Discrete Applied Mathematics 160 (2012) 1285–1290

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

On the connectivity of p-diamond-free vertex transitive graphs[☆]

Yingzhi Tian [*], Jixiang Meng, Zhao Zhang

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, People's Republic of China

ARTICLE INFO

Article history:
Received 17 August 2010
Received in revised form 13 June 2011
Accepted 17 November 2011
Available online 17 December 2011

Keywords:
Connectivity
Superconnected
Vertex transitive graphs
Diamond-free graphs
p-diamond-free graphs

ABSTRACT

Let G be a graph of order $n(G)$, minimum degree $\delta(G)$ and connectivity $\kappa(G)$. We call the graph G *maximally connected* when $\kappa(G) = \delta(G)$. The graph G is said to be *superconnected* if every minimum vertex cut isolates a vertex.

For an integer $p \geq 1$, we define a *p-diamond* as the graph with $p + 2$ vertices, where two adjacent vertices have exactly p common neighbors, and the graph contains no further edges. Usually, the 1-diamond is triangle and the 2-diamond is *diamond*. We call a graph *p-diamond-free* if it contains no p-diamond as a (not necessarily induced) subgraph. A graph is *vertex transitive* if its automorphism group acts transitively on its vertex set.

In this paper, we give some sufficient conditions for vertex transitive graphs to be maximally connected. In addition, superconnected p-diamond-free $(1 \leq p \leq 3)$ vertex transitive graphs are characterized.

1. Introduction

We shall follow [2] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a finite graph without loops and multiple edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. For each vertex $v \in V(G)$, the *neighborhood* $N(v)$ of v is defined as the set of all vertices adjacent to v and $d(v) = |N(v)|$ is the *degree* of v. We denote by $\delta(G)$ the *minimum degree* and by $n(G) = |V(G)|$ the *order* of G.

The underlying topology of an interconnection network is often modeled by a graph $G = (V, E)$. The *connectivity* of G, denoted by $\kappa(G)$, has long been used to measure the reliability of the network. Because of $\kappa(G) \leq \delta(G)$, we call a graph G *maximally connected* when $\kappa(G) = \delta(G)$. Many authors have presented sufficient conditions for graphs to be maximally connected; see the survey paper [7]. For further study, Boesch proposed the concept of superconnected graphs [1]. A graph G is said to be *superconnected* if any minimum cut of G isolates a vertex. There are many studies on this subject; see [5,6,11] for examples.

Graph symmetry is another factor that should be taken into account in the design of an interconnection network. A graph G is *vertex transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$, and is *edge transitive* if $\mathrm{Aut}(G)$ acts transitively on $E(G)$. Since each vertex of a vertex transitive graph has the same degree, we define the degree of a vertex transitive graph as this same degree. Several known results indicate that graphs with high symmetry have high connectivities. For instance, connected vertex transitive graphs (digraphs) have maximum edge (arc) connectivity [10] and connected edge (arc) transitive graphs (digraphs) have maximum vertex connectivity [15]. In view of the above results, it is natural to ask what is the relationship between the symmetry of graphs and its superconnectivity. In [11], Meng characterized the superconnected vertex and edge transitive graphs. Zhang and Meng [18] characterized the superconnected irreducible edge transitive graphs (where irreducible means that the graph has no vertices with the same neighbor set) and gave some sufficient

☆ The research is supported by NSFXJ (2010211A06), NSFC (10971255), the Key Project of Chinese Ministry of Education (208161), the Program for New Century Excellent Talents in University, and the project sponsored by SRF for ROCS, SEM.
* Corresponding author.
E-mail addresses: tianyzhxj@163.com (Y. Tian), mjx@xju.edu.cn (J. Meng), zhzhao@xju.edu.cn (Z. Zhang).

0166-218X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2011.11.016

conditions for reducible edge transitive graphs to be superconnected. It seems difficult to characterize superconnected vertex transitive graphs. The authors in [9] characterized superconnected vertex transitive bipartite graphs and the authors in [13] characterized superconnected vertex transitive line graphs. Superconnected vertex transitive graphs with degree 3, 4 (5) and 6 were characterized by those authors in [16,14,12], respectively.

