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A uniform bound on the nilpotency degree of certain subalgebras of Kac–Moody algebras

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Abstract

Let $\mathfrak{g}$ be a Kac–Moody algebra and $\mathfrak{b}_1, \mathfrak{b}_2$ be Borel subalgebras of opposite signs. The intersection $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2$ is a finite-dimensional solvable subalgebra of $\mathfrak{g}$. We show that the nilpotency degree of $[\mathfrak{b}, \mathfrak{b}]$ is bounded above by a constant depending only on $\mathfrak{g}$. This confirms a conjecture of Y. Billig and A. Pianzola [Y. Billig, A. Pianzola, Root strings with two consecutive real roots, Tohoku Math. J. (2) 47 (3) (1995) 391–403].

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1. Introduction

Let A be a generalized Cartan matrix, let $\mathfrak{g}$ be a Kac–Moody algebra of type A with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and W be its Weyl group. For each $w \in W$ we set $\Delta(w) = \{\alpha \in \Delta_+ \mid w.\alpha < 0\}$ and $\mathfrak{g}_w = \langle \mathfrak{g}_\alpha \mid \alpha \in \Delta(w) \rangle$. It is known that $\Delta(w)$ is finite and that $\mathfrak{g}_w$ is a finite-dimensional nilpotent subalgebra of $\mathfrak{g}$. The main result of this paper is the following:

Theorem 1.1. *The nilpotency degree of $\mathfrak{g}_w$ is bounded above by a constant depending on A, but not on w.*

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This statement was conjectured in [BP95, Conjecture 1]. In view of [Tit87, Proposition 1], it is equivalent to its group theoretic counterpart, which can be stated as follows. Let G be the complex simply connected Kac–Moody group of type A. For each $\alpha \in {}^{\mathrm{re}}\Delta$, let U_α be the one-parameter subgroup of G with Lie algebra $\mathfrak{g}_\alpha$ and for all $w \in W$, let $U_w = \langle U_\alpha \mid \alpha \in \Delta(w)\rangle$.

Theorem 1.2. *The nilpotency degree of U_w is bounded above by a constant depending on A, but not on w.*

The latter statement holds not only for the complex Kac–Moody group G, but in fact for any split or almost split Kac–Moody group over an arbitrary field: this follows from a reformulation of Theorem 1.1 in terms of root systems (see Proposition 2.2 below) together with the description of commutation relations in Kac–Moody groups [Mor87, Theorem 2]. The following consequence of Theorem 1.2 was pointed out to me by B. Rémy:

Corollary 1.3. *Let G be a split or almost split Kac–Moody group over a finite field $\mathbb{F}_q$. Then there exists a constant $N \in \mathbb{N}$, depending only on G, such that every element of G is either of infinite order or of order smaller than N.*

It is useful to keep in mind the group-side of the theory. For example, in the case where A is of finite or affine type, Theorem 1.2 is an immediate consequence of the fact that the group G is linear (modulo center). On the other hand, for any other type of generalized Cartan matrix, the group G is known to be nonlinear [Cap06a, Theorem 7.1]. Our proof of Theorem 1.1 is based on a reduction to the affine case. The main tools are, on the one hand, the classification of pairs of real roots whose sum is a real root, due to Y. Billig and A. Pianzola [BP95], and, on the other hand, on some sufficient conditions on a set of roots to generate an affine subsystem, which were established in [Cap06b].

We remark that the bound on the nilpotency degree of $\mathfrak{g}_w$ that could be extracted from the proof below is far from sharp. It is likely that a much sharper bound could be expressed as an affine function of the maximal size of a Cartan submatrix of finite type of A. In fact, the proof below supports the intuition that the sharp bound depends actually on the size of submatrices of finite type of A, rather than on the size of A itself.

2. Nilpotent sequences in root systems

2.1. Definition

A set of roots $\Phi \subset \Delta$ is called *prenilpotent* if there exist $w, w' \in W$ such that $w.\Phi \subset \Delta_+$ and $w'.\Phi \subset \Delta_-$. In particular $\Phi \subset {}^{\mathrm{re}}\Delta$. The set Φ is called *closed* if for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta$ is a root, we have $\alpha + \beta \in \Phi$. Since the intersection of any collection of closed subsets is closed, it makes sense to consider the closure of a set of roots. The following lemma is obvious:

Lemma 2.1. *The closure of any prenilpotent set of roots is prenilpotent.*

Proof. The closure of a set Φ is contained in $(\sum_{\alpha\in\Phi} \mathbb{Z}_+\alpha) \cap \Delta$. $\square$

A sequence of roots $(\beta_k)_{k=1,\dots,n}$ is called *nilpotent* if it satisfies the following conditions:

(NS1) The set $\{\beta_1, \dots, \beta_k\}$ is prenilpotent.
(NS2) For each $k = 1, \dots, n$, we have $\sum_{i=1}^{k} \beta_i \in \Delta$.

