Beurling type quotient modules over the bidisk and boundary representations

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Abstract
This paper mainly concerns Beurling type quotient modules of $H^2(D^2)$ over the bidisk. By establishing a theorem of function theory over the bidisk, it is shown that a Beurling type quotient module is essentially normal if and only if the corresponding inner function is a rational inner function having degree at most $(1, 1)$. Furthermore, we apply this result to the study of boundary representations of Toeplitz algebras over quotient modules. It is proved that the identity representation of $C^*(S_z, S_w)$ is a boundary representation of $B(S_z, S_w)$ in all nontrivial cases. This extends a result of Arveson to Toeplitz algebras on Beurling type quotient modules over the bidisk (cf. [W. Arveson, Subalgebras of $C^*$-algebras, Acta Math. 123 (1969) 141–224; W. Arveson, Subalgebras of $C^*$-algebras II, Acta Math. 128 (1972) 271–308]). The paper also establishes $K$-homology defined by Beurling type quotient modules over the bidisk.

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1. Introduction

Let $D^2 = \{(z, w) \in \mathbb{C}^2: |z| < 1, |w| < 1\}$ be the unit polydisk in $\mathbb{C}^2$, and $T^2 = \{(z, w): |z| = 1, |w| = 1\}$ be the distinguished boundary of $D^2$. The Hardy space $H^2(D^2)$ is the closure of all polynomial in $L^2(T^2, \frac{1}{(2\pi)^2} d\theta_1 d\theta_2)$.

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It is well known that $H^2(\mathbb{D}^2)$ is a functional Hilbert space over the bidisk $\mathbb{D}^2$ with the reproducing kernel $K_{(z_1, z_2)}(z, w) = \frac{1}{(1 - \overline{z_1}z)(1 - \overline{z_2}w)}$. The study of operator theory and function theory over $H^2(\mathbb{D}^2)$ has a long history, and plays an important role in multivariable operator theory.

Douglas and Paulsen [18] introduced Hilbert modules as one natural approach to multivariable operator theory. For a Hilbert space $H$ and a tuple of commuting operators $T = (T_1, \ldots, T_d)$ acting on $H$, one naturally makes $H$ into a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$. The framework of Hilbert modules provides a new viewpoint and new questions in operator theory. For a Hilbert space $H$ and a tuple of commuting operators $T = (T_1, \ldots, T_d)$ acting on $H$, one naturally makes $H$ into a Hilbert module over the polynomial ring $C[z_1, \ldots, z_d].$
The $C[z_1, \ldots, z_d]$-module structure is defined by

$$p \cdot \xi = p(T_1, \ldots, T_d)\xi, \quad p \in C[z_1, \ldots, z_d], \quad \xi \in H.$$  

The framework of Hilbert modules provides a new viewpoint and new questions in operator theory. A Hilbert module $H$ is said to be essentially normal if the cross-commutator $[T^*_i, T_j]$ is compact for $1 \leq i, j \leq d$. In this case, if the tuple $T = (T_1, \ldots, T_d)$ is irreducible, one has a natural exact sequence

$$0 \rightarrow \mathcal{K} \hookrightarrow C^*(H) \xrightarrow{\pi} C(\sigma_e(H)) \rightarrow 0,$$

where $\sigma_e(H) = \sigma_e(T_1, \ldots, T_d)$ and $C^*(H)$ is $C^*$-algebra generated by $T_1, \ldots, T_d$. Furthermore, by the celebrated BDF-theory [10], the exact sequence yields a $K_1$-cycle over $\sigma_e(H)$, which is a natural module invariant and establishes important connections between operator theory, algebraic geometry, homology theory and complex analysis. In the case of unit ball of $\mathbb{C}^d$, most of natural Hilbert modules are essentially normal—including the $d$-shift Hilbert module $[3,4]$, the Hardy and Bergman modules of the unit ball. Arveson conjectured that graded submodules over the unit ball inherit this property and seeks an affirmative answer [5–7]. Much work has been done along this line, such as [5–7,14–16,19–21,24,22,27].

The paper will be devoted to the essential normality and boundary representations of quotient modules over the bidisk. Let $(M_z, M_w)$ be the coordinate operator tuple acting on the Hardy space $H^2(\mathbb{D}^2)$, then $H^2(\mathbb{D}^2)$ is a Hilbert module over the polynomial ring $\mathbb{C}[z, w]$, and is called Hardy module over the bidisk. By a submodule $M$ we mean that it is invariant for $M_z, M_w$. Considering the quotient $N = H^2(\mathbb{D}^2)/M$, and naturally identified with $H^2(\mathbb{D}^2) \ominus M$, the quotient $N$ is endowed with a $\mathbb{C}[z, w]$-module structure by

$$p \cdot f = p(S_z, S_w)f, \quad p \in \mathbb{C}[z, w], \quad f \in N,$$

where $S_z = P_N M_z|_N$ and $S_w = P_N M_w|_N$.

It is interesting and important to study essential normality of quotient modules. One motivation is from an attempt to understand operator theory, function theory and related geometric analysis over the bidisk. There is a large literature concerning the study of essential normality over the bidisk, see [12,11,17,18,23,25,28], here we have made no attempt to compile a comprehensive list of references.

Another motivation comes from the study of boundary representations of Toeplitz algebras on Hardy quotient modules over the unit disk [1,2]. First, let us recall some notations and definitions. For a $C^*$-algebra $B$ and a subset $A$ of $B$ which satisfies $B = C^*(A)$, an irreducible representation $\omega$ of $B$ is called a boundary representation for $A$ if $\omega|_A$ has a unique completely positive linear extension to $B$, namely $\omega$ itself. In other words, $\omega$ is determined by $A$ in the sense of complete positivity. We refer the reader to the references [1,2] for more information, and
Theorem 1.2. characterizes essential normality of Beurling type quotient modules. Let $E$ be the set of all boundary representations for $A$, we say that $E$ is sufficient for $A$ if $\|a\| = \sup_{w \in E} \|w(a)\|$ for any $a \in M_n(B)$ (this definition comes from [1], but is a little different to [8]). One natural question is whether or not there exists sufficiently many boundary representations. In the case of $B$ containing all compact operators, Arveson proved that it is equivalent to whether or not the identity representation is boundary representation [2].

By well-known Beurling theorem, any submodule $M$ of $H^2(\mathbb{D})$ is generated by an inner function $\eta$, that is $M = [\eta] = \eta H^2(\mathbb{D})$. Consider the quotient $N = H^2(\mathbb{D}) \ominus M$, and let $S = P_N M_z |_N$ be the compression of the coordinate operator $M_z$ on the quotient module. It is easy to verify that $S$ is essentially normal. Write $B(S)$ and $C^*(S)$ for the Banach algebra and $C^*$-algebra, respectively, generated by the identity operator and $S$. Then $C^*(S)$ contains all compact operators. Arveson showed that boundary representation of $B(S)$, as a subalgebra of $C^*(S)$, is completely characterized by the zero set of $\eta$.

Theorem 1.1. (See [1,2].) The identity representation of $C^*(S)$ is a boundary representation for the subalgebra $B(S)$ if and only if $Z_\eta$ is a proper subset of $\mathbb{T}$, where $Z_\eta$ consists of all points $\lambda$ on $\mathbb{T}$ for which $\eta$ cannot be continued analytically from $\mathbb{D}$ to $\lambda$.

To extend Arveson’s theorem to Hardy quotient modules over the bidisk, this paper considers the case of Beurling type submodules. We say that a submodule $M$ of $H^2(\mathbb{D}^2)$ is of Beurling type if $M$ is generated by an inner function $\eta \in H^\infty(\mathbb{D}^2)$, that is, $M = [\eta] = \eta H^2(\mathbb{D}^2)$.

To state results, we need the following notation. For a polynomial $p(z, w)$ in two variables, write $\deg_z p, \deg_w p$ for degrees of $p$ in variable $z$ and $w$, respectively. Let $r(z, w) = \frac{p(z, w)}{q(z, w)}$ be a rational function in two variables, where $p$ and $q$ have no common factor. Define $\deg_z r = \max(\deg_z p, \deg_z q)$, $\deg_w r = \max(\deg_w p, \deg_w q)$, and define the degree of $r(z, w)$ by the tuple $(\deg_z r, \deg_w r)$. We say that a rational function $r(z, w)$ has degree at most $(m, n)$ if $\deg_z r \leq m$, and $\deg_w r \leq n$.