For an integer $p \geq 1$, we define a *p-diamond* as the graph with $p+2$ vertices, where two adjacent vertices have exactly p common neighbors, and the graph contains no further edges. Usually, the 1-diamond is triangle and the 2-diamond is *diamond*. We call a graph *p-diamond-free* if it contains no p-diamond as a (not necessarily induced) subgraph. In [4], Dankelmann, Hellwig and Volkmann proved that a connected diamond-free graph of order $n(G) \leq 3\delta(G)$ is maximally connected when $\delta(G) \geq 3$. Holtkamp and Volkmann [8] gave minimum degree condition for p-diamond-free graphs to be maximally local connected. In this paper, we will give some sufficient conditions for vertex transitive graphs to be maximally connected. In addition, we will characterize superconnected p-diamond-free ($1 \leq p \leq 3$) vertex transitive graphs.

2. Preliminary

For a subset A of $V(G)$, let $N(A) = N_G(A) = \{u \in V(G) \setminus A : \text{ there exists } v \in A, \text{ such that } (u, v) \in E(G)\}$ be the *neighbor set* of A, $N[A] = N_G[A] = A \cup N(A)$ and $R(A) = R_G(A) = V \setminus N[A]$. For a nonempty vertex set A of V, if $R(A) \neq \emptyset$, then $N(A)$ is a vertex cut of G, and thus $|N(A)| \geq \kappa(G)$. A vertex set A is said to be a *fragment* of G, if $N(A)$ is a minimum vertex cut of G. A fragment of minimum cardinality is called an *atom* of G. A fragment A with $2 \leq |A| \leq |V(G)| - \kappa(G) - 2$ is called a *strict fragment* of G. If there exists a strict fragment in G, then G is said to be *degenerate*. Clearly, if A is a strict fragment of G, so is $R(A)$. A smallest strict fragment of G is called a *superatom* of G, whose cardinality is denoted by $\omega(G)$.

Recall that an *imprimitive block* for a permutation group Φ on a set T is a proper, non-trivial subset A of T such that if $\varphi \in \Phi$ then either $\varphi(A) = A$ or $\varphi(A) \cap A = \emptyset$. A subset A of $V(G)$ is called an *imprimitive block* for G if it is an imprimitive block for $\mathrm{Aut}(G)$ on $V(G)$.

Theorem 2.1 ([15]). *Let A be an imprimitive block of a vertex transitive graph G, then the induced subgraph $G[A]$ is also vertex transitive.*

Let G and H be two graphs. The *lexicographic product* of G by H, denoted by $G(H)$, is the graph with vertex set $V(G) \times V(H)$ and, for two vertices (u_1, v_1) and (u_2, v_2) of $G(H)$, (u_1, v_1) and (u_2, v_2) are adjacent if and only if either u_1 and u_2 are adjacent in G or $u_1 = u_2$ and v_1 and v_2 are adjacent in H. Denote by C_n the cycle of length n, N_n the empty graph of order n, K_n the complete graph of order n and Q_3 the 3-cube.

For vertex transitive graphs, the following theorem is well-known.

Theorem 2.2 ([17]). *If G is a connected vertex transitive graph with degree k, then $\kappa(G) \geq 2(k+1)/3$.*

For connected vertex transitive graphs with degree at most 6, Chen [3] obtained the following result.

Theorem 2.3 ([3]). *Let G be a k-regular connected vertex transitive graph with $k \leqslant 6$, then $\kappa(G) = k$ unless $k = 5$ and $G \cong C_n(K_2)$ for some $n \geq 4$.*

The *line graph* of G, denoted by $L(G)$, is a graph with vertex set $E(G)$ and $e_1, e_2 \in E(G)$ are adjacent if and only if they are incident in G. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. Let H be a group and S a subset of $H \setminus \{1_H\}$ with $S = S^{-1}$, where 1_H is the identity of H and $S^{-1} = \{s^{-1} : s \in S\}$. Define the *Cayley graph* $C(H, S) = (V, E)$, where $V = H, E = \{\{h, hs\} : h \in H, s \in S\}$. It is well known that Cayley graph is vertex transitive. In [11], Meng characterized superconnected graphs which are both vertex transitive and edge transitive.

Theorem 2.4 ([11]). *Let G be a connected graph which is both vertex transitive and edge transitive. Then G is not superconnected if and only if $G \cong C_n(N_m)$ $(n \geq 6)$ or $G \cong L(Q_3)(N_m)$.*

In the following, superconnected vertex transitive graphs with degree $k \leq 6$ were characterized.

