeomorphism of \( X \), then by the approximate homeomorphism lifting property there exists a self-homeomorphism \( \tilde{h} \) of \( \tilde{X} \) such that \( \tilde{h} \) projects to a self-homeomorphism of \( X \) within \( \varepsilon \) of \( h \). Since \( \tilde{h} \) is by assumption a composition of self-homeomorphisms of \( \tilde{X} \) each within \( \delta \) of the identity, the projections of these are homeomorphisms each within \( \varepsilon \) of the identity whose composition is a self-homeomorphism of \( X \) within \( \varepsilon \) of \( h \).

Applications of the above exist for circle-like and finitely cyclic continua.

The **circle of pseudo-arcs** [6] is a circle-like continuum admitting a continuous decomposition into pseudo-arcs with decomposition space a simple closed curve. A **solenoid of pseudo-arcs** [24] is a circle-like continuum admitting a continuous decomposition into pseudo-arcs with decomposition space a solenoid.

**Theorem 20** [11,16]. Every nondegenerate homogeneous circle-like continuum is either a pseudo-arc, a simple closed curve, a solenoid, a circle of pseudo-arcs or a solenoid of pseudo-arcs. There exists a unique circle of pseudo-arcs or solenoid of pseudo-arcs corresponding to the circle or each solenoid, with each such continuum being homogeneous.

For each of the circle of pseudo-arcs and each solenoid of pseudo-arcs, the decomposition into pseudo-arcs is given by the aposyndetic decomposition [25], or equivalently by the decomposition into maximal terminal subcontinua. The projection map associated with
this decomposition has the homeomorphism lifting property and every homeomorphism of a circle of pseudo-arcs or solenoid of pseudo-arcs respects this decomposition \[18\].

**Corollary 21.** The only nondegenerate circle-like continuum having a locally generated homeomorphism group is the pseudo-arc.

**Proof.** This follows from the previous results and the observation that the pseudo-arc has a locally generated homeomorphism group while the circle and solenoids do not. \[\square\]

A continuum \(X\) is said to be **finitely cyclic** if \(X\) is homeomorphic to an inverse limit of graphs, with a bound on the genus of the graphs. For finitely cyclic homogeneous continua a classification similar to the above holds, replacing the pseudo-arc by tree-like continua and without the known result that the decomposition has the (approximate) homeomorphism lifting property.

**Theorem 22** \[14\]. Every homogeneous finitely cyclic continuum is either tree-like, a simple closed curve, a solenoid or admits a decomposition into maximal terminal subcontinua such that the decomposition elements are mutually homeomorphic, homogeneous, tree-like continua and the decomposition space is either a simple closed curve or a solenoid.

**Corollary 23.** If \(X\) is a finitely cyclic homogeneous continuum with locally generated homeomorphism group, then \(X\) is tree-like if and only if the maximal terminal decomposition mapping \(\pi : X \to X/D\) has the approximate homeomorphism lifting property.

**Proof.** If \(X\) is tree-like and homogeneous then it is hereditarily indecomposable \[13\]. In this case the maximal terminal decomposition consists of a single element with decomposition space a singleton.

If the maximal terminal decomposition mapping \(\pi : X \to X/D\) has the approximate homeomorphism lifting property, then \(X/D\) has locally generated homeomorphism group. The only candidate for such among finitely cyclic homogeneous continua are tree-like continua. \[\square\]

It remains an open question whether every nondegenerate homogeneous tree-like continuum is a pseudo-arc \[21,26,27\].

There is another sense in which the homeomorphisms of the pseudo-arc in an arbitrarily small neighborhood of the identity contain information about all homeomorphisms of the pseudo-arc.

**Theorem 24** \[20\]. Let \(h : P \to P\) be a self-homeomorphism of the pseudo-arc and let \(\varepsilon > 0\). There exists a self-homeomorphism \(g : P \to P\) such that \(g^{-1} \circ h \circ g\) is a self-homeomorphism of the pseudo-arc with \(\text{dist}(p, g(p)) < \varepsilon\) for each \(p \in P\), i.e., every self-homeomorphism of the pseudo-arc is conjugate to a homeomorphism arbitrarily close to the identity.

Considerations of orientation or distinguished subsets again prevent this property being shared by self-homeomorphisms of the arc, simple closed curve, product of pseudo-arcs
or many other continua. The considerations which prevent self-homeomorphisms of the Menger curve from being locally generated do not so readily appear to prevent them from being conjugate to \( \varepsilon \)-homeomorphisms.

**Question 25.** Does the Menger curve have the property that every self-homeomorphism of it is conjugate to one arbitrarily close to the identity?

**Question 26.** Does the Hilbert Cube have the property that every self-homeomorphism of it is conjugate to one arbitrarily close to the identity?

**Question 27.** What relation if any is there between a continuum having locally generated homeomorphism group and having the property that every self-homeomorphism is conjugate to one arbitrarily close to the identity?

We conclude by repeating the question which initiated these investigations.

**Question 28.** Does the space of self-homeomorphisms of the pseudo-arc have any nondegenerate connected subsets?

## References