Set $\phi_a(z) = \frac{z^a}{1-z^2}$ for $a \in \mathbb{D}$. We can now state the following theorem which completely characterizes essential normality of Beurling type quotient modules.

Theorem 1.2. Given an inner function $\eta \in H^\infty(\mathbb{D}^2)$, then the quotient $H^2(\mathbb{D}^2) \ominus [\eta]$ is essentially normal if and only if $\eta$ is a rational inner function having degree at most $(1, 1)$. This turns out to be equivalent to that $\eta$ has one of the following forms:

1. $\eta = \beta \phi_a(z)$ or $\eta = \beta \phi_a(w)$, for some $|a| < 1, |\beta| = 1$;
2. $\eta = \beta \phi_a(z) \phi_b(w)$, for some $|a| < 1, |b| < 1, |\beta| = 1$;
3. $\eta = \beta \frac{z^{aw+az+bw+c}}{1+\bar{a}w+\bar{b}z+\bar{c}zw}$ for some $|\beta| = 1$ and $c \neq ab$.

The proof of Theorem 1.2 is considerably technical. It essentially comes from a theorem of function theory over the bidisk which is establish in Appendix A.

Write $B(S_z, S_w)$ and $C^*(S_z, S_w)$ for the Banach algebra and $C^*$-algebra, respectively, generated by the identity operator, $S_z$ and $S_w$. Under the conditions of Theorem 1.2, then $C^*(S_z, S_w)$ contains all compact operators.

We can extend Arveson’s Theorem 1.1 to the following version.

Theorem 1.3. Under the conditions of Theorem 1.2, we have

1. if $\eta = \beta \phi_a(z)$ or $\eta = \beta \phi_a(w)$, for some $|a| < 1, |\beta| = 1$, then the identity representation of $C^*(S_z, S_w)$ is not a boundary representation for $B(S_z, S_w)$;
2. if \( \eta = \beta \phi_a(z)\phi_b(w) \) for some \(|\beta| = 1, |a| < 1, |b| < 1\), then the identity representation of \( C^*(S_z, S_w) \) is a boundary representation for \( \mathcal{B}(S_z, S_w) \);

3. if \( \eta = \beta \frac{az+bw+cz}{1+aw+hz+c\bar{w}} \) for some \(|\beta| = 1 \) and \( c \neq ab \), then the identity representation of \( C^*(S_z, S_w) \) is a boundary representation for \( \mathcal{B}(S_z, S_w) \).

2. Essential normality of quotient modules

Let \( M = [\eta] = \eta H^2(\mathbb{D}^2) \) be a Beurling type submodule generated by an inner function \( \eta \), and \( P_\eta \) be the projection to the quotient \( N = H^2(\mathbb{D}^2) \ominus [\eta] \). Since \( \eta \) is inner, \( P_\eta = I - M_\eta M_\eta^* \). Set \( S_z = P_\eta M_z|_N \), \( S_w = P_\eta M_w|_N \), the compressions of \( M_z \) and \( M_w \) on the quotient module \( N = H^2(\mathbb{D}^2) \ominus [\eta] \), then by Fuglede–Putnam theorem, the quotient module \( N \) is essentially normal if and only if the commutators both [\( S_z, S_z^* \)] and [\( S_w, S_w^* \)] are compact.

**Theorem 2.1.** Given a nonconstant inner function \( \eta \), then the quotient module \( H^2(\mathbb{D}^2) \ominus [\eta] \) is essentially normal if and only if \( \eta \) is a rational inner function having degree at most \((1, 1)\). This turns out to be equivalent to that \( \eta \) has one of the following forms:

1. \( \eta = \beta \phi_a(z) \) or \( \eta = \beta \phi_a(w) \), for some \(|a| < 1, |\beta| = 1\);
2. \( \eta = \beta \phi_a(z)\phi_b(w) \), where \(|\beta| = 1\), and \(|a| < 1, |b| < 1\);
3. \( \eta = \beta \frac{az+bw+cz}{1+aw+hz+c\bar{w}} \), where \(|\beta| = 1 \) and \( c \neq ab \).

In this case, the quotient module yields the following exact sequence

\[
0 \to K \to C^*(S_z, S_w) \to C(\sigma_e(S_z, S_w)) \to 0. \tag{2.1}
\]

Firstly, let us recall some notations and some well-known facts of function theory over bidisk. Given \( f \in H^2(\mathbb{D}^2) \), clearly \( f(\cdot, w) \in H^2(\mathbb{D}) \) for any \( w \in \mathbb{D} \). Moreover, since \( f \in L^2(\mathbb{T}^2) \), we have \( f(\cdot, w) \in \mathbb{L}^2(\mathbb{T}) \) for almost every \( w \in \mathbb{T} \). In fact, a further analysis shows that for almost every \( w \in \mathbb{T} \), \( f(\cdot, rw) \) weakly converges to \( f(\cdot, w) \) in \( \mathbb{L}^2(\mathbb{T}) \) as \( r \to 1 \) and \( f(\cdot, w) \in H^2(\mathbb{D}) \).

Let \( P \) be the orthogonal projection from \( \mathbb{L}^2(\mathbb{T}^2) \) onto \( H^2(\mathbb{D}^2) \). The Toeplitz operator \( T_f : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) with symbol \( f \in \mathbb{L}^\infty(\mathbb{T}^2) \) is defined by \( T_f(h) = P(fh) \) for any \( h \in H^2(\mathbb{D}^2) \). The Hankel operator \( H_f \) with symbol \( f \) is defined by \( H_fh = (I - P)(fh) \). For \( f, g \in \mathbb{L}^\infty(\mathbb{T}^2) \), Toeplitz and Hankel operators are connected by the following formula

\[
Tfg - TfTg = H_f^*H_g. \tag{2.2}
\]

When \( f \in \mathbb{L}^\infty(\mathbb{D}^2) \), \( T_f \) is a multiplication operator defined by \( f \), and denoted by \( M_f \).

**Lemma 2.2.** If \([S_z, S_z^*] \) is compact on the quotient module \( N \), then there exists a measurable subset \( E \subseteq \mathbb{T} \) of positive measure such that for \( w \in E \), \( \eta(\cdot, w) \) is a rational function of one variable with degree at most \( 1 \).

**Proof.** Let \( P^w \) be the orthogonal projection from \( H^2(\mathbb{D}^2) \) onto the subspace \( H^2_w = \text{span}\{1, w, w^2, \ldots\} \), it is easy to verify \( P^w = I - M_zM_z^* \) and

\[
(P^w f)(z, w) = f(0, w), \quad f \in H^2(\mathbb{D}^2). \tag{2.3}
\]
Since
\[ P_\eta = I - M_\eta M^*_\eta, \]  
we have
\[
\begin{align*}
[S_z, S^*_\eta] &= S_z S^*_\eta - S^*_z S_z = P_\eta M_z P_\eta M^*_\eta P_\eta - P_\eta M_z^* P_\eta M_z P_\eta \\
&= P_\eta [M_z M^*_\eta - (I - M^*_\eta M_\eta M^*_\eta)] P_\eta \\
&= -P_\eta P^w P_\eta + P_\eta M_z^* M_\eta M^*_\eta M_z P_\eta.
\end{align*}
\]  
(2.5)

To achieve the desired conclusion, we begin by calculating \( \langle [S_z, S^*_\eta] k_\lambda, k_\lambda \rangle \), where
\[
k_\lambda(z, w) = \frac{K_\lambda(z, w)}{\|K_\lambda(z, w)\|} = \frac{\sqrt{(1 - |\lambda_1|^2)(1 - |\lambda_2|^2)}}{(1 - \lambda_1 z)(1 - \lambda_2 w)}, \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2,
\]
is the normalized reproducing kernel of \( H^2(\mathbb{D}^2) \). By (2.5), we see
\[
\langle [S_z, S^*_\eta] k_\lambda, k_\lambda \rangle = \| M^*_\eta M_z P_\eta k_\lambda \|^2 - \| P^w P_\eta k_\lambda \|^2.
\]  
(2.6)

Since \( M^*_\eta k_\lambda = \eta(\lambda) k_\lambda \), by (2.4)
\[
P_\eta k_\lambda = k_\lambda - M_\eta M^*_\eta k_\lambda = (1 - \eta(\lambda)\eta) k_\lambda.
\]  
(2.7)

