On the Spectra of Gaussian Matrices

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ABSTRACT

We give a simple characterization of the moduli of the eigenvalues of a complex Gaussian matrix in terms of χ^2 distributions. We also show that the spectral radius of a $k \times k$ complex Gaussian matrix is stochastically smaller than the norm of a $k \times (k+1)$ real Gaussian matrix.

INTRODUCTION

Theorem 1.1 below gives a simple characterization of the moduli of the eigenvalues of a $k \times k$ complex Gaussian matrix in terms of χ^2 distributions. It states that the squared moduli of the eigenvalues behave like independent χ^2_{2k} distributions as i runs from one to k. The argument is similar to that of Ginibre [3], but the emphasis there is on the spectral radius. Theorem 2.2 gives a relationship between the distribution of the spectral radius of a $k \times k$ complex Gaussian matrix, the norm of a $k \times (k+1)$ real Gaussian matrix, and the χ^2_{2k} distribution. In particular, we establish a stochastic ordering for these three random variables. We begin with basic definitions and notation.

Definition. A real Gaussian matrix is a matrix whose elements are independent standard Gaussian variables. A complex Gaussian matrix is a matrix whose real and imaginary parts are independent real Gaussian matrices.

Notation. $M^{(1)}_{k,n}$ will denote a $k \times n$ real Gaussian matrix. $M^{(2)}_{k,n}$ will denote a $k \times n$ complex Gaussian matrix. We will assume that $k \leqslant n$.

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NOTATION. For any matrix M, $\| M \|_2$ will denote the operator norm of M with respect to the Euclidean norm. For any square matrix M, $\sigma(M)$ will denote the spectral radius of M, and per M will denote the permanent of M.

NOTATION. $\{X_i\}$ will denote independent nonnegative random variables such that X_i^2 has a χ_i^2 distribution.

1. GAUSSIAN MATRICES AND X^2 DISTRIBUTIONS

THEOREM 1.1. *The collection of moduli of the eigenvalues of $M_{k,k}^{(2)}$ has the same distribution as the collection of random variables* $\{X_{2i}\}_{i=1,\cdots,k}$.

Theorem 1.1 will follow immediately from Lemma 1.4 and Lemma 1.5. From this theorem we have the immediate

COROLLARY 1.2. $\operatorname{Prob}[\sigma(M_{k,k}^{(2)}) > z] = \operatorname{Prob}[\max\{X_{2i}\}_{i=1,k} > z]$. ∎

For a discussion of $\sigma(M_{k,k}^{(1)})$, see Geman [2]. We will also make use of an analog of Corollary 1.2:

THEOREM 1.3. $\operatorname{Prob}[\| M_{k,n}^{(\beta)} \|_2 > z] \geqslant \operatorname{Prob}[\max\{X_{\beta(n+k-2i+1)}\}_{i=1,\cdots,k} > z]$.

Proof. This follow from inspection of the columns of the semidiagonalization of Gaussian matrices discussed in Silverstein [5]. Silverstein only proves the real case, but as mentioned by Edelman [1], his argument can be generalized. ∎

LEMMA 1.4. *Let $r_1 \geqslant \cdots \geqslant r_k$ be the moduli of the eigenvalues of* $\mathrm{M}_{k,k}^{(2)}$. *Then the joint density of $(r_i)_{i=1,\ldots,k}$ is given by*

$$\bar{B}_k \operatorname{per}\left[r_i^{2j-1}\right]_{i,j=1,\ldots,k} \exp\left[-\frac{1}{2}\sum_{i=1}^{k} r_i^2\right], \qquad \text{where} \quad \bar{B}_k = 2^k \prod_{j=1}^{k} \Gamma(j)^{-1}.$$

Proof. Let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of $M_{k,k}^{(2)}$, $|\lambda_1| \geqslant \cdots \geqslant |\lambda_k|$. Then the joint density of $(\lambda_i)_{i=1,\ldots,k}$ is given by

$$B_k \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\left[-\frac{1}{2}\sum_{i=1}^{k} |\lambda_i|^2\right], \qquad \text{where} \quad B_k = \pi^{-k} \prod_{j=1}^{k} \Gamma(j)^{-1}$$

