Finite transition matrices for permutations avoiding pairs of length four patterns

Darla Kremer\textsuperscript{a}, Wai Chee Shiu\textsuperscript{b,1}

\textsuperscript{a}Department of Mathematics and Computer Science, Gettysburg College, Gettysburg, PA 17325-1486, USA
\textsuperscript{b}Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

Received 6 March 2000; received in revised form 20 February 2001; accepted 11 December 2002

Abstract

We show that the four classes of pattern avoiding permutations, $S_n(1234;3214)$, $S_n(4123;3214)$, $S_n(2341;2143)$ and $S_n(1234;2143)$ are enumerated by the formula $(4^n - 1 + 2)/3$. In an electronic appendix we provide finite transition matrices for the number $|S_n(u,v)|$ of permutations avoiding pairs $(u,v)$ of length four patterns where $u$ contains the subsequence 123 and $v$ contains the subsequence 321 as well as a transition matrix for $|S_n(1234,43215)|$.

\textcopyright 2003 Elsevier Science B.V. All rights reserved.

Keywords: Forbidden subsequences; Pattern avoiding permutations; Restricted permutations

1. Introduction

Let $S_n$ denote the symmetric group on $[n] = \{1,2,\ldots,n\}$. For $\pi \in S_{n-1}$ and $j \in [n]$, let $\pi_j$ be the permutation in $S_n$ obtained from $\pi$ by inserting $n$ into the $j$th position. That is,

$$\pi_j = \left( \begin{array}{c} i \\ \pi_j(i) \end{array} \right) = \left( \begin{array}{cccccccc} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(j-1) & \pi(j) & \pi(j+1) & \cdots & \pi(n-1) \end{array} \right).$$

\textsuperscript{1}A portion of this research was completed while on sabbatical at Massachusetts Institute of Technology in 1998.

E-mail address: dkremer@gettysburg.edu (D. Kremer).
Writing only the bottom line of the above two-line notation, we view permutations as sequences.

**Definition 1.** Let $\gamma \in S_k$. A permutation $\pi \in S_n$, is said to be $\gamma$-avoiding if there is no sequence of integers $i_1, i_2, \ldots, i_k$ such that $1 \leq i_{(1)} < i_{(2)} < \cdots < i_{(k)} \leq n$ and $\pi(i_{(1)}) < \pi(i_{(2)}) < \cdots < \pi(i_{(k)})$. The subsequence $\{\pi(i_{(j)})\}_{j=1}^k$ is said to have type $\gamma$. We write $S_n(\gamma)$ for the set of $\gamma$-avoiding permutations of length $n$. More generally, if $\Gamma$ is a set of permutations, let $S_n(\Gamma) = \bigcap_{\gamma \in \Gamma} S_n(\gamma)$ be the set of all permutations in $S_n$ which avoid every $\gamma \in \Gamma$.

**Definition 2.** If $\pi \in S_{n-1}(\Gamma)$, then we call $j$ an active site if $\pi_j \in S_n(\Gamma)$. $\pi_j$ will be referred to as a child of $\pi \in S_{n-1}(\Gamma)$.

Note that if $\pi \notin S_{n-1}(\Gamma)$ then $\pi_j \notin S_n(\Gamma)$ for any $j$. Thus, if $k(\pi)$ is equal to the number of active sites in $\pi$, then $|S_n(\Gamma)| = \sum_{\pi \in S_{n-1}(\Gamma)} k(\pi)$.

For example, if $\Gamma = \{132, 231\}$, then $S_4(132, 231)$ is the set of permutations in $S_4$ none having a three element subsequence in which the middle element is the largest. $S_4(132, 231) = \{4321, 3214, 4213, 2134, 4312, 3124, 4123, 1234\}$, and $|S_4(132, 231)| = 8$. The active sites (indicated by arrows) of $^123^1$ are sites 1 and 4. Schmidt and Simion [10] enumerated $S_n(\Gamma)$ for all $\Gamma \subseteq S_3$. In the case where $\Gamma$ consists of a single permutation in $S_3$, the reader is referred to work of Babson and West [2], Bona [3,4], Regev [9], Stankova [11], and West [12–14]. In [13], a complete enumeration of pairs of patterns $(u,v)$ where $u \in S_3$ and $v \in S_4$ is given.

The technique of West, which we employ, is that of producing transition matrices for $|S_n(\Gamma)|$. This is an inductive procedure which assigns to each permutation $\pi$ in $S_{n-1}(\Gamma)$ a label encoding both the number $k$ of active sites in $\pi$ and a recursive rule for determining the label on each $\pi_j \in S_n(\Gamma)$, where $1 \leq j \leq k$ and $1 \leq i_1 < \cdots < i_k \leq n$. Given labels on $\pi \in S_n(\Gamma)$ for some initial value $m$, the number of permutations in $S_n(\Gamma)$ having label $(x)$ (denoted $(x)_m$) is recursively determined. $|S_n(\Gamma)|$ is obtained by summing $(x)_m$ over all distinct labels $x$.