For all $\alpha, \beta \in \Delta$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Furthermore for each $w \in W$, the prenilpotent set $\Delta(w)$ is closed. Therefore, the subalgebra $\mathfrak{g}_w$ splits as a direct sum $\mathfrak{g}_w = \bigoplus_{\alpha\in\Delta(w)} \mathfrak{g}_\alpha$. It is easy to deduce from these basic facts that the nilpotency degree of $\mathfrak{g}_w$ coincides with the maximal possible length of a nilpotent sequence of roots contained in $\Delta(w)$. Therefore, the following statement is equivalent to Theorems 1.1 and 1.2 and can be viewed as its root system version:

Proposition 2.2. *The supremum of the set of lengths of nilpotent sequences of roots in Δ is finite.*

The proof of Proposition 2.2 is deferred to Section 2.3. We first need to collect a series of subsidiary results: this is the purpose of the next subsection.

2.2. *On infinite root systems and their geometric realizations*

We freely use the standard notation and terminology on infinite root systems which can be found in [MP95, Chapter 5]. We view Δ as the root system of a set of root data $\mathscr{D} = (A, \Pi, \Pi^\vee, V, V^\vee, \langle\cdot,\cdot\rangle)$ over $\mathbb{R}$ and denote by W its Weyl group. We assume that the set Π is finite. Moreover, we need to consider a geometric realization of ${}^{\mathrm{re}}\Delta$; we henceforth denote by X the interior of the Tits cone $\mathfrak{X}^\vee \subset V^\vee$. Recall that X is W-invariant and that the induced action is properly discontinuous [MP95, Chapter 5, Proposition 15]. For each root $\alpha \in {}^{\mathrm{re}}\Delta$, we set $D(\alpha) = \{x \in X \mid \langle\alpha, x\rangle > 0\}$ and $\partial\alpha = \{x \in X \mid \langle\alpha, x\rangle = 0\}$. The set $\partial\alpha$ is called a *wall*; it is the trace on X of a hyperplane of $V^\vee$ and it cuts X into two nonempty convex open cones, called *half-spaces*, namely $D(\alpha)$ and $D(-\alpha)$. Note that walls and half-spaces are convex. The notion of convexity will be crucial to our purposes.

Remark. Instead of the interior of the Tits cone, we might equally use the Davis complex associated with the Weyl group W. This also provides a convenient geometric realization of ${}^{\mathrm{re}}\Delta$, which has no linear structure but is instead equipped with a W-invariant $\mathsf{CAT}(0)$-metric. This allows to define walls and half-spaces and yields an appropriate notion of (geodesic) convexity. The 1-skeleton of the Davis complex is nothing but the Cayley graph of W with respect to its canonical Coxeter generating set S. This graph may be embedded in the interior of the Tits cone by considering as vertex set the W-orbit of a point in the interior of the fundamental Weyl chamber and this makes it easy to pass from one viewpoint to the other. In the present note, we keep the Tits cone viewpoint throughout, but we will be led to quote references which use rather the Davis complex as a preferred geometric realization.

The following lemma collects a few basic facts on pairs of roots:

Lemma 2.3. *Let $\alpha, \beta \in {}^{\mathrm{re}}\Delta$.*

(i) *The subsystem generated by α and β is finite if and only if ∂α meets ∂β.*
(ii) *$D(\alpha) \subset D(\beta)$ or $D(\alpha) \supset D(\beta)$ if and only if $\langle\alpha, \beta^\vee\rangle\langle\beta, \alpha^\vee\rangle \geqslant 4$ and $\langle\alpha, \beta^\vee\rangle > 0$.*
(iii) *The pair $\{\alpha, \beta\}$ is prenilpotent if and only if one of the following assertions holds*:

- $D(\alpha) \subset D(\beta)$,
- $D(\alpha) \supset D(\beta)$,
- $\alpha \neq -\beta$ *and the subsystem generated by* α *and* β *is finite.*

Proof. (i) follows from [MP95, Chapter 5, Proposition 14] and the fact that any finite subgroup of W fixes a point of X.

(ii) We may assume $\alpha \neq \pm\beta$, otherwise the desired assertion is obvious. In that case, we have $D(\alpha) \subset D(\beta)$ or $D(\alpha) \supset D(\beta)$ only if the subsystem generated by α and β is infinite in view of (i). Since $\alpha \neq \pm\beta$, this in turn is equivalent to $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \geqslant 4$. Now, if $\langle \alpha, \beta^\vee \rangle < 0$, then $\langle \beta, \alpha^\vee \rangle < 0$ by [MP95, Chapter 5, Proposition 8] and it readily follows that $D(\alpha) \cap D(\beta) \subset D(r_\beta(\alpha)) \cap D(r_\alpha(\beta))$. On the other hand, if $D(\alpha) \subset D(\beta)$, then, transforming by r_β, we obtain $D(r_\beta(\alpha)) \subset D(-\beta)$ whence $D(r_\beta(\alpha)) \cap D(\alpha) = \emptyset$. Similarly, if $D(\alpha) \supset D(\beta)$ then $D(r_\alpha(\beta)) \cap D(\beta) = \emptyset$. This shows that if $D(\alpha) \subset D(\beta)$ or $D(\alpha) \supset D(\beta)$ then $\langle \alpha, \beta^\vee \rangle > 0$. The converse statement follows because, if $D(\alpha) \not\subset D(\beta)$ and $D(\alpha) \not\supset D(\beta)$, then $D(-\alpha) \subset D(\beta)$ or $D(-\alpha) \supset D(\beta)$.