Theorem 2.5 ([16]). *Let G be a connected vertex transitive cubic graph with $|V(G)| \geq 7$. Then G is superconnected if and only if G is 1-diamond-free (that is girth $(G) \geq 4$).*

Theorem 2.6 ([14]). *Let G be a connected vertex transitive 4-regular graph with $|V(G)| \geq 8$. Then G is superconnected if and only if G is diamond-free, $G \ncong C_n(N_2)$ $(n \geq 6)$ and $G \ncong L(Q_3)$.*

Theorem 2.7 ([14]). *Let G be a connected vertex transitive 5-regular graph. If $|V(G)| \geq 9$, then G is superconnected if and only if G is 3-diamond-free.*

Theorem 2.8 ([12]). *Let G be a connected vertex transitive 6-regular graph with $|V(G)| \geq 10$. Then G is superconnected if and only if G is 4-diamond-free, $\omega(G) \neq 3$ and G is not isomorphic to one of the following graphs: $C_n(N_3)$ $(n \geq 6)$, $L(C(Z_6, \{-1, -2, 1, 2\}))$ and $H(N_2)$, where H is a connected 3-regular vertex transitive graph with girth $g(H) = 3$ and $|V(H)| \geq 7$.*

Motivated by the above results, we will characterize superconnected p-diamond-free $(1 \leq p \leq 3)$ vertex transitive graphs in Section 4.

3. Connectivity of p-diamond-free vertex transitive graphs

By the definition of atom, we easily obtain the following lemma.

Lemma 3.1. *Let G be a connected graph. Then* (i) $\kappa(G) = \delta(G)$ *if and only if every atom of G has cardinality* 1*; and* (ii) *if $\kappa(G) < \delta(G)$, then each atom has cardinality at most $\lfloor(|V(G)| - \kappa(G))/2\rfloor$ and induces a connected subgraph of G.*

Proof. The first conclusion is obvious. (ii) Let A be an atom of G. Since $R(A) = V \setminus N[A]$ is a fragment of G, it must have cardinality greater than or equal to $|A|$. Since $V(G) = A \cup N(A) \cup R(A)$ and $|N(A)| = \kappa(G)$, it follows that $|A| \leq \lfloor(|V(G)| - \kappa(G))/2\rfloor$. Suppose to the contrary that the induced subgraph $G[A]$ is not connected. Let A_1 be a component of $G[A]$. But then $N(A_1) \subseteq N(A)$ and hence A_1 is a fragment of G contained in A and $A_1 \neq A$, which contradicts the definition of atom. $\square$

In [10], Mader proved the following basic theorem.

Theorem 3.2 ([10])**.** *Let G be a connected graph which is not a complete graph and let A be an atom of G. If B is a fragment of G with $A \cap B \neq \emptyset$, then $A \subset B$.*

Theorem 3.2 implies that atoms are imprimitive blocks for the graph G. The following corollary is also observed from Theorem 3.2.

Corollary 3.3. *Let G be a connected graph and A be an atom of G. If C is a minimum vertex cut of G, then A is either disjoint from C or a subset of C.*

Now, we give the following sufficient conditions for vertex transitive graphs to be maximally connected.

Theorem 3.4. *Let G be a connected vertex transitive graph with degree k and let $p = (k+3)/2$ if k is odd and $p = (k+6)/2$ if k is even. If G is p-diamond-free, then G is maximally connected.*

Proof. Suppose to the contrary that $\kappa(G) < k$. Let A be an atom in G. Then $|A| \geq 2$ and $G[A]$ is connected by Lemma 3.1. If $\alpha \in \mathrm{Aut}(G)$, then $\alpha(A)$ is also an atom, and so by Theorem 3.2, either $A = \alpha(A)$ or $A \cap \alpha(A) = \emptyset$. Hence A is an imprimitive block for $\mathrm{Aut}(G)$ on $V(G)$, and its translates partition $V(G)$ by the vertex transitivity of G. Corollary 3.3 now yields that $N(A)$ is partitioned by translates of A, and therefore $|N(A)| = t|A|$ for some integer $t \geq 2$ ($t = 1$ implies that $V(G) = A \cup N(A)$, which is impossible).

If k is odd, then we have $|N(A)| \leq k - 1$ and $|A| \leq (k-1)/2$. Thus $|N[A]| = |A| + |N(A)| \leq 3(k-1)/2$. Since $|A| \geq 2$ and $G[A]$ is connected, we can select an edge $e = uv \in E(G[A])$. But then

$$|N[A]| \geq |N[\{u, v\}]| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2k - (k+1)/2 > 3(k-1)/2,$$

a contradiction (the second inequality follows since G is $(k+3)/2$-diamond-free).