**Definition 3.** For fixed $m \geq 1$, a labeling on $\{\pi \in S_n(\Gamma) | n \geq m\}$ consists of:

1. An assignment of labels $(x)$ to each $\pi \in S_n(\Gamma)$.
2. For each label assigned to $\pi \in S_{n-1}(\Gamma)$ a succession rule describing the number of children of $\pi$ contained in $S_n(\Gamma)$ and the label on each child.

We remark that a labeling on a set of pattern avoiding permutations may not exist. If a labeling exists, then we form transition matrices, defined in **Definition 4**. A transition matrix may be finite or infinite, depending on the number of distinct labels used in a labeling scheme.

**Definition 4.** Let $l$ be the number of distinct labels assigned to permutations in $\{\pi \in S_n(\Gamma) | n \geq m \geq 1\}$. A transition matrix $A$ is the $l \times l$ matrix whose rows and columns are indexed by the labels and whose $(a,b)$th entry specifies the number
of permutations in \( S_n(\Gamma) \) having label \((x_r)\) produced by a permutation in \( S_{n-1}(\Gamma') \) having label \((x_c)\). Let \( e_c \) be the number of permutations in \( S_n(\Gamma) \) having label \((x_r)\), for some fixed (small) \( m \). Then, the matrix transition equation

\[
\begin{pmatrix}
(x_1) & (x_2) & \cdots & (x_n)
\end{pmatrix}^T = A^{n-m}(e_1 \quad e_2 \quad \cdots \quad e_l)^T
\]

gives the number of permutations in \( S_n(\Gamma) \) having the labels \((x_r)\), \( 1 \leq r \leq l \).

A simple example is given by \( S_n(132,231) \). Any permutation in \( S_{n-1}(132,231) \) has two active sites, since both the first and the last site of \( \pi \) are active and for \( n \geq 3 \), insertion of \( n \) into any other site creates a forbidden subsequence. Thus, the label on \( 1 \in S(132,231) \) is \((2)\) and the recursive rule is that \( \pi_1 \) and \( \pi_2 \) both have label \((2)\). The transition matrix is the \( 1 \times 1 \) matrix \( A = (2) \) and \( e_1 = 1 \). Thus \( |S_n(132,231)| = (2)^n = 2^{n-1} \).

It is convenient to define the reversal, \( \tilde{\sigma} \in S_n \) of a permutation \( \sigma \in S_n \) as \( \tilde{\sigma}(i) = \sigma(n+1-i) \), and the complement, \( \sigma^* \in S_n \) of \( \sigma \) as \( \sigma^* = n+1-\sigma(i) \). The following lemma (taken from [10]) limits the number of cases which need to be enumerated. In [14], Lemma 5 is formulated in terms of an action of the dihedral group \( D_4 \) on permutation matrices.

**Lemma 5.** For any set \( \Gamma \) of permutations in \( S_k \), let \( \tilde{\Gamma} = \{ \tilde{\gamma} \in \Gamma \}, \Gamma^* = \{ \gamma^* : \gamma \in \Gamma \} \), and \( \Gamma^{-1} = \{ \gamma^{-1} : \gamma \in \Gamma \} \). Then,

\[ \pi \in S_n(\Gamma) \iff \tilde{\pi} \in S_n(\tilde{\Gamma}) \iff \pi^* \in S_n(\Gamma^*) \iff \pi^{-1} \in S_n(\Gamma^{-1}) \].

Thus, \( |S_n(\Gamma)| = |S_n(\tilde{\Gamma})| = |S_n(\Gamma^*)| = |S_n(\Gamma^{-1})| \).

For the purpose of enumerating \( S_n(\Gamma) \), we say that the sets \( \Gamma, \tilde{\Gamma}, \Gamma^*, \) and \( \Gamma^{-1} \) as defined in Lemma 5 are equivalent.

In this paper, we are concerned with the enumeration of permutations which avoid pairs of length four patterns. There are 56 inequivalent classes of permutations to consider, listed in Table 1. Some of these distinct classes are known to be equinumerous. In [5,11] Bona and Stankova find five permutation classes which are enumerated by the generating function for the class of \((1324,2143)\)-avoiding permutations, the so-called smooth permutations. A permutation \( \pi \) in this class has the property that the Schubert cell indexed by \( \pi \) is smooth. Gire, Kremer and West [7,8,12] give ten inequivalent classes which are enumerated by the Schröder numbers. Numerical data (see Table 1) provided by West for \( |S_n(u,v)|, (u,v) \in S_4 \times S_4, \) and \( n < 11 \) suggests that of the remaining 41 classes, 36 of them are enumeratively distinct. The data provides (for \( n \leq 11 \)) three exceptions:

1. \( |S_n(1243,2134)| = |S_n(1342,3124)| \),
2. \( |S_n(1342,2143)| = |S_n(1432,2413)| \), and
3. \( |S_n(1234,2314)| = |S_n(4123,3214)| = |S_n(2341,2143)| = |S_n(1234,2143)| \).