Combining the formula (2.3) and (2.7) shows
\[
P^w P_\eta k_\lambda = P^w \left( [1 - \eta(\lambda)\eta] k_\lambda \right) = (1 - |\lambda_1|^2)\left(1 - \eta(\lambda)\eta_0\right) k^*_\lambda,
\]  
(2.8)

where \( k^*_\lambda(z, w) = k_{(0, \lambda_2)}(z, w) = \frac{\sqrt{1 - |\lambda_2|^2}}{1 - \lambda_2 w} \) and \( \eta_0(w) = \eta(0, w) \). From the formula (2.2), we see
\[
\begin{align*}
M^*_{1 - \eta(\lambda)\eta_0} M_{1 - \eta(\lambda)\eta_0} &= M^*_{1 - \eta(\lambda)\eta_0} M_{1 - \eta(\lambda)\eta_0} + H^*_{1 - \eta(\lambda)\eta_0} H_{1 - \eta(\lambda)\eta_0} \\
&= M^*_{1 - \eta(\lambda)\eta_0} M_{1 - \eta(\lambda)\eta_0} + |\eta(\lambda)|^2 H^*_{\eta_0} H_{\eta_0}.
\end{align*}
\]  
(2.9)

Combining (2.8) and (2.9) we have
\[
\| P^w P_\eta k_\lambda \|^2 = \left(1 - |\lambda_1|^2\right) \left|M^*_{1 - \eta(\lambda)\eta_0} M_{1 - \eta(\lambda)\eta_0} k^*_\lambda, k^*_\lambda\right|^2 \\
= \left(1 - |\lambda_1|^2\right) \left|\eta(\lambda)\eta_0(\lambda_2)\right|^2 + \left|\eta(\lambda)\right|^2 \left\| H^*_{\eta_0} k^*_\lambda\right\|^2.
\]  
(2.10)

Let \( \eta_1(z, w) = \frac{\eta(z, w) - \eta_0(w)}{z} \). Clearly \( \eta_1 \in H^\infty(\mathbb{D}^2) \) and
\[
\eta(z, w) = \eta_0(w) + z\eta_1(z, w).
\]  
(2.11)
Furthermore, since $[M^*_f(w), M_z] = 0$ for any bounded analytic functions $f(w)$ and $I - M_z M^*_z = P^w$, we have

$$[M^*_\eta, M_z] = M^*_\eta M_z - M_z M^*_\eta = [M^*_{\eta + z\eta_1}, M_z] = [M^*_{\eta z}, M_z - M_z M^*_\eta] = P^w M^*_\eta.$$  

This implies that

$$M^*_\eta M_z P^w k_\lambda = M^*_\eta M_z (I - M^*_{\eta M_z}) k_\lambda = (M^*_\eta M_z - M^*_\eta M_z M^*_\eta) k_\lambda$$  

$$= (M^*_\eta M_z - M_z M^*_\eta) k_\lambda = P^w M^*_\eta k_\lambda$$  

$$= P^w (\eta_1(\lambda) k_\lambda) = \eta_1(\lambda_1) (1 - |\lambda_1|^2)^k k_\lambda^*,$$

and hence

$$\| M^*_\eta M_z P^w k_\lambda \|^2 = (1 - |\lambda_1|^2)^2 \eta_1(\lambda_1)^2.$$

It follows from (2.6), (2.10) and (2.12) that

$$\langle [S_z, S^*_z] k_\lambda, k_\lambda \rangle = (1 - |\lambda|^2) (|\eta_1(\lambda)|^2 - |1 - \eta(\lambda)\eta_0(\lambda)|^2 - |\eta(\lambda)|^2 \| H^\eta k^*_z \|^2).$$

Now fixing $\lambda_2 \in \mathbb{D}$, consider the function

$$f_{\lambda_2}(z) = \frac{\eta(z, \lambda_2) - \eta_0(\lambda_2)}{1 - \eta_0(\lambda_2)\eta(z, \lambda_2)} = \phi_{\alpha}(\eta(z, \lambda_2)),$$

where $\alpha = \eta_0(\lambda_2)$. Then $f_{\lambda_2} \in \mathcal{H}(\mathbb{D})$, $f_{\lambda_2}(0) = 0$ and $\|f_{\lambda_2}\|_\infty \leq 1$. By Schwarz lemma, $|f_{\lambda_2}(z)| \leq |z|$ for any $z \in \mathbb{D}$. Since $\eta_1(z, \lambda_2) = \eta(z, \lambda_2)$, we have that for $\lambda_1, \lambda_2 \in \mathbb{D}$

$$|\eta_1(\lambda_1, \lambda_2)| = \left| f_{\lambda_2}(\lambda_1) [1 - \eta_0(\lambda_2)\eta(\lambda_1, \lambda_2)] \right|_\lambda_1$$  

$$\leq |1 - \eta_0(\lambda_2)\eta(\lambda_1, \lambda_2)|$$  

$$= |1 - \eta(\lambda)\eta_0(\lambda_2)|.$$  

Furthermore, since $[S_z, S^*_z]$ is compact and $k_\lambda$ is weakly convergent to 0 as $\lambda \to \partial \mathbb{D}$, we have

$$\langle [S_z, S^*_z] k_\lambda, k_\lambda \rangle \to 0.$$  

By formulas (2.13), (2.14) and (2.15), as $\lambda = (\lambda_1, \lambda_2) \to \partial \mathbb{D}$, we have

$$(1 - |\lambda_1|^2)|\eta(\lambda)|^2 \| H^\eta k^*_\lambda \| \to 0;$$

$$(2.16)$$
and
\[(1 - |\lambda_1|^2)(|\eta_1(\lambda)|^2 - |1 - \overline{\eta_0(\lambda_2)}\eta(\lambda)|^2) \to 0.\] (2.17)

Thus, for fixed $\lambda_1 \in \mathbb{D}$, if $\lambda_2 \to \partial \mathbb{D}$, we have
\[|1 - \eta_0(\lambda_2)\eta(\lambda)|^2 - |\eta_1(\lambda)|^2 \to 0.\] (2.18)

This implies that, for $\lambda_1 \in \mathbb{D}$ and almost every $\lambda_2 \in \mathbb{T}$,
\[|1 - \eta_0(\lambda_2)\eta(\lambda_1, \lambda_2)| = \left|\eta(\lambda_1, \lambda_2) - \eta_0(\lambda_2)\right|/\lambda_1.\] (2.19)

Set $\eta_{\lambda_2}(z) = \eta(z, \lambda_2)$. Then there exists a measurable subset $E \subseteq \mathbb{T}$ satisfying $m(E) = 1$ such that for each $\lambda_2 \in E$, $\eta_{\lambda_2}$ is well defined. Clearly, $\eta_{\lambda_2} \in H^\infty(\mathbb{D})$, $\|\eta_{\lambda_2}\|_\infty \leq 1$ for $\lambda_2 \in E$. Moreover, by (2.19), if $\lambda_2 \in E$,
\[|1 - \eta_{\lambda_2}(0)\eta_{\lambda_2}(z)| = \left|\eta_{\lambda_2}(z) - \eta_{\lambda_2}(0)/z\right|, \quad \text{for } z \in \mathbb{D}.\] (2.20)

To complete the proof, it is enough to show that $\eta_{\lambda_2}$ is a rational function with the degree at most 1 for each $\lambda_2 \in E$.

Fix $\lambda_2 \in E$. If $|\eta_{\lambda_2}(0)| = 1$, then $\eta_{\lambda_2}$ is a constant because $\|\eta_{\lambda_2}\|_\infty \leq 1$. Therefore it suffices to show the case $|\eta_{\lambda_2}(0)| < 1$. Set
\[f_{\lambda_2}(z) = \frac{\eta_{\lambda_2}(z) - \eta_{\lambda_2}(0)}{1 - \eta_{\lambda_2}(0)\eta_{\lambda_2}(z)}, \quad z \in \mathbb{D},\]
then $\|f_{\lambda_2}\|_\infty \leq 1$, and $f_{\lambda_2}(0) = 0$. By (2.20), $|f_{\lambda_2}(z)| = |z|$ for $z \in \mathbb{D}$. Using Schwarz lemma shows $f_{\lambda_2}(z) = c_{\lambda_2}z$, where $c_{\lambda_2}$ is a constant with modulo 1. This means that for $\lambda_2 \in E$,
\[\eta(z, \lambda_2) = \frac{c_{\lambda_2}z - \eta_{\lambda_2}(0)}{1 - \eta_{\lambda_2}(0)c_{\lambda_2}z}\]
is a rational function with degree 1, completing the proof. \(\square\)

If $[S_w^*, S_w]$ is compact, the same argument as above shows that there exists a subset $E \subseteq \mathbb{T}$ of positive measure such that for $z \in E$, $f(z, \cdot)$ is a rational function with degree at most 1.