—see Ginibre [3] for a proof and discussion. The quantity $\prod_{i<j} |\lambda_i - \lambda_j|^2$ is the squared modulus of the Vandermonde determinant:

$$\prod_{i<j} |\lambda_i - \lambda_j|^2 = \left| \sum_{\sigma \in S_k} (-1)^{\operatorname{sign}(\sigma)} \prod_{i=1}^{k} \lambda_i^{\sigma(i)-1} \right|^2,$$

where S_k denotes the permutation group on k symbols. Write $\lambda_i = r_i e^{i\theta_i}$. If $\sigma(j) \neq \sigma'(j)$ for some j, then

$$\int_0^{2\pi} \left(\prod_{i=1}^{k} r_i^{\sigma(i)-1} e^{i(\sigma(i)-1)\theta_i} \right) \overline{\left(\prod_{i=1}^{k} r_i^{\sigma'(i)-1} e^{i(\sigma'(i)-1)\theta_i} \right)} \, d\theta_j = 0.$$

Thus

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i<j} |\lambda_i - \lambda_j|^2 \, d\theta_1 \cdots d\theta_k = (2\pi)^k \sum_{\sigma \in S_k} \prod_{i=1}^{k} r_i^{2\sigma(i)-2}$$
$$= (2\pi)^k \operatorname{per}\left[r_i^{2j-2} \right]_{i,j=1,\dots,k}.$$

Multiplying this by

$$\left(\prod_{i=1}^{k} r_i \right) \exp\left[-\frac{1}{2} \sum_{i=1}^{k} r_i^2 \right]$$

establishes the lemma. ■

LEMMA 1.5. *Assume we are given an ordered k-tuplet of independent random variables* $(A_i)_{i=1,\dots,k}$, *with corresponding densities* $(\rho_i)_{i=1,\dots,k}$. *Define a new k-tuplet of random variables*, $(B_i)_{i=1,\dots,k}$, *as a random permutation of the* (A_i), *each permutation considered equal in probability. Then the joint density of the random vector* $(B_i)_{i=1,\dots,k}$ *is given by* $(1/k!)$ $\operatorname{per}[\rho_i(B_j)]_{i,j=1,\dots,k}$. ■

2. BARGMANN-MONTGOMERY-VON NEUMANN TYPE ESTIMATES

LEMMA 2.1.

$$\operatorname{Prob}\left[X_{\beta(n+k-1)} > z \right] \leqslant \operatorname{Prob}\left[\|M_{k,n}^{(\beta)}\|_2 > z \right]$$
$$\leqslant \frac{\Gamma(\beta/2)\Gamma(\beta(n+k-1)/2)}{\Gamma(\beta k/2)\Gamma(\beta n/2)} \operatorname{Prob}\left[X_{\beta(n+k-1)} > z \right].$$

Furthermore the right hand side is asymptotic to equality as $z \to \infty$.

It follows that this bound is optimal among all bounds of the form

$$C_1 \operatorname{Prob}\left[X_j > z \right] \leqslant \operatorname{Prob}\left[\|M_{k,n}^{(\beta)}\|_2 > z \right] \leqslant C_2 \operatorname{Prob}\left[X_j > z \right],$$

where C_1, C_2, and j depend on n, k, and β, but not on z.

Proof. The left hand inequality follows immediately from theorem 1.3. The proof of the right hand inequality can be found in Goldstine and von Neumann [4, II.8.2]. Goldstine and von Neumann restrict their attention to the real square Gaussian matrices, but the argument can be generalized. See Edelman [1] for a discussion. ■

THEOREM 2.2.

$$\begin{aligned} \operatorname{Prob}\left[X_{2k} > z \right] &\leqslant \operatorname{Prob}\left[\sigma\left(M_{k,k}^{(2)}\right) > z \right] \\ &\leqslant \operatorname{Prob}\left[\|M_{k,k+1}^{(1)}\|_2 > z \right] \leqslant 2^{k-1} \operatorname{Prob}\left[X_{2k} > z \right]. \end{aligned}$$

Furthermore the left and right hand inequalities are asymptotic to equality as $z \to \infty$.

Proof. The left hand inequality follows from Corollary 1.2. It can be seen to by asymptotic by inspection of the density given in Lemma 1.4. The central inequality follows from Corollary 1.2 and Theorem 1.3. The right hand inequality is a special case of Lemma 2.1. ■

REFERENCES

1 A. Edelman, Eigenvalues and condition numbers of random matrices, *SIAM J. Matrix Anal. Appl.* 9:543–560 (1988).

2 S. Geman, The spectral radius of large random matrices, *Ann. Probab.* 14:1318–1328 (1986).

3 J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, *J. Math. Phys.* 6:440–449 (1965).

4 H. Goldstine and J. von Neumann, Numerical inverting of matrices of high order, in J. von Neumann, *Collected Works*, Vol. V, Pergamon, Oxford, 1963.

5 J. W. Silverstein, The smallest eigenvalue of a large dimensional Wishart matrix, *Ann. Probab.* 13:1364–1368 (1985).

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