Of the distinct classes, we know of only one other enumerative result. Atkinson [1] showed that the number of skew-merged, or \((2143,3412)\)-avoiding permutations is given by the formula

\[
\binom{2n}{n} - \sum_{m=0}^{n-1} 2^{n-m-1} \binom{2m}{m}.
\]
Table 1
Data on (4,4)-pairs

| $\Gamma$ | $|S_n(\Gamma)|$, $n = 5, 6, 7, 8, 9, 10, 11$ | Reference |
|----------|-----------------------------------|-----------|
| 1234,4321 | 86,306,882,1764,1764,0,0 | Lemma 6 (Erdős and Szekeres) |
| 1234,3421 | 86,321,1085,3266,8797,21478,48206 | Example 18 |
| 1234,3241 | 86,330,1198,4087,13185,40619,120636 | Example 19 |
| 1243,3421 | 86,330,1206,4174,13726,43134,130302 | Example 14 |
| 1234,4231 | 86,332,1217,4140,12934,37088,98115 | Open |
| 1234,3412 | 86,333,1235,4339,14443,45770,138988 | Open |
| 1234,4231 | 86,335,1266,4598,16016,53579,172663 | Open |
| 1234,3412 | 86,335,1271,4680,16766,58656,201106 | Open |
| 1234,4231 | 86,336,1282,4758,17234,61242,214594 | Open |
| 1243,3214 | 86,336,1290,4870,18164,67234,247786 | Example 17 |
| 1243,3412 | 86,337,1295,4854,17760,63594,223488 | Open |
| 1234,3241 | 86,337,1299,4910,18228,66640,240550 | Example 16 |
| 1234,4213 | 86,338,1314,5046,19190,72482,272530 | Open |
| 1243,3241 | 86,338,1318,5106,19718,76066,293398 | Example 13 |
| 1234,3214 | 86,338,1318,5110,19770,76466,295810 | Example 15 |
| 1243,3412 | 86,340,1340,5254,20518,79932,311028 | Atkinson [1] |
| 1243,3214 | 86,342,1366,5462,21846,87382,349526 | Theorem 8 |
| 2143,2143 | 87,348,1374,5335,20462,77988,296787 | Open |
| 1432,3214 | 87,352,1428,5768,23156,92416,367007 | Open |
| 1432,4123 | 87,352,1434,5861,24019,98677,406291 | Open |
| 1432,2134 | 87,354,1459,6056,25252,105632,442916 | Open |
| 1432,3124 | 88,363,1507,6241,25721,105485,430767 | Open |
| 1432,2431 | 88,363,1508,6255,25842,106327,435965 | Open |
| 1243,2431 | 88,365,1540,6568,28269,122752,537708 | Open |
| 1243,2143 | | Bona [5] |
| 1342,2134 | | |
| 1243,3241 | 88,366,1556,6720,29396,129996,580276 | Open |
| 1243,2413 | 88,367,1568,6810,29943,132958,595227 | Open |
| 1243,3124 | 88,367,1571,6861,30468,137229,625573 | Open |
| 1243,2143 | 88,368,1584,6968,31192,141656,651136 | Open |
| 1432,2413 | 89,376,1611,6901,29375,123996,518971 | Open |
| 1243,2341 | 89,379,1664,7460,33977,156727,730619 | Open |
| 1234,1432 | 89,380,1677,7566,34676,160808,752608 | Open |
| 1234,1342 | 89,380,1678,7584,34875,162560,766124 | Open |
| 1432,2143 | 89,381,1696,7781,36572,175277,853410 | Open |
| 1243,1432 | 89,382,1711,7922,37663,182936,904302 | Open |
| 2413,3142 | | |
| 1342,2314 | | Gire [7] |
| 1342,2341 | | |
Table 1 (continued)

| $\Gamma$        | $|S_n(\Gamma)|$, $n = 5, 6, 7, 8, 9, 10, 11$ | Reference          |
|-----------------|--------------------------------------------|--------------------|
| 3124,3214       | 90,394,1806 .8558,41586,206098,1037718     | Kremer [8]         |
| 3142,3214       |                                            | West [8]           |
| 3412,3421       |                                            |                    |
| 1324,2134       |                                            |                    |
| 3124,2314       |                                            |                    |
| 2134,3124       |                                            |                    |
| 2143,2413       | 90,395,1823,8741,43193,218704,1129944      | Open              |
| 1234,1324       | 90,396,1837,8864,44074,224352,1163724      | Zeilberger$^a$     |

$^a$Via the package WILF (see http://www.math.temple.edu/~zeilberger).

The Maple package WILF, created by Doron Zeilberger to automatically generate algorithms for the enumeration of pattern avoiding permutations, found an enumeration scheme of depth 4 for $|S_n(1234,1324)|$. This is example 9 on the package’s homepage http://www.math.temple.edu/~zeilberger. WILF obtained an enumeration scheme of depth 4 for $(1234,1243)$ (equivalent to $(1234,2134)$ in Table 1), one of depth 5 for the pair $(1234,3214)$ and one of depth 7 for the pair $(1234,43215)$, but was unable to obtain enumeration schemes of depth $\leq 7$ for any of the other open pairs. An exposition on the package WILF is given in [15].