To complete the proof of Theorem 2.1, we need the following theorem about the function theory over bidisk, which is interesting in itself. We will give the proof of this theorem in Appendix A.

**Theorem 2.3.** For $f \in H^\infty(\mathbb{D}^2)$, if there exist two subsets $E_1, E_2 \subseteq \mathbb{T}$ of positive measure such that for $w \in E_1$, $f(\cdot, w)$ is a rational function with degree at most $r_1$, and for $z \in E_2$, $f(z, \cdot)$ is a rational function with degree at most $r_2$, then $f(z, w)$ is a rational function with degree at most $(r_1, r_2)$. 
Proof of Theorem 2.1. Suppose quotient module $H^2(\D^2) \ominus [\eta]$ is essentially normal, that is, both $[S^*_z, S_z]$ and $[S^*_w, S_w]$ are compact. Combining Lemma 2.2 and Theorem 2.3 shows that $\eta(z, w)$ is a rational inner function of degree at most $(1, 1)$.

If $\eta$ is a rational function of degree $(1, 0)$ or $(0, 1)$, then $\eta$ is independent of $w$ or independent of $z$, respectively. An easy reasoning shows that $\eta$ has the form $\eta = \beta \phi_a(z)$ or $\eta = \beta \phi_a(w)$ for some $|\beta| = 1, |a| < 1$.

Now we consider the case that $\eta$ is a rational inner function of degree $(1, 1)$. By [26, Theorem 5.2.5], $\eta$ has the following form

$$\eta(z, w) = \frac{d}{d} zw + az + bw + c.$$  

Since $\eta$ is of degree $(1, 1)$, we have that $0 \leq k, l \leq 1$.

If $k = 1$, since $\eta(z, w)$ is a rational function of degree $(1, 1)$, we see the inner function $\eta(z, w) z$ is of degree $(0, 1)$ and hence $\eta(z, w) z = \beta \phi_b(w)$ for some $|\beta| = 1$ and $|b| < 1$. This implies that $\eta(z, w) = \beta \phi_b(w)$, as desired. In the case of $l = 1$, the similar argument is valid.

It remains to show the case $k; l = 0$ and $\eta(z, w) = \frac{d}{d} zw + az + bw + c$. Since $\eta$ is a rational inner function of degree $(1, 1)$, then $d \neq 0$. Therefore,

$$\eta(z, w) = \frac{d}{d} z + b' w + a' z + c' z w.$$

where $a' = \frac{a}{d}, b' = \frac{b}{d}, c' = \frac{c}{d}$. If $c' = a' b'$, then

$$\eta(z, w) = \frac{d}{d} z + b' w + a' z + c' z w.$$

Since $\eta(z, w)$ is inner, it follows that both $|a'| < 1$ and $|b'| < 1$, and in this case, we have

$$\eta(z, w) = \frac{d}{d} \phi_{-b'}(z) \phi_{-a'}(w).$$

The remaining case is (3) of Theorem 2.1.

In the opposite direction, it is easy to show that in the cases $\eta = \beta \phi_a(z), \eta = \beta \phi_a(w)$ and $\eta = \beta \phi_a(z) \phi_b(w)$, the quotient modules $H^2(\D^2) \ominus [\eta]$ are essentially normal. Below, we consider the case $\eta = \frac{z w + az + bw + c}{1 + a w + d z + c w}$, satisfying $c \neq ab$. Since $zw + az + bw + c = (z + b)(w + a) + c - ab \in [\eta]$, this shows that

$$S_{z+b}S_{w+a} = c - ab,$$

and hence

$$S_{z+b} = (c - ab)S_{w+a}^{-1}. \quad (2.21)$$

From (2.21), a simple reasoning gives

$$(c - ab)[S_{w+a}, S^*_{w+a}] = S_{w+a}[S^*_{w+a}, S_{z+b}]S_{w+a}. \quad (2.22)$$
By Corollary 2.5 and Theorem 2.6 in [28], $[S_z, S_w^*]$ belongs to Hilbert–Schmidt class. Using Eq. (2.22) shows that the self-commutator $[S_w, S_w^*]$ also belongs to Hilbert–Schmidt class. Similarly, the self-commutator $[S_z, S_z^*]$ is in Hilbert–Schmidt class. This implies that $H^2(\mathbb{D}^2) \ominus [\eta]$ is essentially normal.

Moreover, if the quotient is essentially normal, by [10], we have the following exact sequence

$$0 \rightarrow \mathcal{K} \hookrightarrow C^*(S_z, S_w) \rightarrow C\left(\sigma_e(S_z, S_w)\right) \rightarrow 0,$$

completing the proof. □

### 3. Boundary representations

In [1,2], Arveson introduced a theory of boundary representations and studied boundary representations of Toeplitz algebras on quotient modules of $H^2(\mathbb{D})$, and established Theorem 1.1 (see Introduction). In this section, we will investigate boundary representations of Toeplitz algebras on Hardy quotient modules over the bidisk.

For an irreducible set $A$ of operators, there is a general criterion for determining whether or not the identity representation of $C^*(A)$ is a boundary representation for $A$. It is the following Arveson’s boundary representation theorem.

**Theorem 3.1.** (See [2].) Let $A$ be an irreducible set of operators on a Hilbert space $H$, such that $A$ contains the identity and $C^*(A)$, the $C^*$-algebra generated by $A$, contains the algebra $K(H)$ of all compact operators on $H$. Then the identity representation of $C^*(A)$ is a boundary representation for $A$ if and only if the quotient map $Q : B(H) \rightarrow B(H)/K(H)$ is not completely isometric on the linear span of $A \cup A^*$, where $B(H)$ is the algebra of all bounded linear operators.

**Below we will come to the proof of Theorem 1.3.**

**Case 1.** If $\eta(z, w) = \phi_a(z)$ for some $|a| < 1$, then on the quotient module $N = H^2(\mathbb{D}^2) \ominus [\phi_a(z)]$, $S_z = a$, and $S_w$ is a unilateral shift. Applying Theorem 3.1 shows that in this case, the identity representation of $C^*(S_z, S_w)$ is not a boundary representation for $B(S_z, S_w)$. The same argument is applied in the case $\eta(z, w) = \phi_a(w)$.

**Case 2.** Now we consider the case $\eta(z, w) = \phi_a(z)\phi_b(w)$ for some $a, b$ with $|a| < 1, |b| < 1$. By a Möbius transformation $\Phi(z, w) = (\phi_{-a}(z), \phi_{-b}(w))$ on the bidisk, this case is reduced to $a = b = 0$ and $\eta = zw$. Below we consider this simple situation. We claim that $\|S_z + w\| > \|S_z + w\|e$. By the fact that the quotient module $N = H^2(\mathbb{D}^2) \ominus [\eta]$ has a canonical orthonormal basis

$$\{\ldots, w^n, \ldots, w, 1, z, \ldots, z^n, \ldots\},$$

it follows that

$$\|S_z + w\| > \|z + w\| = \sqrt{2}.$$
On the other hand, let $H^2_w = \operatorname{span}\{1, w, w^2, \ldots\}$ and $H^2_{(z)} = \operatorname{span}\{z, z^2, \ldots\}$, then $H^2(\mathbb{D}^2) \ominus [\eta] = H^2_{(z)} \oplus H^2_w$. By a computation on the canonical orthonormal basis, both $H^2_{(z)}$ and $H^2_w$ are reducing subspaces of $S_{z+w} - 1 \otimes z$, where $1 \otimes z$ is an operator of rank 1 defined by

$$(1 \otimes z)h = \langle h, 1 \rangle z.$$ 

Moreover, the operator $S_{z+w} - 1 \otimes z$, restricted on both $H^2_{(z)}$ and $H^2_w$, is a unilateral shift. This implies

$$\|S_{z+w}\|_e = \|S_{z+w} - 1 \otimes z\|_e = 1,$$

and hence $\|S_{z+w}\| > \|S_{z+w}\|_e$, as desired.