(iii) We may assume that the subsystem generated by $\{\alpha, \beta\}$ is infinite, otherwise the desired assertion is easy. For any root $\alpha \in {}^{\mathrm{re}}\Delta$, we have $\alpha > 0$ if and only if the half-space $D(\alpha)$ contains the Weyl chamber. Now the claim readily follows. $\square$

Lemma 2.4. *There exists a constant* K*, depending only on the generalized Cartan matrix* A*, such that the following condition holds. Given a prenilpotent pair* $\{\alpha, \beta\} \subset {}^{\mathrm{re}}\Delta$ *such that* $\alpha + \beta$ *is a root, then* $\langle \alpha, \beta^\vee \rangle \leqslant K$.

Proof. Follows from [BP95, Proposition 1 and Theorem 1]. $\square$

Lemma 2.5. *For any integer* n*, there exists a constant* $L(n)$*, depending on the generalized Cartan matrix* A*, such that any prenilpotent set of at least* $L(n)$ *roots contains a subset* $\{\alpha_1, \ldots, \alpha_n\}$ *of cardinality* n *such that* $D(\alpha_1) \subsetneq D(\alpha_2) \subsetneq \cdots \subsetneq D(\alpha_n)$.

Proof. It is shown in [NR03, Lemma 3] that there exists a constant $L(2)$ such that any set of more than $L(2)$ walls contains a pair of parallel walls (i.e. nonintersecting walls). Combining this with Ramsey's theorem (see [GRS80, Section 1.1, Theorem 1]), it follows that for any integer n, there exists a constant $L(n)$ such that any set of more than $L(n)$ walls contains a set of n pairwise parallel walls. Let now Φ be a prenilpotent set of roots of cardinality greater than $L(n)$. Hence Φ contains a subset Φ_0 of cardinality n such that the elements of $\partial\Phi_0 = \{\partial\alpha \mid \alpha \in \Phi_0\}$ are pairwise parallel. Since Φ_0 is prenilpotent, it follows from Lemma 2.3(iii) that the elements of $\{D(\alpha) \mid \alpha \in \Phi_0\}$ are totally ordered by inclusion. Thus they form a chain, as desired. $\square$

Lemma 2.6. *There exists a constant* M*, depending on the generalized Cartan matrix* A*, such that the following property holds. Let* $\alpha, \alpha', \beta_0, \ldots, \beta_n \in {}^{\mathrm{re}}\Delta$ *be real roots such that*:

(1) *The subsystem generated by* $\{\alpha, \alpha', \beta_0\}$ *is finite of rank* 2. (*Equivalently*: *we have* $\emptyset \neq \partial\alpha \cap \partial\alpha' \subset \partial\beta_0$.)
(2) $D(\beta_0) \subsetneq D(\beta_1) \subsetneq \cdots \subsetneq D(\beta_n)$.
(3) *For each* $i = 1, \ldots, n$*, the subsystem generated by* $\{\alpha, \beta_i\}$ (*respectively* $\{\alpha', \beta_i\}$) *is finite.* (*Equivalently*: *the wall* $\partial\beta_i$ *meets both* $\partial\alpha$ *and* $\partial\alpha'$.)

If $n \geqslant M$, then the subsystem generated by $\{\alpha, \alpha', \beta_0, \ldots, \beta_n\}$ is of irreducible affine type; furthermore, it is contained in (a conjugate of) a parabolic subsystem of affine type of Δ.

In particular the subsystem generated by $\{\beta_0, \ldots, \beta_n\}$ is of affine type and rank 2.

Proof. See [Cap06b, Theorem 8]. □

We will need to appeal to Lemma 2.6 several times in the proof of Proposition 2.2. The following lemma will be helpful when checking that the hypotheses of Lemma 2.6 are satisfied.

Lemma 2.7. *We have the following*:

(i) *Let $\{\alpha, \alpha', \gamma\} \subset \Delta$ be a prenilpotent set such that $D(\alpha) \subsetneq D(\alpha')$ or $D(\alpha) \supsetneq D(\alpha')$ and $\langle \alpha, \gamma^\vee \rangle < 0$. If $\partial r_\gamma(\alpha)$ meets $\partial\alpha'$, then so does $\partial\gamma$.*

(ii) *Let $\{\alpha, \alpha', \beta, \beta'\} \subset \Delta$ be a prenilpotent set such that $D(\alpha) \subsetneq D(\alpha')$ and $D(\beta) \subsetneq D(\beta')$. If $\partial\beta$ and $\partial\beta'$ both meet $\partial\alpha'$ and if $\partial\beta'$ meets $\partial\alpha$, then $\partial\beta$ meets $\partial\alpha$. Similarly, if $\partial\beta$ and $\partial\beta'$ both meet $\partial\alpha$ and if $\partial\beta$ meets $\partial\alpha'$, then $\partial\beta'$ meets $\partial\alpha'$.*

Proof. (i) Up to replacing $\{\alpha, \alpha', \gamma\}$ by the prenilpotent set $\{-\alpha, -\alpha', -\gamma\}$, we may assume without loss of generality that $D(\alpha') \subsetneq D(\alpha)$. Since $\langle \alpha, \gamma^\vee \rangle < 0$, it readily follows that $D(\alpha) \cap D(\gamma) \subset D(r_\gamma(\alpha))$. Assume now that $\partial\gamma$ does not meet $\partial\alpha'$. Then we have $D(\alpha') \subsetneq D(\gamma)$ by Lemma 2.3(iii) because, in view of Lemma 2.3, the wall $\partial\gamma$ meets $\partial\alpha$ and moreover we have $D(\alpha') \subsetneq D(\alpha)$. It follows that $D(\alpha') \subset D(\alpha) \cap D(\gamma) \subsetneq D(r_\gamma(\alpha))$. In particular the wall $\partial\alpha'$ does not meet $\partial r_\gamma(\alpha)$.