The first two equivalences suggested by the data, $|S_n(1234,2134)| = |S_n(1342,3214)|$ and $|S_n(1342,2143)| = |S_n(1432,2413)|$, remain conjectures. In Section 2, we prove that

$$|S_n(1234,3214)| = |S_n(4123,3214)| = |S_n(2341,2143)|$$

$$= |S_n(1234,2143)| = \frac{4^{n-1} + 2}{3}.$$

The pairs $(1234,3214)$ and $(4123,3214)$ are particularly tractable since $u$ contains an increasing subsequence of length three and $v$ contains a decreasing subsequence of length three. A well-known result [6, p. 160] of Erdős and Szekeres given in Lemma 6 implies that any $\pi$ in the set of permutations avoiding such pairs has at most five active sites.

**Lemma 6** (Erdős and Szekeres). *Any sequence of $ml + 1$ real numbers has either an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $l + 1$.*

In particular, Lemma 6 shows that $|S_n(1234,4321)| = 0$ for $n \geq 10$. For $n = 1, 2, \ldots, 9$, the sequence $1, 2, 6, 22, 86, 306, 882, 1764, 1764$, provided by West, enumerates $S_n(1234,4321)$.

In the electronic appendix we give transition matrices for the remaining seven classes of permutations which avoid $(u,v) \in S_4 \times S_4$ where $u$ contains the subsequence 123 and $v$ contains the subsequence 321. In each case, these matrices are finite. This idea is extended to enumerate one additional class $|S_n(1234,43215)|$. Unfortunately, each case
seems to require a unique argument. It is our hope that a comparative study of the results here as well as those in [13] will yield a more general approach to these enumeration problems.

Lemma 7 gives recursive upper and lower bounds for certain \(|S_n(u,v)|\).

**Lemma 7.** Let \(u = u_1 \cdots u_m\) and \(v = v_1 \cdots v_l\) be patterns of length \(m\) and \(l\), respectively, with \(m \leq l\).

1. If \(u_t = m\) and \(v_s = l\) then any permutation which avoids both \(u\) and \(v\) has at least \(\min\{t-1, s-1\} + \min\{m-t, l-s\}\) active sites. For \(n > m\), \(|S_n(u,v)| \geq k|S_{n-1}(u,v)|\), where \(k = \min\{t-1, s-1\} + \min\{m-t, l-s\}\).

2. If \(u\) contains the subsequence \(12 \cdots (m-1)\) and \(v\) contains the subsequence \((l-1)(l-2)\cdots 21\), then any permutation which avoids both \(u\) and \(v\), has at most \((m-2)(l-2) + 1\) active sites. In this case, \(|S_n(u,v)| \leq [(m-2)(l-2) + 1] |S_{n-1}(u,v)|\).

**Proof.** Let \(\pi \in S_{n-1}(u,v)\). Because the maximum element in the pattern to be avoided has either \(t-1\) or \(s-1\) numbers to its left the \(\min\{t-1, s-1\}\) sites at the beginning of \(\pi\) are always active. Similarly, the \(\min\{m-t, l-s\}\) sites at the end of \(\pi\) are active, proving part 1.

If \(\pi = p_1 p_2 \cdots p_{n-1}\) has \((m-2)(l-2) + 2\) or more active sites, then there exists a subsequence of \(\pi\), \(q_1 q_2 \cdots q_{(m-2)(l-2)+1}\), with active sites as indicated. By Lemma 6, \(q_1 \cdots q_{(m-2)(l-2)+1}\) has a subsequence of length \(m-1\) or a decreasing subsequence of length \(l-1\). Insertion of \(n\) into one or more active sites creates either a forbidden subsequence of type \(u\) or one of type \(v\). \(\square\)

2. Permutations enumerated by the formula \((4^n-1+2)/3\)

**Theorem 8.** For \(n \geq 2\),

\[
|S_n(1234,3214)| = |S_n(4123,3214)| = |S_n(2341,2143)| = |S_n(1234,2143)| = \frac{4^n-1+2}{3}.
\]

We prove Theorem 8 case by case in Propositions 9–12. Finite transition matrices are provided for the pairs \((1234,3214)\) and \((4123,3214)\). The succession rules for the pair \((1234,2143)\) imply an infinite transition matrix, and the pair \((2341,2143)\) is enumerated without actually describing the succession rules. A recursively defined bijection between \(S_n(1234,3214)\) and \(S_n(4123,3214)\) is provided, but the succession rules do not make clear a bijection between any of the other sets of permutations enumerated by \((4^n-1+2)/3\).