Therefore, the quotient map $Q : B(N) \rightarrow B(N)/K(N)$ is not completely isometric on $B(S_z, S_w)$. By Theorem 3.1, the identity representation of $C^*(S_z, S_w)$ is a boundary representation for $B(S_z, S_w)$.

To complete the proof of Theorem 1.3, the remaining will be devoted to the case $\eta = zw^2 + az + bw + c$, where $c \neq ab$. This is the next theorem.

**Theorem 3.2.** Let $\eta = zw^2 + az + bw + c$, where $c \neq ab$, then on the quotient module $H^2(\mathbb{D}^2) \ominus [\eta]$, the identity representation of $C^*(S_z, S_w)$ is a boundary representation for $B(S_z, S_w)$.

The proof of Theorem 3.2 is divided into several steps. We need several lemmas and propositions.

**Lemma 3.3.** Let $\eta = \frac{zw - a}{1 - azw}$, where $0 < a < 1$, then

$$\ldots, (1 + a\eta)w^2, (1 + a\eta)w, (1 + a\eta), (1 + a\eta)z, (1 + a\eta)z^2, \ldots$$

is an orthogonal basis of the quotient submodule $H^2(\mathbb{D}^2) \ominus [\eta]$.

**Proof.** We claim that

$$H^2(\mathbb{D}^2) \ominus [\eta] = \operatorname{span}\{\ldots, (1 + a\eta)w^2, (1 + a\eta)w, (1 + a\eta), (1 + a\eta)z, (1 + a\eta)z^2, \ldots\}.$$

Since $H^2(\mathbb{D}^2)$ has a canonical orthogonal basis $\{z^n w^m\}_{n,m \geq 0}$, this shows that

$$\operatorname{span}\{P_\eta(z^n w^m)\}_{n,m \geq 0} = H^2(\mathbb{D}^2) \ominus [\eta],$$

where $P_\eta$ is the projection from $H^2(\mathbb{D}^2)$ onto $H^2(\mathbb{D}^2) \ominus [\eta]$. Moreover, for any nonnegative integer $n$ and $f \in H^2(\mathbb{D}^2)$, we have

$$\langle (1 + a\eta)w^n, \eta f \rangle = \langle w^n, \eta f \rangle + \langle a\eta w^n, \eta f \rangle = \langle w^n, \eta f \rangle + \langle aw^n, f \rangle$$

$$= \langle w^n, (\eta + a)f \rangle = \left\langle w^n, \frac{(1 - a^2)zw}{1 - azw} f \right\rangle = 0.$$
This implies that

\[(1 + a\eta)w^n = P_\eta((1 + a\eta)w^n) = P_\eta w^n.\]  \hspace{1cm} (3.2)

Similarity, \(P_\eta z^n = (1 + a\eta)z^n\). From \(zw = a + \eta(1 - azw)\), we have that if \(n \geq m\),

\[z^n w^m = z^{n-m}(a^m + \eta f)\]

for some \(f \in H^2(\mathbb{D})\), and hence

\[P_\eta(z^n w^m) = a^m (1 + a\eta)z^{n-m}.\]  \hspace{1cm} (3.3)

Similarly, if \(m \geq n\), we have

\[P_\eta(z^n w^m) = a^n (1 + a\eta)w^{m-n}.\]  \hspace{1cm} (3.4)

Combining (3.1) and (3.3), (3.4) shows that the claim is true.

To complete the proof, it remains to show orthogonality. Let us consider the case \(((1 + a\eta)w^n, (1 + a\eta)z^m)\), where \(n; m \geq 1\). By (3.2), \((1 + a\eta)w^n\) is in \(H^2(\mathbb{D}) \ominus [\eta]\), and hence

\[\langle (1 + a\eta)w^n, (1 + a\eta)z^m \rangle = \langle (1 + a\eta)w^n, z^m \rangle = 0.\]

The remaining cases are verified by the same way, completing the proof. \(\square\)

**Proposition 3.4.** If \(\eta = \frac{zw - a}{1 - azw}\) for some \(0 < a < 1\), then the identity representation of \(C^*(S_z, S_w)\) is a boundary representation for \(B(S_z, S_w)\).

**Proof.** By Theorem 3.1, it suffices to show \(\|S_{z^n+w^n}\| > \|S_{z^n+w^n}\|_e\) for sufficiently large \(n\). From (3.2),

\[\|P_\eta (w^n)\|^2 = \|w^n\|^2 - \|a\eta w^n\|^2 = 1 - a^2.\]

Similarly, \(\|P_\eta (z^n)\|^2 = 1 - a^2\) for \(n \geq 0\), and hence \(\|P_\eta 1\| = \sqrt{1 - a^2}\). By Lemma 3.3 and its proof, \(P_\eta (z^n) \perp P_\eta (w^n)\) for \(n \geq 1\), this implies that

\[\|P_\eta (z^n + w^n)\| = \sqrt{2(1 - a^2)}.\]

It follows that

\[\|S_{z^n+w^n}\| > \frac{\|S_{z^n+w^n}(P_\eta 1)\|}{\|P_\eta 1\|} = \frac{\|P_\eta (z^n + w^n)\|}{\|P_\eta 1\|} = \sqrt{2}.\]

On the other hand, since \(S_\eta = 0\) and on the quotient module \(N\),

\[I - S_z S^*_z - S_w S^*_w + S_z S_w S^*_w S^*_z = P_\eta \left( I - M_z M^*_z - M_w M^*_w + M_z M_w M^*_w M^*_z \right) P_\eta\]

\[= P_\eta (1 \otimes 1) P_\eta\]
is compact, using formula (2.1) shows that \( \sigma_e(S_z, S_w) \subseteq Z(\eta) \cap \partial \mathbb{D}^2 \). An easy computation gives that

\[
Z(\eta) \cap \partial \mathbb{D}^2 = \left\{ (z, az^{-1}) : |z| = 1 \right\} \cup \left\{ (aw^{-1}, w) : |w| = 1 \right\}.
\]

Using formula (2.1) again shows that

\[
\|S_{z^n}w^n\|_e \leq \left\| (z^n + w^n)|_{Z(\eta) \cap \partial \mathbb{D}^2} \right\| = 1 + a^n,
\]

which tends to 1 as \( n \to \infty \). Therefore, the following holds

\[
\|S_{z^n}w^n\| > \|S_{z^n}w^n\|_e
\]

for sufficiently large natural number \( n \), completing the proof. \( \square \)

We also need the following estimate.

**Lemma 3.5.** Suppose both \( b \) and \( c \) are positive, and \( c \neq 1 \). If \( b^2 + 1 < 2c \), then \( \sup_{|z|=1} \left| \frac{z-b}{z-c} \right| = \frac{1+b}{1+c} \), and the maximum is attained at \( z = -1 \).

**Proof.** Writing \( z = e^{i\theta} \), a direct computation shows that

\[
\left| \frac{z-b}{z-c} \right|^2 = \frac{(b - \cos \theta)^2 + \sin^2 \theta}{(c - \cos \theta)^2 + \sin^2 \theta} = \frac{b^2 + 1 - 2b \cos \theta}{c^2 + 1 - 2c \cos \theta} = \frac{b}{c} \left( 1 + \frac{b^2 + 1}{2b} - \frac{c^2 + 1}{2c} - \cos \theta \right).
\]

Since \( \frac{b^2 + 1}{2b} < \frac{c^2 + 1}{2c} \), the above function of \( \cos \theta \) is monotonic decreasing. Thus the maximum is attained at \( \cos \theta = -1 \), completing the proof. \( \square \)

**Proposition 3.6.** Let \( \eta = \frac{zw - az - (1-a)w}{1-aw - (1-a)z} \), where \( 0 < a < 1 \). Then the identity representation of \( C^*(S_z, S_w) \) is a boundary representation for \( \mathcal{B}(S_z, S_w) \).