(ii) Suppose that $\partial\beta$ and $\partial\beta'$ both meet $\partial\alpha'$ and that $\partial\beta'$ meets $\partial\alpha$. Now assume in order to obtain a contradiction that $\partial\beta$ does not meet $\partial\alpha$. Then, by Lemma 2.3 we have $D(\alpha) \subset D(\beta)$ or $D(\alpha) \supset D(\beta)$. Since $\partial\beta$ meets $\partial\alpha'$ and $D(\alpha) \subset D(\alpha')$, we have in fact $D(\alpha) \subset D(\beta)$. On the other hand, since $\partial\beta'$ meets $\partial\alpha$, it follows that $D(\alpha) \cap D(-\beta')$ is nonempty. We deduce $\emptyset \neq D(\alpha) \cap D(-\beta') \subset D(\alpha) \subset D(\beta)$. This contradicts the fact that $D(\beta) \subset D(\beta')$ which implies $D(\beta) \cap D(-\beta') = \emptyset$.

The other assertion follows by considering the prenilpotent set $\{-\alpha, -\alpha', -\beta, -\beta'\}$. □

The next lemma provides useful sufficient conditions which ensure that a root belongs to a given parabolic subsystem of affine type:

Lemma 2.8. *Let $\Phi \subset \Delta$ be a parabolic subsystem of affine type and let $\alpha \in \Delta$. Then we have $\alpha \in \Phi$ provided that one of the following conditions is fulfilled*:

(i) *There exist $\beta, \beta' \in \Phi$ such that $D(\beta) \subsetneq D(\alpha) \subsetneq D(\beta')$.*

(ii) *There exist $\beta_1, \ldots, \beta_8, \gamma \in \Phi$ such that the elements of $\{\partial\beta_1, \ldots, \partial\beta_8\}$ are pairwise parallel, the wall $\partial\alpha$ meets $\partial\beta_i$ for all $i = 1, \ldots, 8$ and $\langle \alpha, \gamma^\vee \rangle \neq 0$.*

(iii) *There exist $\beta_1, \ldots, \beta_n \in \Phi$ such that $D(\alpha) \subsetneq D(\beta_1) \subsetneq \cdots \subsetneq D(\beta_n)$ and $\langle \beta_n, \alpha^\vee \rangle < \frac{n}{2}$.*

Proof. (i) follows from [Cap06b, Proposition 16 and Lemma 17].

(ii) follows from [Cap06b, Lemma 11, Proposition 16 and Lemma 22].

(iii) Assume that $\beta_1, \ldots, \beta_n \in \Phi$ are roots such that $D(\alpha) \subsetneq D(\beta_1) \subsetneq \cdots \subsetneq D(\beta_n)$ and suppose that $\alpha \notin \Phi$. We must prove that $\langle \beta_n, \alpha^\vee \rangle \geqslant \frac{n}{2}$.

Since Φ is of affine type, the condition $D(\beta_1) \subsetneq \cdots \subsetneq D(\beta_n)$ implies that the group $\langle r_{\beta_1}, \ldots, r_{\beta_n} \rangle$ is infinite dihedral [Cap06b, Theorem D and Proposition 14] and, hence, the subsystem Φ_0 generated by $\{\beta_1, \ldots, \beta_n\}$ is affine of rank 2. Let $\beta_0 \in \Phi_0$ be the root such that $\{-\beta_0, \beta_1\}$ is a basis of Φ_0, and define inductively $\beta_{-k} = -r_{\beta_{-k+1}}(\beta_{-k+2})$ for all $k > 0$. Thus, for all $k \geqslant 0$, we have $\beta_{-k} \in \Phi$, $D(\beta_{-k}) \subset D(\beta_{-k+1})$ and $\{-\beta_{-k}, \beta_{-k+1}\}$ is a basis of Φ_0. Note moreover that for all $k > 0$, we have $\langle \beta_k, \alpha^\vee \rangle > 0$ by Lemma 2.3(ii). We claim that there exists $k \geqslant 0$ such that $\langle \beta_{-k}, \alpha^\vee \rangle \leqslant 0$.