Now we assume that k is even. Since $N(A)$ is partitioned by translates of A, we have (1) $|N(A)| \geq 3|A|$ when $|N(A)| = k-1$; and (2) $|N(A)| \geq 2|A|$ when $|N(A)| \leq k-2$. That is, $|A| \leq (k-1)/3$ when $|N(A)| = k-1$ and $|A| \leq (k-2)/2$ when $|N(A)| \leq k-2$. Thus $|N[A]| \leq 4(k-1)/3$ when $|N(A)| = k-1$ and $|N[A]| \leq 3(k-2)/2$ when $|N(A)| \leq k-2$. We can assume that $k \geq 8$ by Theorem 2.3. If $k = 8$, then $|N(A)|$ cannot be $k-1$ (for otherwise, since 7 is a prime number, we deduce that $|A| = 1$, which is impossible), and thus $|N[A]| \leq 3(k-2)/2$. If $k \geq 10$, we also obtain that $|N[A]| \leq 3(k-2)/2$ from $3(k-2)/2 \geq 4(k-1)/3$. On the other hand, since $|A| \geq 2$ and $G[A]$ is connected, we can select an edge $e = uv \in E(G[A])$. But then

$$|N[A]| \geq |N[\{u, v\}]| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2k - (k+4)/2 > 3(k-2)/2,$$

a contradiction (the second inequality follows since G is $(k+6)/2$-diamond-free). $\square$

If G is a connected vertex transitive graph with degree $k \leq 4$, then by Theorem 2.3 $\kappa(G) = k$. Thus the following corollary is obtained by Theorems 2.3 and 3.4.

Corollary 3.5. *Let G be a connected vertex transitive graph. If G is* 4*-diamond-free, then G is maximally connected. Furthermore, each connected* 5*-diamond-free vertex transitive graph G is maximally connected unless $G \cong C_n(K_2)$ for some $n \geq 4$.*

4. Superconnected 3-diamond-free vertex transitive graphs

In [6], Hamidoune proved the following useful results on strict fragments and superatoms of vertex transitive graphs.

Theorem 4.1 ([6])**.** *Let G be a k-regular vertex transitive graph with $\kappa(G) = k$.*

(i) *Let A be a superatom and B be a strict fragment such that $A \cap B \neq \emptyset$ and $A \not\subseteq B$. Then $|A \cap B| = 1$, $A \cup B$ is a fragment and $N[A \cap B] = N[A] \cap N[B]$.*
(ii) *If $\omega(G) \geq 3$, then the intersection of three distinct superatoms of G is empty.*

By the definition of superatoms, we observe the following lemma.

Lemma 4.2. *Let G be a connected degenerate graph and A be a superatom of G. If $\omega(G) \geq 3$, then the induced subgraph $G[A]$ is connected.*

Proof. Suppose to the contrary that $G[A]$ is not connected. Let $A_1, A_2, \ldots, A_t$ $(t \geq 2)$ be the components of $G[A]$. If there exists a component A_i with $|A_i| \geq 2$, then, by $N(A_i) \subseteq N(A)$, we have that A_i is a strict fragment with cardinality less than A, a contradiction. Thus, assume that $|A_i| = 1$ for $1 \leq i \leq t$. By $\omega(G) \geq 3$, we see that $t \geq 3$. Similarly, we argue that $A_1 \cup A_2$ is a strict fragment with cardinality less than A, also a contradiction. □

We only consider vertex transitive graphs with degree greater than 6, since superconnected vertex transitive graphs with degree less than or equal to 6 were characterized by Theorems 2.5–2.8. As in the case with atoms, we wish to prove that superatoms of G are imprimitive blocks of G. This fundamental fact is the following lemma.

Lemma 4.3. *Let G be a connected degenerate vertex transitive graph with degree $k \geq 7$. Assume that $\omega(G) \geq 3$, A is a superatom of G and B is a strict fragment of G. If G is 3-diamond-free and $A \cap B \neq \emptyset$, then $A \subseteq B$.*

Proof. Since G is 3-diamond-free, we have $\kappa(G) = k$ by Corollary 3.5.

Suppose to the contrary that $A \nsubseteq B$. Then by Theorem 4.1 (i), we have $|A \cap B| = 1$. Let $A \cap B = \{a\}$. By Theorem 4.1(i), we deduce that

$$N(a) = (N(A) \cap B) \cup (A \cap N(B)) \cup (N(A) \cap N(B)). \tag{1}$$

Denote $R_1 = A \cap N(B)$, $R_2 = N(A) \cap B$, $R_3 = N(A) \cap N(B)$, $R_4 = N(A) \cap R(B)$ and $R_5 = R(A) \cap N(B)$. By careful consideration, we get the following two claims.