**Proposition 9.** For \(n > 1\), \(|S_n(1234,3214)| = (4^n+1)/3\). A matrix transition equation for \(|S_n(1234,3214)|\) is given by
Proof. Let \( \pi = p_1 p_2 \cdots p_{n-1} \in S_{n-1}(1234,3214) \) have \( k \) active sites. If \( p_1 < p_2 \), we give \( \pi \) the label \((ki)\). If \( p_1 > p_2 \), label \( \pi \) with \((kd)\). By Lemma 7, \( 3 \leq k \leq 5 \). Since 4 occurs at the end of both forbidden patterns, these are the first \( k \) sites. This allows us to reduce our study of these permutations to the pattern formed by \( p_1 p_2 p_3 p_4 \). In particular:

- If \( \pi \) has label \((5i)\), then \( p_1 p_2 p_3 p_4 \) is of type 2413 or 3412.
- If \( \pi \) has label \((5d)\), then \( p_1 p_2 p_3 p_4 \) is of type 3124 or 2143.
- If \( \pi \) has label \((4i)\), then \( p_1 p_2 p_3 \) is of type 132 or 231.
- If \( \pi \) has label \((4d)\), then \( p_1 p_2 p_3 \) is of type 213 or 312.
- If \( \pi \) has label \((3i)\), then \( p_1 p_2 \) has type 12, and
- If \( \pi \) has label \((3d)\), then \( p_1 p_2 \) has type 21.

Let \( \pi_j \in S_n(1234,3214) \) obtained from \( \pi \) by inserting \( n \) into the \( j \)th site of \( \pi \). The following tables summarize the recursive rules for labeling permutations. The reader is encouraged to check that active sites in \( \pi \) which become inactive in \( \pi_j \) (i.e., an \( \downarrow \) is replaced by a \( \times \)) occur to the right of an increasing subsequence of length three ending in \( n \) or a decreasing subsequence of length three beginning with \( n \).

<table>
<thead>
<tr>
<th>Active sites</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((3i))</td>
</tr>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((4i))</td>
</tr>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((5i))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Active sites</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((3i))</td>
</tr>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((4i))</td>
</tr>
<tr>
<td>( p_1 p_2 p_3 \cdots p_{n-1} )</td>
<td>((5i))</td>
</tr>
</tbody>
</table>
The matrix transition equation results in the following system of equations:

\[
\begin{align*}
(3i)_n &= (3i)_{n-1} + (4i)_{n-1} + (5i)_{n-1} \\
(3d)_n &= (3d)_{n-1} + (4d)_{n-1} + (5d)_{n-1} \\
(4i)_n &= (3i)_{n-1} + (3d)_{n-1} + 2(4i)_{n-1} + 2(5i)_{n-1} \\
(4d)_n &= |S_{n-1}(1234,3214)| \\
(5i)_n &= (4d)_{n-1} + (5i)_{n-1} + (5d)_{n-1} \\
(5d)_n &= (4d)_{n-1} + 2(5d)_{n-1} \\
(ki)_n &= (kd)_n \text{ for } k = 3, 4, 5.
\end{align*}
\]

Let \((k)_n = (ki)_n + (kd)_n\). It is easy to see that

\[
\begin{align*}
|S_n(1234,3214)| &= 3(3)_{n-1} + 4(4)_{n-1} + 5(5)_{n-1} \\
&= 4(|S_{n-1}(1234,3214)|) + (5)_{n-1} - (3)_{n-1} \\
&= 4(|S_{n-1}(1234,3214)|) - 2.
\end{align*}
\]

The last equality follows from observing that \((5)_{n} - (3)_{n} = (5)_{n-1} - (3)_{n-1}\), so is constant and equal to \((5)_{3} - (3)_{3} = -2\).

Solving the recurrence \(|S_n(1234,3214)| = 4(|S_{n-1}(1234,3214)|) - 2\) with the initial condition \(|S_1| = 1\) gives the enumerative result \(S_n(1234,3214) = (4^{n-1} + 2)/3\). \(\square\)

**Proposition 10.** For \(n > 1\), \(|S_n(4123,3214)| = (4^{n-1}+2)/3\). There is a bijection between \(S_n(4123,3214)\) and \(S_n(1234,3214)\). A matrix transition equation for \(|S_n(4123,3214)|\) is given by

\[
\begin{pmatrix}
(3i)_n \\
(3d)_n \\
(4i)_n \\
(4d)_n \\
(5i)_n \\
(5d)_n
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & \cdots & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 2 & 0 & 1 & 1 \\
1 & 1 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}^{n-2}
\begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Before proving Proposition 10, we introduce some notation. Let \(\pi \in S_{n-1}(\Gamma)\), for some \(\Gamma \subset S_r\). Let \(\pi' \in S_r(\Gamma)\) be obtained from \(\pi\) by inserting \(n\) into the \(j\)th active site of \(\pi\). From the proof of Proposition 9, we see that if \(\pi \in S_{n-1}(1234,3214)\), then the active sites will be the leftmost \(k\) sites, so \(\pi_j = \pi'_j, \ 1 \leq j \leq k\).
Proof (of Proposition 10). Let \( \pi \in S_{n-1}(4123, 3214) \) have \( k \) active sites. As in Proposition 9, \( 3 \leq k \leq 5 \). In this case, the inactive sites will be at the beginning and/or at the end of \( \pi \), and forbidden subsequences occur either to the left of an increasing subsequence of length three ending in \( n \), or to the right of a decreasing subsequence of length three beginning with \( n \). Let \( \bar{p}_1 \bar{p}_2 \cdots \bar{p}_{k-1} \) represent the subsequence of \( \pi \) occurring to the right of the first active site and to the left of the last active site. That is, \( \bar{p}_i \) is to the right of the \( i \)th active site of \( \pi \), \( 1 \leq i \leq k-1 \). Give \( \pi \) the label \((ki)\) if \( \bar{p}_i < \bar{p}_2 \) and \((kd)\) if \( \bar{p}_i > \bar{p}_2 \). The recursive rules are those of Proposition 9 with the following modifications:

<table>
<thead>
<tr>
<th>Label on ( \pi \in S_{n-1}(\Gamma) )</th>
<th>Descendant of ( \pi ) in ( S_n(\Gamma) )</th>
<th>Label on ( \pi' = \pi_j ) in ( S_n(1234, 3214) )</th>
<th>Label on ( \pi' ) in ( S_n(4123, 3214) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4i))</td>
<td>( \pi^3 )</td>
<td>( (3i) )</td>
<td>( (4i) )</td>
</tr>
<tr>
<td>((4i))</td>
<td>( \pi^4 )</td>
<td>( (4i) )</td>
<td>( (4d) )</td>
</tr>
<tr>
<td>((4d))</td>
<td>( \pi^4 )</td>
<td>( (4d) )</td>
<td>( (3i) )</td>
</tr>
<tr>
<td>((5i))</td>
<td>( \pi^3 )</td>
<td>( (3i) )</td>
<td>( (5i) )</td>
</tr>
<tr>
<td>((5i))</td>
<td>( \pi^4 )</td>
<td>( (4i) )</td>
<td>( (5d) )</td>
</tr>
<tr>
<td>((5i))</td>
<td>( \pi^5 )</td>
<td>( (5i) )</td>
<td>( (3i) )</td>
</tr>
<tr>
<td>((5d))</td>
<td>( \pi^4 )</td>
<td>( (4d) )</td>
<td>( (4i) )</td>
</tr>
<tr>
<td>((5d))</td>
<td>( \pi^5 )</td>
<td>( (5d) )</td>
<td>( (4d) )</td>
</tr>
</tbody>
</table>

This table provides a bijection between \( S_n(1234, 3214) \) and \( S_n(4123, 3214) \) defined recursively on the \( k \) descendants of \( \pi \in S_n(\Gamma) \) (for example, \([(4i), \pi^3] \in S_n(1234, 3214) \leftrightarrow [(4d), \pi^4] \in S_n(4123, 3214)] \) Each \([(kx), \pi']\] not listed (in the first two columns) will have the same recursive rule in both sets of pattern avoiding permutations (the last two columns will agree). \( \square \)

Proposition 11. For \( n > 1 \), \(|S_n(1234, 2143)| = (4^{n-1} + 2)/3\). The rules for generating \( S_n(1234, 2143) \) are given by

- \( \pi = 1 \) is labeled \((2, 3)\).
- \( \pi \) is labeled \((k, x)\) where \( k \geq 2 \) and \( 2 < x \leq k + 1 \),

\[
\begin{array}{|c|c|c|c|}
\hline
\pi & \pi^1 & \pi^2 & \pi^3, \ldots, \pi^k \\
\hline
(k, 2) & (k + 1, 4) & (k + 1, 2) & (3, 2) \\
\hline
\end{array}
\]

and for \( k \geq 2 \) and \( 2 < x \leq k + 1 \),

\[
\begin{array}{|c|c|c|c|}
\hline
\pi & \pi^1 & \pi^2 & \pi^3, \pi^4, \ldots, \pi^{x-1}, \pi^x, \ldots, \pi^k \\
\hline
(k, x) & (k + 1, x + 1) & (k + 1, 2) & (k, 3), (k - 1, 3), \ldots, (k - x + 4, 3) \\
\hline
\end{array}
\]

(2, 3)

Proof. Give \( \pi \in S_{n-1}(1234, 2143) \) the label \((k, x)\) where \( k \geq 2 \) is the number of active sites in \( \pi \) and \( x \) is defined according to the position of the first ascent in \( \pi \). If \( p_1 < p_2 \),
then let \( x = 2 \). If \( p_1 > p_2 \), and there exists some \( p_{i-1} < p_i \) with an active site to the right of \( p_i \), then let \( x \) be the first such active site (\( 3 \leq x \leq k + 1 \)). If \( \pi \) is strictly decreasing, or there are no active sites to the right of the first ascent, then let \( x = k + 1 \).

The first two sites are active and insertion of \( n \) into these sites will not create a subsequence of type 123 or 213 ending in \( n \), so \( \pi_1 \) and \( \pi_2 \) have \( k + 1 \) active sites. Inserting \( n \) at the beginning of \( \pi \) increases the position of the first ascent, if it exists, by one. For \( x = 2 \), \( \pi_1 \) is labeled \((k + 1, 4)\) and for \( x > 2 \), \( \pi_1 \) is labeled \((k + 1, x + 1)\). \( \pi_2 \) is labeled \((k + 1, 2)\) regardless of the value of \( x \), since \( p_1 < n \).