Suppose the contrary. If the wall $\partial\alpha$ meets at least 8 elements of $\{\partial\beta_{-k} \mid k > 0\}$, then we obtain $\alpha \in \Phi$ by (ii). Therefore, we may assume that $\partial\alpha$ is parallel to almost every element of $\{\partial\beta_{-k} \mid k > 0\}$. Suppose now that $\partial\beta_{-k} \subset D(\alpha)$ for some $k > 0$. Since $\langle \beta_{-k}, \alpha^\vee \rangle > 0$, we have $D(\beta_{-k}) \subset D(\alpha) \subset D(\beta_1)$ by Lemma 2.3(ii) and, hence, we obtain $\alpha \in \Phi$ by (i). Therefore, we may assume that $D(\alpha) \subset D(\beta_{-k}) \subset D(\beta_1)$ for almost every $k > 0$. Let $x \in \partial\alpha$ and $y \in \partial\beta_1$ be any points. It follows from the above that the segment $[x, y]$ meets $\partial\beta_{-k}$ for almost every $k \geqslant 0$, which contradicts [MP95, Chapter 5, Propositions 6 and 7]. This proves the claim.

Let now $i = \min\{k \geqslant 0 \mid \langle \beta_{-k}, \alpha^\vee \rangle \leqslant 0\}$. Hence $\{-\beta_{-i}, \beta_{-i+1}\}$ is a basis of Φ_0 and moreover $\langle \beta_{-i}, \alpha^\vee \rangle \geqslant 0$ and $\langle \beta_{-i+1}, \alpha^\vee \rangle \geqslant 1$. Define $\phi_1 = \beta_{-i+1}$, $\phi'_1 = r_{\beta_{-i+1}}(\beta_i)$ and for all $k > 0$, set $\phi_{k+1} = r_{\beta_{-i+1}} r_{\beta_{-i}}(\phi_k)$ and $\phi'_{k+1} = r_{\beta_{-i+1}} r_{\beta_{-i}}(\phi'_k)$. Thus we have $D(\beta_{-i+1}) = D(\phi_1) \subsetneq D(\phi'_1) \subsetneq D(\phi_2) \subsetneq D(\phi'_2) \subsetneq \cdots$ and moreover $\{\phi \in \Phi_0 \mid D(\beta_{-i+1}) \subset D(\phi)\} = \{\phi_k, \phi'_k \mid k > 0\}$. In particular, we have $\beta_n \in \{\phi_k, \phi'_k\}$ for some $k \geqslant \frac{n}{2}$.

Since $\{-\beta_{-i}, \beta_{-i+1}\}$ is a basis of Φ_0, we may write $\phi_k = x_k.(-\beta_{-i}) + y_k.\beta_{-i+1}$ for some nonnegative integers x_k, y_k; similarly $\phi'_k = x'_k.(-\beta_{-i}) + y'_k.\beta_{-i+1}$ for some $x'_k, y'_k \in \mathbb{Z}_+$. Since Φ_0 is of affine type, an easy computation shows that $y_{k+1} - y_k$ (respectively $y'_{k+1} - y'_k$) is a constant positive integer (it is independent of k). In particular, the sequence (y_k) (respectively (y'_k)) is a linear function of k with positive integral coefficient and, hence, we have $y_k \geqslant k$ (respectively $y'_k \geqslant k$) for all $k > 0$. It follows that

$$\begin{aligned}\langle \phi_k, \alpha^\vee \rangle &= x_k.\langle -\beta_{-i}, \alpha^\vee \rangle + y_k.\langle \beta_{-i+1}, \alpha^\vee \rangle \\ &\geqslant y_k \\ &\geqslant k.\end{aligned}$$

Similarly, we obtain $\langle \phi'_k, \alpha^\vee \rangle \geqslant k$. Since $\beta_n \in \{\phi_k, \phi'_k\}$ for some $k \geqslant \frac{n}{2}$, we finally obtain $\langle \beta_n, \alpha^\vee \rangle \geqslant \frac{n}{2}$. □

2.3. *Proof of Proposition 2.2*

Assume, in order to obtain a contradiction, that Δ contains nilpotent sequences of arbitrarily large length. Therefore, by Lemma 2.5, given any integer n, there exists a nilpotent sequence $(\beta_i)_{i=1,\ldots,k}$ such that the set $\{D(\beta_1), \ldots, D(\beta_k)\}$ contains a chain of half-spaces of length n. Assume now that k is fixed and let $I \subset \{1, \ldots, k\}$ be a subset *of maximal possible cardinality* such that $\{D(\beta_i) \mid i \in I\}$ is a chain of half-spaces. Provided that k is large enough, we may assume that $|I|$ is arbitrarily large.

For every $i \in \{1, \ldots, k\}$, we set

$$I(i) = \{j \in I \mid j < i\}$$

and

$$J(i)=\{j \mid 1 \leqslant j<i,\ \langle\beta_j,\beta_i^\vee\rangle<0\}.$$

Note that for all $j \in J(i)$, the wall $\partial\beta_j$ meets $\partial\beta_i$ by Lemma 2.3, since the pair $\{\beta_i, \beta_j\}$ is prenilpotent.

Claim 1. *For each $i \in I$, we have $|J(i)| \geqslant \frac{|I(i)|-K}{3}$, where K is the constant of Lemma* 2.4.