Claim 1. $|R_1| \leq |R_2|$. If $B \cap R(A) = \emptyset$, then $|R_1| \leq |R_2|$ by $|A| \leq |B|$. Otherwise, assume that $B \cap R(A) \neq \emptyset$ and $|R_1| > |R_2|$. Since $N(B \cap R(A)) \subseteq R_2 \cup R_3 \cup R_5$, $|R_1| + |R_3| + |R_5| = k$, $|R_1| > |R_2|$ and $R(B) \subseteq R(B \cap R(A))$, we have $N(B \cap R(A))$ is a vertex cut with $|N(B \cap R(A))| \leq |R_2| + |R_3| + |R_5| < |R_1| + |R_3| + |R_5| = k$, which contradicts to $\kappa(G) = k$.

Claim 2. $|A \cap R(B)| \leq 1$. For otherwise, assume that $|A \cap R(B)| \geq 2$. Since $|N(A \cap R(B))| \leq |R_1| + |R_3| + |R_4| \leq |R_2| + |R_3| + |R_4| = k$ and $R(A) \subseteq R(A \cap N(B))$, we see that $A \cap R(B)$ is a strict fragment with cardinality less than A, a contradiction. Thus $|A \cap R(B)| \leq 1$. In the following, we consider two cases.

Case 1. $|A \cap R(B)| = 1$.

Let $\{u\} = A \cap R(B)$. Since $|R_1| \leq |R_2|$, $|R_2| + |R_3| + |R_4| = k$, and $N(u) \subseteq R_1 \cup R_3 \cup R_4$, we have that $N(u) = R_1 \cup R_3 \cup R_4$ and $|R_1| = |R_2|$. By (1), we know $N(a) = R_1 \cup R_2 \cup R_3$. For each vertex $v \in R_1$, we verify that $N(v) \subseteq \{u, a\} \cup R_1 \cup R_2 \cup R_3 \cup R_4 = \{u, a\} \cup N(u) \cup N(a)$. Thus $N(v) = \{u, a\} \cup (N(v) \cap N(u)) \cup (N(v) \cap N(a))$ is obtained. Since G is 3-diamond-free, we get $|N(v) \cap N(u)| \leq 2$ and $|N(v) \cap N(a)| \leq 2$. But then $k = d(v) \leq |\{a, u\}| + |N(v) \cap N(u)| + |N(v) \cap N(a)| \leq 6$, which contradicts to $k \geq 7$.

Case 2. $|A \cap R(B)| = 0$.

Now we have $A = R_1 \cup \{a\}$. Obviously, $|R_1| \geq 2$, and thus $|R_2| \geq |R_1| \geq 2$. Let $w \in R_1$. We verify that $N(w) \subseteq \{a\} \cup R_4 \cup (N(w) \cap N(a))$ (because $N(a) = R_1 \cup R_2 \cup R_3$ by (1)). Since G is 3-diamond-free and $|R_2| + |R_3| + |R_4| = k$, we have $|N(w)| \leq 1 + |R_4| + |N(w) \cap N(a)| \leq 3 + k - |R_2| - |R_3|$. If $|R_2| + |R_3| \geq 4$, then $|N(w)| \leq k - 1$, a contradiction. Thus, we assume that $|R_2| + |R_3| \leq 3$. But then $k = |N(a)| = |R_1| + |R_2| + |R_3| \leq 2(|R_2| + |R_3|) \leq 6$, also a contradiction. □

Lemma 4.4. *Let G be a connected degenerate transitive graph with degree $k \geq 7$. If G is 3-diamond-free, then $\omega(G) = 2$. Furthermore, each superatom of G is an independent set.*

Proof. Clearly, $\omega(G) \geq 2$. Suppose $\omega(G) \geq 3$, we will derive a contradiction. Let A be a superatom in G. Then $|A| \geq 3$ and $G[A]$ is connected by Lemma 4.2. If $\alpha \in \mathrm{Aut}(G)$, then $\alpha(A)$ is also a superatom, and so by Lemma 4.3, either $A = \alpha(A)$ or $A \cap \alpha(A) = \emptyset$. Hence A is an imprimitive block for $\mathrm{Aut}(G)$ on $V(G)$, and its translates partition $V(G)$ by the vertex transitivity of G. Lemma 4.3 now yields that $N(A)$ is partitioned by translates of A, and therefore $|N(A)| = t|A|$ for some integer $t \geq 2$ ($t = 1$ implies that $V(G) = A \cup N(A)$, which is impossible). Then we have $|A| \leq k/2$. Thus $|N[A]| = |A| + |N(A)| \leq 3k/2$. Since $|A| \geq 3$ and $G[A]$ is connected, we can select an edge $e = uv \in E(G[A])$. But then

$$|N[A]| \geq |N[\{u, v\}]| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2k - 2 > 3k/2,$$

a contradiction (the second inequality follows from G is 3-diamond-free). Thus $\omega(G) = 2$ is obtained.