If \( x = 2 \) and \( j = 3 \), then \( p_1 p_2 p_3 \) has type 123, so \( \pi_3 \) is labeled \((3, 2)\). For \( x = 2 \) and \( j > 3 \), \( p_3 \) must be less than \( p_2 \). \( p_1 p_2 n \) has type 123, so all sites to the right of \( n \) are inactive and \( p_2 p_3 n \) is of type 213, so all sites between \( p_3 \) and \( n \) are inactive. Thus \( \pi' \) is also labeled \((3, 2)\).

If \( x > 2 \) and \( 3 \leq j \leq x - 1 \), then \( p_1 p_2 n \) forms a subsequence of type 213, so all sites between \( p_2 \) and \( n \) are inactive in \( \pi' \). No new subsequence of type 123 is formed in this case and \( n \) will be the first ascent. \( \pi' \) is labeled \((k - j + 3, 3)\).

If \( x > 2 \) and \( j \leq x \leq k + 1 \), then \( p_1 p_2 n \) forms a subsequence of type 213 and \( abn \) is of type 123, where \( a < b \) represents the first ascent. In this case, only the first two sites are active, and \( \pi' \) gets label \((2, 3)\).

To enumerate \(|S_n(1234, 2143)|\), let \((k)_n\) denote the number of permutations in \(S_n(1234, 2143)\) having \( k \) children. Then, \(|S_n(1234, 2143)| = \sum_{k=2}^{n+1} (k)_n = \sum_{k=2}^{n+1} k(k)_{n-1}\).

Careful inspection of the recursive rules for generating these permutations yields the following information:

For \( 4 \leq k \leq n + 1 \),

\[
(k)_n = 2(k - 1)_{n-1} + (k - 1)_{n-2} + (k - 1)_{n-3} + \cdots + (k - 1)_{k-2}. \tag{1}
\]

To see this, we observe

\[
(k, 2)_n = (k - 1)_{n-1},
\]

\[
(k, 3)_n = \sum_{x=4}^{k+1} (k, x)_{n-1} + \sum_{x=5}^{k+2} (k + 1, x)_{n-1} + \cdots + \sum_{x=n+4-k}^{n+1} (n, x)_{n-1}
\]

\[
= (k - 1)_{n-2} + (k - 1)_{n-3} + \cdots (k - 1)_{k-2}
\]

and

\[
\sum_{x=4}^{k+1} (k, x) = (k - 1)_{n-1}.
\]

\[
(3)_n = 2(2)_{n-1} + (2)_{n-2} + \cdots + (2)_1 + |S_{n-1}(1234, 2143)| - 1. \tag{2}
\]

Let \((3, 2)_n\) denote the number of permutations having label \((3, 2)_n\) generated via rule 1 (by the label \((k, 2)_{n-1}, k \geq 3\)), and let \((3, 2)^c_n\) denote the number generated via rule
2, so \((3, 2)_n^2 = (2, 3)_{n-1} = (2)_{n-1}\).

\[
(3, 2)_n^1 = \sum_{k=3}^{n} (k - 2)(k, 2)_{n-1}
\]

\[
= (3, 2)_n^1 + (3, 2)_{n-1}^2 + \sum_{k=4}^{n} (k - 2)(k, 2)_{n-1}
\]

\[
= (3, 2)_n^1 + (2)_{n-2} + \sum_{k=4}^{n} (k - 1)(k - 1)_{n-2} - \sum_{k=4}^{n} (k - 1)_{n-2}
\]

\[
= (3, 2)_n^1 + |S_{n-1}(1234, 2143)| - |S_{n-2}(1234, 2143)|
\]

\[
= |S_{n-1}(1234, 2143)| - 1.
\]

This last equality uses the initial conditions \((3, 2)_2^1 = 1, \ |S_2| = 2\). Eq. (2) follows from

\[
(3)_n = (3, 2)_n + (3, 3)_n + (3, 4)_n
\]

\[
= (3, 2)_n^1 + 2(2)_{n-1} + (2)_{n-2} + \sum_{k=3}^{n} (k, k + 1)_{n-2}
\]

\[
= (3, 2)_n^1 + 2(2)_{n-1} + (2)_{n-2} \cdots + (2)_1.
\]

\[
(2)_n = 2|S_{n-2}(1234, 2143)| + (2)_{n-1} - 2.
\]

\[
(2)_n = \sum_{k=3}^{n} \sum_{x=3}^{k} (k + 1 - x)(k, x)_{n-1}
\]

\[
= \sum_{k=3}^{n} (k - 2)(k, 3)_{n-1} + \sum_{k=4}^{n} (k - 3)(k, 4)_{n-1} + \sum_{k=5}^{n} \sum_{x=5}^{k} (k + 1 - x)(k, x)_{n-1}
\]

\[
= \sum_{k=3}^{n} [(k - 1) - 1][(k - 1)_{n-3} + (k - 1)_{n-2} + \cdots (k - 1)_1]
\]

\[
+ \sum_{k=4}^{n} ((k - 1) - 2)(k - 1, 2)_{n-2} + \sum_{k=4}^{n} (k - 3)(k - 1, 3)_{n-2}
\]

\[
+ \sum_{k=5}^{n} \sum_{x=5}^{k} ((k - 1) + 1 - x)(k - 1, x)_{n-2}
\]
Proposition 12. For \( n \geq 1 \), \( |S_n(2341,2143)| = (4^{n-1} + 2)/3 \).