By (NS1) and Lemma 2.4, we have $\sum_{j=1}^{i-1}\langle\beta_j,\beta_i^\vee\rangle \leqslant K$. On the other hand, for each $j \in I$ we have $\langle\beta_j,\beta_i^\vee\rangle \geqslant 1$ by Lemma 2.3(ii) and for all $j \in J(i)$, we have $\langle\beta_j,\beta_i^\vee\rangle \geqslant -3$. We deduce successively:

$$\begin{aligned}
K &\geqslant \sum_{j\in I(i)}\langle\beta_j,\beta_i^\vee\rangle+\sum_{j<i,\, j\notin I}\langle\beta_j,\beta_i^\vee\rangle\\
&\geqslant |I(i)|+\sum_{j\in J(i)}\langle\beta_j,\beta_i^\vee\rangle\\
&\geqslant |I(i)|-3|J(i)|.
\end{aligned}$$

Claim 2. *Let m be an integer such that $|I| > 4m$. Suppose that I possesses an element i such that $|J(i)| \geqslant L(4m)$ and $m > M$, where L (respectively M) is as in Lemma* 2.5 (*respectively Lemma* 2.6). *Then there is a set $I_{\mathrm{aff}} \subset \{1,\dots,k\}$ of cardinality m, such that $\{D(\beta_i) \mid i \in I_{\mathrm{aff}}\}$ is a chain of half-spaces and the subsystem generated by $\{\beta_i \mid i \in I_{\mathrm{aff}}\}$ is of affine type and rank* 2, *and is contained in a parabolic subsystem of affine type.*

By assumption, there exist $4m$ elements $\lambda_1,\dots,\lambda_{4m} \in J(i)$ such that $D(\beta_{\lambda_1}) \subsetneq \cdots \subsetneq D(\beta_{\lambda_{4m}})$. Let $\gamma_1=\beta_i+\langle\beta_i,\beta_{\lambda_1}^\vee\rangle\beta_{\lambda_1}$ and $\gamma_{4m}=\beta_i+\langle\beta_i,\beta_{\lambda_{4m}}^\vee\rangle\beta_{\lambda_{4m}}$. Note that, by Lemma 2.1, the set $\{\beta_1,\dots,\beta_k\} \cup \{\gamma_1,\gamma_{4m}\}$ is prenilpotent. Let also $I_-=\{j \in I \mid D(\beta_j) \subsetneq D(\beta_i)\}$ and $I_+=\{j \in I \mid D(\beta_i) \subsetneq D(\beta_j)\}$.

Suppose first that there exists a subset $I_0 \subset I_-$ of cardinality m such that for each $j \in I_0$, the wall $\partial\beta_j$ meets $\partial\gamma_1$. Since $\lambda_1 \in J(i)$ and since the pair $\{\beta_i,\beta_{\lambda_1}\}$ is prenilpotent, it follows from Lemma 2.3 that $\{\beta_i,\beta_{\lambda_1}\}$ generates a finite subsystem. Furthermore, by Lemma 2.7(i), the wall $\partial\beta_{\lambda_1}$ meets $\partial\beta_j$ for all $j \in I_0$. Therefore, Lemma 2.6 ensures that $\{\beta_j \mid j \in I_0\}$ generates an affine subsystem of rank 2 which is contained in a parabolic subsystem of affine type. Thus we are done in this case.

Suppose similarly that there exists a subset $I_0 \subset I_+$ of cardinality m such that for each $j \in I_0$, the wall $\partial\beta_j$ meets $\partial\gamma_{4m}$. Then, by the same argument as in the preceding paragraph, we conclude that $\{\beta_j \mid j \in I_0\}$ generates an affine subsystem of rank 2 which is contained in a parabolic subsystem of affine type. Thus we are done in this case as well.

Suppose now that there exists a subset $I_1 \subset I_-$ of cardinality m such that for each $j \in I_1$ and for each $j' \in \{2,\dots,m\}$, the wall $\partial\beta_j$ meets $\partial\beta_{\lambda_{j'}}$. If for some $j' \in \{2,\dots,m\}$, the wall $\partial\gamma_1$ is parallel to $\partial\beta_{\lambda_{j'}}$, then it follows from Lemma 2.7(ii) applied to $\{\beta_i,\beta_j,\beta_{\lambda_1},\gamma_1\}$ that $\partial\gamma_1$ meets $\partial\beta_j$ for all $j \in I_1$. Thus we are reduced to a case which has already been settled. Therefore, we may assume that $\partial\gamma_1$ meets $\partial\beta_{\lambda_{j'}}$ for all $j' \in \{2,\dots,m\}$. In that case, Lemma 2.6 implies that

$\{\beta_{\lambda_1}, \dots, \beta_{\lambda_m}\}$ generates a subsystem of affine type and rank 2 which is contained in a parabolic subsystem of affine type.

Suppose similarly that there exists a subset $I_1 \subset I_+$ of cardinality m such that for each $j \in I_1$ and for each $j' \in \{3m+1, \dots, 4m-1\}$, the wall $\partial\beta_j$ meets $\partial\beta_{\lambda_{j'}}$. Then, by the same argument as in the preceding paragraph using γ_{4m} instead of γ_1, we conclude that $\{\beta_{\lambda_{3m+1}}, \dots, \beta_{\lambda_{4m}}\}$ generates an affine subsystem of rank 2 which is contained in a parabolic subsystem of affine type. Thus we are done in this case as well.