Assume B is a superatom which is not an independent set. Then $|N(B)| = 2k - 2 - |N(u) \cap N(v)| \geq 2k - 4 > k$, a contradiction. □

A graph G is said to be *reducible* if there exist two vertices u and v with the same neighbor set. Otherwise, G is said to be *irreducible*. By Lemma 4.4, we obtain the following corollary.

Corollary 4.5. *Let G be a connected irreducible transitive graph with degree $k \geq 7$. If G is 3-diamond-free, then G is superconnected.*

Proof. Suppose to the contrary that G is not superconnected. Then G is degenerate. By Lemma 4.4, G has a superatom A such that $|A| = 2$ and A is an independent. But then the two vertices in A have the same neighbor set, a contradiction to the irreducibility of G. □

Let G be a connected vertex transitive graph with degree k. Assume that G is both 3-diamond-free and irreducible. Since $\kappa(G) = k$, we obtain that G is superconnected if $|V(G)| \leq k + 3$ (a minimum cutset has k vertices and one of the connected components has cardinality one). If $k = 1$, then $G \cong K_2$. If $k = 2$, then G is not superconnected if and only if $G \cong C_n (n \geq 6)$. If $k = 3$, then G is not superconnected if and only if G has girth 3 and $|V(G)| \geq 7$ by Theorem 2.5. If $k = 4$, then G is not superconnected if and only if $G \cong L(Q_3)$ or G contains diamond as a subgraph and $|V(G)| \geq 8$ by Theorem 2.6. If $k = 5$, then G is superconnected by Theorem 2.7. If $k = 6$, then by Theorem 2.8 we verify that G is not superconnected if and only if $G \cong L(C(Z_6, \{-1, -2, 1, 2\}))$. (If $\omega(G) = 3$ and let A be a superatom, then $|N(A)| \geq 2k - 5 > k$, a contradiction.) The following theorem is thus obtained.

Theorem 4.6. *Let G be a connected irreducible vertex transitive graph. If G is 3-diamond-free, then G is not superconnected if and only if one of the following conditions is satisfied:*

(i) *G is isomorphic to C_n $(n \geq 6)$, or $L(Q_3)$, or $L(C(Z_6, \{-1, -2, 1, 2\}))$; or*
(ii) *G is a connected 3-regular vertex transitive graph with girth $g(G) = 3$ and $|V(G)| \geq 7$; or*
(iii) *G is a connected 4-regular vertex transitive graph with diamond as its subgraph and $|V(G)| \geq 8$.*

Let G be a connected 3-diamond-free vertex transitive graph with degree $k \geq 7$. If G is not superconnected, then it follows from Lemma 4.4 that $\omega(G) = 2$ and each superatom of G consists of exactly two independent vertices. Now define an equivalence relation T on the vertex set of G. For v_1 and v_2 in $V(G)$, $v_1 T v_2 \Leftrightarrow N(v_1) = N(v_2)$.

According to this equivalence, $V(G)$ is partitioned into some non-empty sets, say, $A_1, A_2, \dots, A_t$. Clearly, $|A_i| \geq \omega(G) = 2$ and, for any $u \in A_i$, $A_i = \{v \in V(G) : N(v) = N(u)\}$. Thus, each A_i is an imprimitive block for G. By the vertex transitivity of G, $|A_i|$ is independent of i. It follows that $|A_i| = |V(G)|/t$ for some integer $t \geq 2$.

Now define a quotient graph $\overline{G} = G/T$ of G. The vertices of $\overline{G}$ are A_i, $i = 1, 2, \dots, t$, and A_i and A_j are adjacent in $\overline{G}$ if and only if some vertex in A_i is adjacent to some vertex in A_j in G. The following is obvious.