Proof. Let \((k)_n\) denote the number of \( \pi = p_1 p_2 \cdots p_n \in S_n(2341,2143) \) which have \( k \) active sites. For any \( \pi \in S_n(2341,2143) \), \( 3 \leq k \leq n + 1 \), since 4 is in the third position in both patterns 2341 and 2143. The first two sites and the last site of \( \pi \) will be active.

We compute \( |S_{n+2}(2341,2143)| \) by first generating \( n + 1 \) permutations in \( S_{n+1} \) corresponding to each \( \pi \in S_n(2341,2143) \), two of which have \( k + 1 \) active sites, and the remaining \( n - 1 \) have three active sites. We show that the \((n+1)|S_n(2341,2143)| - |S_{n+1}(2341,2143)|\) of these permutations which are not in \( S_{n+1}(2341,2143) \) have three active sites. Thus,

\[
|S_{n+2}(2341,2143)| = 2\sum_{k=3}^{n+1} (k+1)(k)_n + 3\{(n-1)|S_n(2341,2143)| - [(n+1)|S_n(2341,2143)| - |S_{n+1}(2341,2143)|]\}
\[
= 2(|S_{n+1}(2341,2143)| + |S_n(2341,2143)|) + 3(|S_{n+1}(2341,2143)|) - 2(|S_n(2341,2143)|)
\]
\[
= 5|S_{n+1}(2341,2143)| - 4|S_n(2341,2143)|.
\]

Summing Eqs. (1)–(3)

\[
(2)_n = (2)_{n-1} + 2|S_{n-2}(2341,2143)| - 2,
\]

\[
(3)_n = 2(2)_{n-1} + (2)_{n-2} + (2)_{n-3} + \cdots + (2) + |S_{n-1}(2341,2143)| - 1,
\]

\[
(k)_n = 2(k-1)_{n-1} + (k-1)_{n-2} + (k-1)_{n-3} + \cdots + (k-1)_{k-2},
\]

we conclude that \( |S_n(2341,2143)| = 3|S_{n-1}(2341,2143)| + 3|S_{n-2}(2341,2143)| + |S_{n-3}(2341,2143)| + \cdots + |S_3| - 3 + (2)_{n-1} \). Therefore, \( |S_n(2341,2143)| - |S_{n-1}(2341,2143)| = 3|S_{n-1}(2341,2143)| - 2|S_{n-3}(2341,2143)| + (2)_{n-1} - (2)_{n-2} \).

Since \( (2)_{n-1} = (2)_{n-2} + 2|S_{n-3}(2341,2143)| - 2 \),

\[
|S_n(2341,2143)| - |S_{n-1}(2341,2143)| = 3|S_{n-1}(2341,2143)| - 2.
\]

Proposition 12. For \( n > 1 \), \( |S_n(2341,2143)| = (4^{n-1} + 2)/3 \).
Subtract $4|S_{n+1}(2341,2143)|$ from both sides to see that $|S_{n+2}(2341,2143)| - 4|S_{n+1}(2341,2143)|$ is constant. Since $|S_1| = 1$ and $|S_2| = 2$, $|S_n(2341,2143)| = 4|S_{n-1}(2341,2143)| - 2$.

Let $\pi \in S_n(2341,2143)$ have $k$ active sites. Consider the $n+1$ permutations in $S_{n+1}$, $i\sigma_i$, where $i$ ranges from 1 to $n+1$ and $\sigma_i = q_1 q_2 \cdots q_n$ is the permutation of $[n+1]\setminus \{i\}$ of type $\pi$. If $i\sigma_i \notin S_{n+1}(2341,2143)$ then $i$ must participate in any forbidden subsequence, since $\sigma_i$ has no forbidden subsequences.

When $i = 1$ and when $i = n+1$, $i\sigma_i \in S_{n+1}(2341,2143)$ and has $k+1$ active sites. For each $1 < i < n+1$, $i\sigma_i$ has three active sites. To see this, consider the cases $q_1 > i$ and $q_1 < i$. If $q_1 > i$ then $iq_11$ is of type 231 and all sites between $q_1$ and 1 are inactive. If $q_n = 1$ only sites 1, 2 and $n+1$ are active. If $q_n > 1$ then $i < q_n$ implies $i1q_n$ is of type 213 and $i > q_n$ implies $i(n+1)q_n$ is of type 231, so the sites between 1 and $q_n$ are inactive. If $q_1 < i$, $iq_1(n+1)$ is of type 213, and if $q_n < n+1$, $i(n+1)q_n$ is of either of type 231 or $i1q_n$ is of type 213. In any case, only sites 1, 2 and $n+1$ are active in $i\sigma_i$. □

Acknowledgements

We are grateful to Timothy Chow for initiating this collaboration, for encouraging us along the way, and for reading preliminary versions of this paper. We also thank Julian West for supplying the data found in Table 1.

References