Let us now define $I' \subset I$ to be the subset consisting of all those js such that $\partial\beta_j$ meets $\partial\beta_{\lambda_{j'}}$ for some $j' \in \{m+1, \dots, 3m\}$. By Lemma 2.7(ii), if $j \in I' \cap I_-$ then $\partial\beta_j$ meets $\partial\beta_{\lambda_{j'}}$ for all $j' \in \{1, \dots, m\}$. It follows that we may assume $|I' \cap I_-| < m$, otherwise we are reduced to a case which has already been settled. Similarly, if $j \in I' \cap I_+$ then $\partial\beta_j$ meets $\partial\beta_{\lambda_{j'}}$ for all $j' \in \{3m+1, \dots, 4m\}$ and, as above, we may assume that $|I' \cap I_+| < m$. Since $I = I_- \cup \{i\} \cup I_+$, it follows that $I' = (I' \cap I_-) \cup \{i\} \cup (I' \cap I_+)$ and, hence, the last case which remains to be treated is when $|I'| < 2m$. Note that, by definition, the set $\{\partial\beta_j \mid j \in I \setminus I'\} \cup \{\partial\beta_{\lambda_{m+1}}, \dots, \partial\beta_{\lambda_{3m}}\}$ consists of pairwise parallel walls. Therefore, the set $\{D(\beta_j) \mid j \in I \setminus I'\} \cup \{D(\beta_{\lambda_{m+1}}), \dots, D(\beta_{\lambda_{3m}})\}$ is a chain of half-spaces of length $|I| - |I'| + 2m > |I|$. This contradicts the maximality property of I, thereby showing that this last case does not occur.

Claim 3. *Suppose that there exists a set $I_{\mathrm{aff}} = \{\lambda_1, \dots, \lambda_n\} \subset \{1, \dots, k\}$ of cardinality $n > 6.L(8) + 2K + 6$ such that $D(\beta_{\lambda_1}) \subsetneq \cdots \subsetneq D(\beta_{\lambda_n})$ and $\{\beta_{\lambda_1}, \dots, \beta_{\lambda_n}\}$ generates an affine subsystem of rank 2 which is contained in a parabolic subsystem of affine type, where L is the function of Lemma 2.5. Then there exists a nilpotent sequence $(\beta'_j)_{j=1,\dots,k'}$, such that the set $\{\beta'_1, \dots, \beta'_{k'}\}$ contains $\{\beta_{\lambda_j} \mid j = x+1, x+2, \dots, n-x\}$, where $x = 6.L(8) + 2K + 6$, and is contained in a parabolic subsystem of affine type of Δ.*

We make the following definitions:

$$i := \min\{\lambda_j \mid j = x+1, x+2, \dots, n-x\}, \qquad k' := k - i + 2,$$

$$\beta'_1 := \beta_i, \qquad \beta'_2 := \sum_{j=1}^{i-1} \beta_j$$

and $\beta'_j := \beta_{i+j-2}$ for all $j = 3, \dots, k'$.

The set $\{\beta'_1, \dots, \beta'_{k'}\}$ is prenilpotent by Lemma 2.1, thus the sequence $(\beta'_j)_{j \leqslant k'}$ satisfies (NS1). It also satisfies (NS2) because so does $(\beta_j)_{j \leqslant k}$. Hence $(\beta'_j)_{j \leqslant k'}$ is a nilpotent sequence. Furthermore, it follows from the definition that the set $\{\beta'_1, \dots, \beta'_{k'}\}$ contains $\{\beta_{\lambda_j} \mid j = x+1, x+2, \dots, n-x\}$. Let $\Phi \subset \Delta$ be the parabolic subsystem of affine type containing $\{\beta_{\lambda_1}, \dots, \beta_{\lambda_n}\}$. We now show by induction on m that $\{\beta'_j \mid j = 1, \dots, m\}$ is contained in Φ. This is true by hypothesis for $m = 1$, hence the induction can start.

Let $m > 1$. Note that by induction, we have $\sum_{j=1}^{m-1} \beta'_j \in \Phi$ and moreover $\langle \sum_{j=1}^{m-1} \beta'_j, \beta'^{\vee}_m \rangle \neq 0$ by (NS2). Therefore, if $\partial\beta'_m$ meets at least 8 elements of $\{\partial\beta_{\lambda_1}, \dots, \partial\beta_{\lambda_n}\}$, then we have $\beta'_m \in \Phi$ by Lemma 2.8(ii) and we are done. We henceforth assume that $\partial\beta'_m$ meets at most 7 elements of $\{\partial\beta_{\lambda_1}, \dots, \partial\beta_{\lambda_n}\}$.

If $\partial\beta'_m$ meets some element of $\{\partial\beta_{\lambda_8}, \dots, \partial\beta_{\lambda_{n-7}}\}$, then by the above the triple $\{\partial\beta_{\lambda_1}, \partial\beta'_m, \partial\beta_{\lambda_n}\}$ consists of pairwise parallel walls since walls are convex. Moreover, since $\partial\beta_{\lambda_j} \subset D(\beta_{\lambda_n}) \setminus D(\beta_{\lambda_1})$ for all $j = 2, \dots, n-1$ and since $\partial\beta'_m$ meets $\partial\beta_{\lambda_j}$ for some such j, we obtain

$\partial\beta'_m \subset D(\beta_{\lambda_n})\backslash D(\beta_{\lambda_1})$. Finally, we deduce from Lemma 2.3(iii) that $D(\lambda_1) \subsetneq D(\beta'_m) \subsetneq D(\lambda_n)$. By Lemma 2.8(i), this implies that $\beta'_m \in \Phi$.