**Proof.** By Theorem 3.1, it suffices to show that \( \|S_{(1-a)z-aw}\| > \|S_{(1-a)z-aw}\|_e \). Since for any polynomial \( f, \langle (zw - az - (1-a)w) f \rangle = 0 \) and

\[
\langle (1-a)z - aw, (zw - az - (1-a)w) f \rangle = -\langle (1-a)z, az f \rangle + \langle aw, (1-a)w f \rangle = -\langle (1-a), af \rangle + \langle a, (1-a) f \rangle = 0,
\]

and hence \( 1, (1-a)z - aw \in H^2(\mathbb{D}^2) \ominus [\eta] \). We obtain that

\[
\|S_{(1-a)z-aw}\| \geq \| (1-a)z - aw \|
\]

\[
= \sqrt{(1-a)^2 + a^2} = \sqrt{1 - 2a(1-a)}.
\]  

(3.5)

On the other hand, a similar argument as in the proof of Proposition 3.4 shows that \( \sigma_e(S_z, S_w) \subseteq Z(\eta) \cap \partial \mathbb{D}^2 \). By an easy computation,
\[ Z(\eta) \cap \partial D^2 = \left\{ \left( z, \frac{az}{z-1+a} \right) : |z| = 1 \right\} \cup \left\{ \left( \frac{(1-a)w}{w-a}, w \right) : |w| = 1 \right\}. \]

It follows from formula (2.1) that
\[
\| S((1-a)z-aw) \|_e \leq \| [ (1-a)z - aw ] \|_{Z(\eta) \cap \partial D^2} \infty = \max \left\{ \sup_{|z|=1} \left| (1-a)z - \frac{a^2z}{z-1+a} \right| , \sup_{|w|=1} \left| \frac{(1-a)^2w}{w-a} - aw \right| \right\}.
\] (3.6)

In what follows we estimate the above maximum by Lemma 3.5. If \(|z|=1\), then
\[
\left| (1-a)z - \frac{a^2z}{z-1+a} \right| = (1-a) \left| \frac{z - \frac{(1-a^2+a^2)}{1-a}}{z - (1-a)} \right|.
\]

Moreover, by a direct computation, we have
\[
\frac{(1-a^2+a^2)^2}{(1-a)} = \frac{(1-a)^2 + 1}{2(1-a)} = -\frac{a^3}{(1-a)^2 + a^2} < 0.
\]

Applying Lemma 3.5 gives that
\[
\sup_{|z|=1} \left| (1-a)z - \frac{a^2z}{z-1+a} \right| = 1 - \frac{2a(1-a)}{2-a}, \quad (3.7)
\]
which is attained at \(z = -1\). The same reasoning yields that
\[
\sup_{|w|=1} \left| (1-a) \frac{(1-a)w}{w-a} - aw \right| = 1 - \frac{2a(1-a)}{1+a}. \quad (3.8)
\]

Combining formula (3.7), (3.8) and (3.6),
\[
\| S((1-a)z-aw) \|_e = \max \left\{ 1 - \frac{2a(1-a)}{1+a} , 1 - \frac{2a(1-a)}{2-a} \right\}. \quad (3.9)
\]

Considering (3.5) and (3.9), and by an easy computation, we see
\[
1 - 2a(1-a) - \left( 1 - \frac{2a(1-a)}{1+a} \right)^2 = \frac{2a(1-a)^3}{(1+a)^2} > 0 \quad (3.10)
\]
and
\[
1 - 2a(1-a) - \left( 1 - \frac{2a(1-a)}{2-a} \right)^2 = \frac{2a^3(1-a)}{(2-a)^2} > 0. \quad (3.11)
\]

Combining formulas (3.9), (3.10), (3.11) and (3.5), we have the desired inequality
\[
\| S((1-a)z-aw) \|_e > \| S((1-a)z-aw) \|_e,
\]
completing the proof. \(\square\)
Before continuing, let’s mention a corollary in [26, p. 167], which says that for any holomorphic automorphism \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \), there exist \( |\beta_1| = 1, |\beta_2| = 1, |a| < 1, |b| < 1 \) such that \( \Phi(z, w) = (\beta_1 \phi_\alpha(z), \beta_2 \phi_\beta(w)) \) or \( \Phi(z, w) = (\beta_1 \phi_\alpha(w), \beta_2 \phi_\beta(z)) \).

**Proposition 3.7.** Let \( \eta(z, w) = \frac{z + az + bw + c}{1 + zw + bw + cz} \) be a rational inner function satisfying \( c \neq ab \), and set \( p(z, w) = zw + az + bw + c \). Then

1. If \( Z(p) \cap \mathbb{T}^2 \neq \emptyset \), there is a holomorphic automorphism \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) such that \( \eta(\Phi(z, w)) = \beta_1^{zw - az - (1 - a)w} \) for some \( |\beta| = 1, 0 < a < 1 \);
2. If \( Z(p) \cap \mathbb{T}^2 = \emptyset \), there is a holomorphic automorphism \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) such that \( \eta(\Phi(z, w)) = \beta_2^{zw - a} \) for some \( |\beta| = 1, 0 < a < 1 \).

**Proof.** Firstly, we claim that \( Z(\eta) \cap \mathbb{D}^2 \neq \emptyset \). Otherwise, we suppose that \( Z(\eta) \cap \mathbb{D}^2 = \emptyset \). Then for any \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2 \), \( \eta(\lambda)(z) = \eta(\lambda_1 z, \lambda_2 z) \) is a rational inner function over the unit disk \( \mathbb{D} \) satisfying \( Z(\eta(\lambda)) \cap \mathbb{D} = \emptyset \). This implies that \( \eta(\lambda) \) is a constant for any \( \lambda \in \mathbb{T}^2 \) and hence \( \eta \) is a constant. This leads to a contradiction. Therefore, the claim is true and \( Z(\eta) \cap \mathbb{D}^2 \neq \emptyset \). This means that there exists \( (a, b) \in \mathbb{D}^2 \) such that \( \eta(a, b) = 0 \) and hence \( \eta(\phi_\alpha(0), \phi_\beta(0)) = 0 \). This reduces the problem to the case \( \eta(0, 0) = 0 \), that is \( \eta(z, w) = \frac{zw + az + bw + c}{1 + zw + bw + cz} \).

(1) In the case \( Z(p) \cap \mathbb{T}^2 \neq \emptyset \), here \( p(z, w) = zw + az + bw \), that is, there exists \( (\theta_1, \theta_2) \in \mathbb{T}^2 \) such that \( p(\theta_1, \theta_2) = 0 \). Defining a holomorphic automorphism \( \Phi(z, w) = (\theta_1 z, \theta_2 w) : \mathbb{D}^2 \to \mathbb{D}^2 \), then \( p(\Phi(z, w)) \) satisfies \( p(\Phi(1, 1)) = 0 \). Let \( a' = -a\theta_2, b' = -b\theta_1 \), then

\[
\eta(\Phi(z, w)) = \theta_1 \theta_2 \frac{zw - a'z - b'w}{1 - \overline{a'}w - \overline{b'}z}.
\]

By \( p(\Phi(1, 1)) = 0 \), \( a' + b' = 1 \). Since \( Z(1 - \overline{a'}w - \overline{b'}z) \cap \mathbb{D}^2 = \emptyset \), it is easy to show \( |a'| + |b'| \leq 1 \). Therefore, \( 0 < a' < 1; 0 < b' < 1 \) and \( a' + b' = 1 \), as desired.

(2) We now consider the case \( Z(p) \cap \mathbb{T}^2 = \emptyset \), here \( p(z, w) = zw + az + bw \). From \( Z(p) \cap \mathbb{T}^2 = \emptyset \), we see

\[
Z(1 + \overline{a}w + \overline{b}z) \cap \mathbb{T}^2 = \emptyset,
\]

and hence by [26, Theorems 4.9.1, 5.2.5]

\[
Z(1 + \overline{a}w + \overline{b}z) \cap \mathbb{D}^2 = \emptyset.
\]

This implies that

\[
Z(p) \cap \{(z, w): |z| \geq 1, |w| \geq 1\} = \emptyset.
\]

Set

\[
E = \{(z, w) \in \mathbb{C}^2: zw + az + bw = 0, 1 + \overline{a}w + \overline{b}z = 0\},
\]

then clearly, \( E \) is nonempty, and
Without loss of generality, suppose $E \cap \{ |z| < 1, |w| > 1 \} \neq \emptyset$, that is, there exist $e_1, e_2$ such that $|e_1| < 1, |e_2| < 1$ and $(e_1, \frac{1}{e_2}) \in E$. Thus,
\[ e_1 + ae_1 \bar{e}_2 + b = 0 \]
and
\[ \bar{e}_2 + \bar{a} + be_1 \bar{e}_2 = 0. \]

Using a direct computation one sees that
\[
\eta(\phi_{-e_1}(z), \phi_{-e_2}(w)) = \frac{z + e_1}{1 + e_1 z} + \frac{w + e_2}{1 + e_2 w} + \frac{a}{1 + e_1} + \frac{b}{1 + e_2}.
\]

Let $\beta = \frac{1 + a \bar{e}_2 + b e_1}{1 + a \bar{e}_2 + b e_1}$, $\alpha = \frac{e_1}{1 + a \bar{e}_2 + b e_1}$, then $\alpha \neq 0$.
\[
\eta(\phi_{-e_1}(z), \phi_{-e_2}(w)) = \beta \frac{zw - \alpha}{1 - \bar{\alpha}z}.
\]

Define $\Phi(z, w) = (\phi_{-e_1}(\theta z), \phi_{-e_2}(w))$, where $\theta = \frac{\alpha}{|\alpha|}$. Then $\Phi$ is a holomorphic automorphism on $\mathbb{D}^2$ and $\eta(\Phi(z, w)) = \beta \theta \frac{zw - |\theta|}{1 - |\theta| z}$, completing the proof. \( \square \)

Now we come to the proof of Theorem 3.2.