Lemma 4.7. *$\overline{G}$ is a connected 2-diamond-free vertex transitive graph.*

Proof. $\overline{G}$ is clearly connected and 2-diamond-free. (If there is a 2-diamond in $\overline{G}$, we can find a 4-diamond in G by $|A_i| \geq 2$ and the definition of $\overline{G}$.) By the vertex transitivity of G and the fact that A_i $(1 \leq i \leq t)$ are imprimitive blocks for G, we can see that $\overline{G}$ is vertex transitive. □

Lemma 4.8. *If G is not superconnected, then $\overline{G}$ is not superconnected.*

Proof. Let k be the degree of regularity of G and $n = |V(G)|$. By definition, if A_i and A_j $(i \neq j)$ are adjacent in $\overline{G}$, then the induced subgraph $G[A_i \cup A_j]$ is a complete bipartite graph. Thus, $\overline{G}$ has degree $k/|A_i| = kt/n$. Let C be a minimum vertex cut of G such that $G - C$ has no isolated vertices and let $G_1, G_2, \dots, G_s$ be the components of $G - C$. Then we have the following claims.

Claim 1. C is a union of some A_i's.

If not, let $u \in A_i \cap C$ and $v \in A_i$ but $v \notin C$. Set $C' = C \setminus \{u\}$. Then, since $N(u) = N(v)$, C' is also a vertex cut, which is impossible.

By a similar argument we have

Claim 2. For any $j (1 \leq j \leq s)$, $V(G_j)$ is a union of some A_i's.

Then, C corresponds to a vertex cut $\overline{C}$ of $\overline{G}$, $|\overline{C}| = |C|/m = k/m$, where $m = |A_i| = n/t$. Since $\overline{G}$ is a connected 3-diamond-free vertex transitive graph, we have $\kappa(\overline{G}) = k/m$. It follows that $\overline{C}$ is a minimum vertex cut of $\overline{G}$. Clearly, $\overline{G} - \overline{C}$ has no isolated vertices. The result follows. □

The following theorem characterizes superconnected 3-diamond-free vertex transitive graphs.

Theorem 4.9. *Let G be a connected vertex transitive graph. If G is 3-diamond-free, then G is not superconnected if and only if one of the following conditions is satisfied:*

(i) *G is isomorphic to one of the following graphs: $C_n(N_m)$ $(n \geq 6$ and $m \geq 1)$, $L(Q_3)(N_m)(m \in \{1, 2\})$ and $L(C(Z_6, \{-1, -2, 1, 2\}))$; or*
(ii) *$G \cong H(N_m)$ $(m \in \{1, 2\})$, where H is a connected 3-regular vertex transitive graph with girth $g(H) = 3$ and $|V(H)| \geq 7$; or*
(iii) *$G \cong K$, where K is a connected 4-regular vertex transitive graph with diamond as its subgraph and $|V(K)| \geq 8$.*

Proof. By Theorem 4.6, $C_n(n \geq 6), L(Q_3), L(C(Z_6, \{-1, -2, 1, 2\}))$, H and K are not superconnected. For $C_n(N_m)$ ($n \geq 6$ and $m \geq 2$), $L(Q_3)(N_2)$ and $H(N_2)$, the results follow from the definition of lexicographic product of graphs and the fact that $C_n(n \geq 6), L(Q_3)$ and H are not superconnected.

Suppose now that G is not superconnected. If G is irreducible, then by Theorem 4.6, one of the following conditions is satisfied: (1) G is isomorphic to one of the following graphs: $C_n(n \geq 6), L(Q_3)$ and $L(C(Z_6, \{-1, -2, 1, 2\}))$; (2) G is a connected 3-regular vertex transitive graph with girth $g(G) = 3$ and $|V(G)| \geq 7$; (3) G is a connected 4-regular vertex transitive graph with diamond as its subgraph and $|V(G)| \geq 8$. Thus, we assume that G is reducible in the following. Since G is not superconnected, we know that G is degenerate. If $k \leq 6$, then by Theorems 2.5–2.8 we can verify that $G \cong C_n(N_m)$ ($n \geq 6$ and $2 \leq m \leq 3$) or $G \cong H(N_2)$, where H is a connected 3-regular vertex transitive graph with girth $g(H) = 3$ and $|V(H)| \geq 7$. Thus, assume $k \geq 7$. Let A be superatom of G. By Lemma 4.4, it follows that $|A| = 2$ and A is an independent set. Now consider the quotient graph $\overline{G}$ defined above. By Lemmas 4.7 and 4.8 we see that $\overline{G}$ is a connected 2-diamond-free vertex transitive graph which is not superconnected. Since $\overline{G}$ is irreducible, Corollary 4.5 yields that the degree $\overline{k}$ of $\overline{G}$ is less than 7. By Theorems 2.5–2.8, G is 3-diamond-free and $\overline{G}$ is 2-diamond-free, we can verify that $\overline{G}$ is isomorphic to one of the following graphs: C_n ($n \geq 6$) and $L(Q_3)$. Therefore G is isomorphic to one of the following graphs: $C_n(N_m)$ ($n \geq 6$ and $m \geq 4$) and $L(Q_3)(N_2)$. (Since $L(Q_3)(N_m)$ contains 3-diamond for $m \geq 3$.) □