**Proof of Theorem 3.2.** By Proposition 3.7, we can reduce the problem to two special cases. We consider the case (2) in Proposition 3.7. The same argument applies in the case (1).

From (2) in Proposition 3.7, without loss of generality, we suppose that there exists a holomorphic automorphism $\Phi(z, w) = (\phi_d(z), \phi_b(w))$ such that
\[
\tau(z, w) = \eta(\Phi(z, w)) = \eta(\phi_d(z), \phi_b(w)) = \frac{zw - t}{1 - tz}.
\]
for some $0 < t < 1$. Define the operator $U : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ by
\[
(Uf)(z, w) = f(\Phi(z, w))k_{(a, b)}(z, w).
\]

It is well known that $U$ is a unitary operator and $UM_f^* U^* = M_{f \circ \Phi}$ for any $f \in H^\infty(\mathbb{D}^2)$. Let $N_1 = H^2(\mathbb{D}^2) \ominus [\eta]$, $N_2 = H^2(\mathbb{D}^2) \ominus [\tau]$, and for $f \in H^\infty(\mathbb{D}^2)$, let $S_{f}^{N_1} = P_{\eta}M_f|_{N_1}$, $S_{f}^{N_2} = P_{\tau}M_f|_{N_2}$ be compressions of $M_f$ on the quotient modules, respectively. Then it is easy to see that $U$ maps $N_1$ onto $N_2$, and $UP_{\eta}U^* = P_{\tau}$. Therefore,
\[
US_{f}^{N_1} U^* = S_{f \circ \Phi}^{N_2}.
\]
By the proof of Proposition 3.4, there exists an integer \( n \) such that
\[
\| S_{z^{n+w^n}}^N \| > \| S_{z^{n+w^n}}^N \| e.
\]
This implies that
\[
\| S_{\phi_1^n + \phi_2^n}^N \| = \| U^* S_{z^{n+w^n}}^N U \| > \| U^* S_{z^{n+w^n}}^N U \| e = \| S_{\phi_1^n + \phi_2^n}^N \| e.
\]
By Theorem 3.1, the identity representation \( C^*(S_z^{N_1}, S_w^{N_1}) \) is a boundary representation of \( \mathcal{B}(S_z^{N_1}, S_w^{N_1}) \). The same argument is valid for case (1), completing the proof. \( \square \)

4. \( K \)-homology

For the submodule \( M = [\eta] = \eta H^2(\mathbb{D}^2) \) generated by an inner function \( \eta \), if the quotient module \( N = H^2(\mathbb{D}^2) \ominus [\eta] \) is essentially normal, then it yields the following exact extension
\[
0 \to \mathcal{K} \hookrightarrow C^*(S_z, S_w) \to C(\sigma_e(S_z, S_w)) \to 0.
\]
From BDF-theory [10], the exact sequence yields a \( K_1 \) cycle of \( \sigma_e(S_z, S_w) \), which is a natural module invariant. A natural question is whether or not this \( K_1 \) cycle is trivial. In this section, we will show that this \( K_1 \) cycle is nontrivial.

**Theorem 4.1.** Let \( M \) be the submodule generated by an inner function \( \eta \). If the quotient module \( M \perp \) is essentially normal, then the exact extension
\[
0 \to \mathcal{K} \hookrightarrow C^*(S_z, S_w) \to C(\sigma_e(S_z, S_w)) \to 0
\]
determines a nontrivial \( K_1 \) cycle.

Let us firstly recall [24, Lemma 5.5], which is implied by the Universal Coefficient Theorem in [9].

**Lemma 4.2.** If a \( C^* \)-algebra extension
\[
0 \to \mathcal{K} \hookrightarrow A \xrightarrow{\pi} C(X) \to 0
\]
is trivial, then for any natural number \( n \) and Fredholm operator \( A \in A \otimes M_n \), we have \( \text{Ind} A = 0 \), where \( M_n \) is the algebra of all complex \( n \times n \) matrices.

In what follows, we will complete the proof of Theorem 4.1. Since \( H^2(\mathbb{D}^2) \ominus [\eta] \) is essentially normal, \( \eta \) has one of the forms in Theorem 1.2.

**Case 1.** If \( \eta = \phi_\alpha(z) \) for \( |\alpha| < 1 \), then \( S_{w} \) is a unilateral shift on the quotient module and hence \( \text{Ind} S_w = -1 \). Applying Lemma 4.2 shows that \( K_1 \) cycle is nontrivial. The same argument is valid in the case \( \eta = \phi_\alpha(w) \).
Case 2. Now we consider the case of \( \eta = \phi_a(z)\phi_b(w) \) for some \( a, b \) satisfying \( |a| < 1, |b| < 1 \). As done in Case 2 of the proof of Theorem 1.3, we will only consider \( a = b = 0 \) and \( \eta = zw \). From the argument in Case 2 of the proof of Theorem 1.3, \( Sz + w - 1 \otimes z \) is a unilateral shift of multiplicity 2 and hence \( \text{Ind} (Sz + w) = -2 \). This implies that \( K_1 \) cycle is nontrivial by Lemma 4.2.

Case 3. If \( \eta = \frac{zw + az + bw + c}{1 + aw + zw + cw} \) with \( c \neq ab \), then the exact sequence

\[
0 \to K_C S_w \to C(S_z, S_w) \to C(S_z, S_w) \to 0
\]

determines a nontrivial \( K_1 \) cycle.

To see this, let’s consider the special case \( \eta(z, w) = zw - a \) for some \( a \) satisfying \( 0 < a < 1 \). By the proof of Proposition 3.6,

\[
\sigma_e(S_z, S_w) \subseteq \{(z, az^{-1}): |z| = 1\} \cup \{(aw^{-1}, w): |w| = 1\}.
\]

This implies \( S_{z-w} \) is Fredholm. We claim \( \text{Ind} (S_{z-w}) \neq 0 \). Indeed, it is easy to see both \( K(\sqrt{a}, \sqrt{a}) \) and \( K(-\sqrt{a}, -\sqrt{a}) \) are in \( \ker S^*_z \cap [\eta]^\perp \). Thus it suffices to show that \( \ker S_{z-w} = 0 \). Since \( (M_{z-w}, M_{zw-ca}) \) is Fredholm for \( 0 \leq c \leq 1 \), by [13],

\[
\text{Ind}(M_{z-w}, M_{zw-a}) = \text{Ind}(M_{z-w}, M_{zw}) = \text{Ind}(M_{z-w}, M_z) + \text{Ind}(M_{z-w}, M_w) = -2.
\]

Furthermore, a direct computation shows that

\[
\ker(M_{z-w}, M_{zw-a}) = \text{span}\{K(\sqrt{a}, \sqrt{a}), K(-\sqrt{a}, -\sqrt{a})\}
\]

and has dimension 2. Considering the fact \( \ker(M_{z-w}, M_{zw-a}) = 0 \), we have the sequence

\[
0 \to H^2(\mathbb{D}^2) \xrightarrow{(M_{zw-a})} H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}^2) \xrightarrow{(M_{z-w}, M_{zw-a})} H^2(\mathbb{D}^2) \to 0
\]

is exact at the step \( H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}^2) \), that is,

\[
\{(f, g): (z-w)f = (zw-a)g\} = \{((zw-a)h, (z-w)h)\}.
\]

Now suppose that \( S_{z-w} f = 0 \) for some \( f \in H^2(\mathbb{D}^2) \). This means that \( (z-w)f = \eta g \) for some \( g \in H^2(\mathbb{D}^2) \), that is,

\[
(z-w)(1-azw)f = (zw-a)g.
\]

From (4.1), there exists \( h \in H^2(\mathbb{D}^2) \) such that

\[
(1-azw)f = (zw-a)h
\]
and hence \( f = \eta h \in [\eta] \). Therefore, \( f = 0 \) and \( \ker S_{z \leftarrow w} = \emptyset \). Hence, the claim is true and \( \text{Ind} S_{z \leftarrow w} = -2 \). By Lemma 4.2, the \( K_1 \) cycle is nontrivial.