If G is diamond-free or 1-diamond-free, then the following two theorems are obtained by Theorem 4.9.

Theorem 4.10. *Let G be a connected vertex transitive graph. If G is diamond-free, then G is not superconnected if and only if $G \cong C_n(N_m)$ ($n \geq 6$ and $m \geq 1$), or $G \cong L(Q_3)$, or $G \cong H$, where H is a 3-regular vertex transitive graph with $g(H) = 3$ and $|V(H)| \geq 7$.*

Theorem 4.11. *Let G be a connected vertex transitive graph. If G is 1-diamond-free (that is $g(G) \geq 4$), then G is not superconnected if and only if $G \cong C_n(N_m)$ ($n \geq 6$ and $m \geq 1$).*

By Theorem 4.11, we can easily obtain the result in [9].

Corollary 4.12 ([9]). *Let G be a connected vertex transitive bipartite graph. Then G is not superconnected if and only if $G \cong C_n(N_m)$ ($n \geq 6$ and $m \geq 1$).*

Acknowledgments

We would like to thank the anonymous referees for their valuable suggestions which helped us a lot in improving the presentation of this paper.

References

[1] F. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, Journal of Graph Theory 10 (1986) 339–352.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[3] Liang Chen, The connectivity of vertex-transitive graphs with small regular degrees, Journal of Xinjiang University (Natural Science Edition) 4 (2000) 5–7.
[4] P. Dankelmann, A. Hellwig, L. Volkmann, On the connectivity of diamond-free graphs, Discrete Applied Mathematics 155 (2007) 2111–2117.
[5] M.A. Fiol, The superconnectivity of large digraphs and graphs, Discrete Mathematics 124 (1994) 67–78.
[6] Y.O. Hamidoune, Subsets with small sums in abelian groups' I: the vosper property, European Journal of Combinatorics 4 (1997) 541–556.
[7] A. Hellwig, L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: A survey, Discrete Mathematics 308 (2008) 3265–3296.
[8] A. Holtkamp, L. Volkmann, On the connectivity of p-diamond-free graphs, Discrete Mathematics 309 (2009) 6065–6069.
[9] X.D. Liang, J.X. Meng, Z. Zhang, Super-connectivity and hyper-connectivity of vertex transitive bipartite graphs, Graphs and Combinatorics 23 (2007) 309–314.
[10] M. Mader, Minimal n-fach Kantenzusammenhängenden Granphen, Mathematische Annalen 191 (1971) 21–28.
[11] Jixiang Meng, Connevtivity of vertex and edge transitive graphs, Discrete Applied Mathematics 127 (2003) 601–613.
[12] Yingzhi Tian, Jixiang Meng, Superconnected and hyperconnected 6-regular transitive graphs, Journal of Xinjiang University (Natural Science Edition) 3 (5) (2008) 253–262.
[13] Yingzhi Tian, Jixiang Meng, On super restricted edge-connectivity of edge-transitive graphs, Discrete Mathematics 310 (2010) 2273–2279.
[14] Yingzhi Tian, Jixiang Meng, Superconnected and hyperconnected small degree transitive graphs, Graphs and Combinatorics 27 (2011) 275–287.
[15] R. Tindell, Connectivity of Cayley graphs, in: D.Z. Du, D.F. Hsu (Eds.), Combinatorial Network Theory, 1996, pp. 41–64.
[16] Dameng Wang, Jixiang Meng, Superconnected and hyperconnected cubic transitive graphs, OR Transactions 5 (4) (2001) 35–40.
[17] M.E. Watkins, Connectivity of transitive graphs, Journal of Combinatorial Theory 8 (1970) 23–29.
[18] Zhao Zhang, Jixiang Meng, Super-connected edge transitive graphs, Discrete Applied Mathematics 156 (2008) 1948–1953.