In the case \( \eta(z, w) = \frac{zw - az - (1-a)w}{1 - aw - (1-a)z} \) for some \( a \) satisfying \( 0 < a < 1 \), the same argument shows that \( \text{Ind} S_z = -1 \) and hence \( K_1 \) cycle is nontrivial.

Using the Möbius transformation and the similar argument in the proof of Theorem 3.2, one sees that in the case \( \eta = \frac{zw + az + bw + c}{1 + aw + bw + c} \) with \( c \neq ab \), the \( K_1 \) cycle is nontrivial, as desired. \( \square \)

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**Appendix A**

This appendix will complete the proof of Theorem 2.3.

**Theorem A.** Consider \( f \in H^\infty(D^2) \). If there exist two measurable subsets \( E_1, E_2 \subseteq \mathbb{T} \) of positive measure such that for every \( w \in E_1 \), \( f(\cdot, w) \) is a rational function, and \( f(z, \cdot) \) is also a rational function for every \( z \in E_2 \), then \( f(z, w) \) is a rational function.

**Proof.** Let

\[
F_n = \{ w \in E_1 : f(\cdot, w) \text{ is a rational function whose denominator has degree } n \}.
\]

It is not difficult to verify that each \( F_n \) is Lebesgue measurable. Since \( \bigcup_n F_n = E_1 \), there exists a natural number \( N \) such that \( F_N \) has positive measure. Set \( E = F_N \). This means that for any fixed \( w \in E \), \( f(z, w) \) has a unique expression

\[
f(z, w) = \frac{\sum_{k=0}^M \psi_k(w) z^k}{\sum_{k=0}^N \psi_k(w) z^k}, \tag{A.0}
\]

where \( \psi_N(w) = 1 \), and the denominator and numerator are coprime.

Let \( f(z, w) = \sum_{n=0}^\infty f_n(w) z^n \) be the expanding of \( f \) in variable \( z \). Since \( f(z, w) \in H^\infty(D^2) \), this means that \( f_n(w) \in H^\infty(D) \) for each \( n \). A direct computation shows that

\[
\sum_{m=0}^\infty \left( \sum_{j=0}^m f_{m-j}(w) \psi_j(w) \right) z^m = \sum_{m=0}^M \varphi_m(w) z^m, \tag{A.1}
\]

where \( \psi_{N+1} = \psi_{N+2} = \cdots = 0 \). Let \( N' = \max(N, M) \). By the formula (A.1), for any \( w \in E \) and \( m > N' \),

\[
f_m(w) \psi_0(w) + \cdots + f_{m-N-1}(w) \psi_{N-1}(w) + f_{m-N}(w) = 0. \tag{A.2}
\]

We claim that for any fixed \( w \in E \), the vector space

\[
V_w = \operatorname{span}\left\{(f_m(w), \ldots, f_{m-N+1}(w)) \in \mathbb{C}^N : m > N' \right\} = \mathbb{C}^N.
\]
To see this, assume there is a nonzero vector \((r_0, \ldots, r_{N-1})\) of \(\mathbb{C}^N\) such that \((r_0, \ldots, r_{N-1}) \perp V_w\), that is,

\[ r_0 f_m(w) + \cdots + r_{N-1} f_{m-N+1}(w) = 0 \]

for any \(m > N'\). Set \(\widetilde{\psi}_N = 1\) and \(\tilde{\psi_i}(w) = \psi_i(w) + r_i\) for \(0 \leq i \leq N - 1\). An easy verifying shows that \(f(z, w) \sum_{k=0}^{N'} \tilde{\psi_k}(w) z^k\) is a polynomial in variable \(z\) with degree at most \(N'\), that is

\[ f(z, w) = \sum_{k=0}^{N'} \tilde{\varphi_k}(w) z^k / \sum_{k=0}^{N} \tilde{\psi_k}(w) z^k, \]

which leads to a contradiction since the expression of (A.0) is unique.

Let \(I = (i_0, i_1, \ldots, i_{N-1})\), where \(N' < i_0 < i_1 < \cdots < i_{N-1}\), and let

\[ E_I = \{ w \in E : \text{span}\{ (f_{i_k}(w), \ldots, f_{i_k-N+1}(w)) : k = 0, 1, \ldots, N - 1 \} = \mathbb{C}^N \}. \]

The above claim shows that

\[ E = \bigcup_{\text{all } I} E_I. \]

It follows from the above equality that there exists a tuple \(M = (m_0, \ldots, m_{N-1})\) satisfying \(N' < m_0 < m_1 < \cdots < m_{N-1}\) such that \(E_M\) has positive measure. Set \(E' = E_M\), then

\[ \text{span}\{ (f_{m_i}(w), \ldots, f_{m_i-N+1}(w)) : i = 0, \ldots, N - 1 \} = \mathbb{C}^N \]

for any \(w \in E'\). This implies that the \(N \times N\) matrix \(A(w) = [f_{m_i-j}(w)]_{0 \leq i, j \leq N-1}\) is nonsingular for each \(w \in E'\). Applying Eq. (A.2) shows that

\[ \psi_k(w) = \frac{g_k(w)}{\det A(w)} \]

for \(0 \leq k \leq N - 1\), where \(g_k(w)\) is a finite sum of finite products of \(f_{m_i-j}(w)\) for \(0 \leq i \leq N - 1, 0 \leq j \leq N\), and hence by Eq. (A.1)

\[ \varphi_k(w) = \frac{h_k(w)}{\det A(w)}, \]

where \(h_k(w)\) is a finite sum of finite products of \(f_0(w), \ldots, f_{m_{N-1}}(w)\), and \(k = 0, 1, \ldots, M\). Let \(g_N(w) = \det A(w)\), then by (A.0) for \(w \in E'\),

\[ f(z, w) = \frac{\sum_{k=0}^{N'} h_k(w) z^k}{\sum_{k=0}^{N} g_k(w) z^k}. \]

By the above reasoning, \(h_k, g_k \in H^\infty(\mathbb{T}^2)\). This implies that
\[ f(z, w) = \frac{\sum_{k=0}^{N'} h_k(w) z^k}{\sum_{k=0}^{N} g_k(w) z^k}, \quad z, w \in \mathbb{D}. \] (A.3)

Write \( f(z, w) = \sum_{n=0}^{\infty} \phi_n(z) w^n \), then \( \phi_n(z) = \frac{1}{n!} \frac{\partial^n f}{\partial w^n} |_{w=0} \). By the formula (A.3), one sees that \( \phi_n(z) \) is a rational function for each \( n \).

Repeating the above proof in variable \( w \), we conclude that there exists a measurable subset \( E'' \subseteq \mathbb{T} \) of positive measure such that for each \( z \in E'' \),

\[ f(z, w) = \frac{\sum_{k=0}^{N''} \alpha_k(z) w^k}{\sum_{k=0}^{N'} \beta_k(z) w^k}, \]

where \( \alpha_k(z), \beta_k(z) \) are finite sums of finite products of \( \phi_0(z), \ldots, \phi_m(z) \) for some natural number \( m \). Since each \( \phi_n(z) \) is rational, this insures that \( f(z, w) \) is a rational function, completing the proof. \( \square \)

Furthermore, a careful analysis of the above proof shows that the next corollary is true.

**Corollary B.** For \( f \in H^\infty(\mathbb{D}^2) \), if there exist two subsets \( E_1, E_2 \subseteq \mathbb{T} \) of positive measure such that for \( w \in E_1 \), \( f(\cdot, w) \) is a rational function with degree at most \( r_1 \), and \( f(z, \cdot) \) is a rational function with degree at most \( r_2 \) for \( z \in E_2 \), then \( f(z, w) \) is a rational function with degree at most \( (r_1, r_2) \).

**References**