It remains to consider the case when $\partial\beta'_m$ meets no element of $\{\partial\beta_{\lambda_8}, \ldots, \partial\beta_{\lambda_{n-7}}\}$. Thus the set $\{D(\beta'_m), D(\beta_{\lambda_8}), \ldots, D(\beta_{\lambda_{n-7}})\}$ is a chain of half-spaces by Lemma 2.3(iii) and we may assume that either $D(\beta'_m) \subsetneq D(\beta_{\lambda_8})$ or $D(\beta'_m) \supsetneq D(\beta_{\lambda_{n-7}})$, otherwise we may conclude again using Lemma 2.8(i). Define $J'(m) := \{j < m \mid \langle \beta'_j, \beta'^{\vee}_m \rangle < 0\}$. Note that $1 \notin J'(m)$ otherwise $\partial\beta'_m$ would meet $\partial\beta'_1$ by Lemma 2.3. Note that for all $j \in J'(m)$, we have $\langle \beta'_j, \beta'^{\vee}_m \rangle \geqslant -3$ in view of Lemma 2.3. Therefore, since $\langle \sum_{j=1}^{m-1} \beta'_j, \beta'^{\vee}_m \rangle \leqslant K$ by (NS2) and Lemma 2.4, we deduce:

$$\begin{aligned}
\langle \beta'_1, \beta'^{\vee}_m \rangle &\leqslant K - \sum_{j=2}^{m-1} \langle \beta'_j, \beta'^{\vee}_m \rangle \\
&\leqslant K - \sum_{j \in J'(m)} \langle \beta'_j, \beta'^{\vee}_m \rangle \\
&\leqslant K + 3\left|J'(m)\right|.
\end{aligned}$$

If $K + 3|J'(m)| < \frac{x-6}{2}$, then we obtain $\beta'_m \in \Phi$ by Lemma 2.8(iii), as desired. Otherwise, we have $|J'(m)| \geqslant \frac{x-2K-6}{6} = L(8)$ by the definition of x. Therefore, the set $J'(m)$ contains 8 elements $j_1, \ldots, j_8$ such that the walls $\partial\beta'_{j_1}, \ldots, \partial\beta'_{j_8}$ are pairwise parallel. By Lemma 2.3, the wall $\partial\beta'_m$ meets $\partial\beta'_j$ for each $j \in J'(m)$, since the pair $\{\partial\beta'_j, \partial\beta'_m\}$ is prenilpotent. Furthermore, we have $\{\beta'_j \mid j \in J'(m)\} \subset \Phi$ by induction. We finally conclude that $\beta'_m \in \Phi$ by Lemma 2.8(ii).

We are now ready to obtain a final contradiction. The above claims show that the existence of nilpotent sequences of arbitrarily large length in Δ implies the existence of nilpotent sequences of arbitrarily large length, entirely contained in parabolic subsystems of affine type of Δ. Note that there are only finitely many orbits of such subsystems under the Weyl group action. Thus there must exist nilpotent sequences of arbitrarily large length, entirely contained in some fixed parabolic subsystem of affine type of Δ. As mentioned in the introduction, this is impossible because it contradicts the fact that Kac–Moody groups of affine type are linear modulo center. Here are some more details.

Linearity of affine Kac–Moody groups follows from their well-known realization as matrix groups over rings of Laurent polynomials: if G is the complex simply connected Kac–Moody group of untwisted affine type $X^{(1)}$ (notation of [Kac90, Chapter 4]) and $\mathbf{G}$ denotes the simple simply connected algebraic group scheme of type X, then there is a central homomorphism $\varphi : G \to \mathbf{G}(\mathbb{C}[t, t^{-1}])$ by [Tit83, Section 7.3]. Using the divisibility of the subgroup $U_w < G$, it follows that the Zariski closure of $\varphi(U_w)$ in $\mathbf{G}(\overline{\mathbb{C}[t, t^{-1}]})$ is connected. Since it is moreover nilpotent, it is contained in a Borel subgroup of $\mathbf{G}$. Therefore, the nilpotency degree of U_w is bounded from above by the solvability degree of Borel subgroups of $\mathbf{G}$, which is of course independent of w. A similar argument also applies to the case of twisted affine groups, since these can be viewed as almost split forms of untwisted affine groups [Tit83, Section 7.3] and, hence, are linear as well.

An alternative way to prove that the length of nilpotent sequences in a root system Δ of affine type is uniformly bounded, is to use the description of real roots in Δ provided by [Kac90, Proposition 6.3].

This result shows that if $(\beta_j)_{j=1,\ldots,k}$ is a nilpotent sequence of Δ, then $(\overline{\beta}_j)_{j=1,\ldots,k}$ is a nilpotent sequence of the finite (possibly nonreduced) root system $\overline{{}^{\mathrm{re}}\Delta}$, where $\alpha \mapsto \overline{\alpha}$ denotes the orthogonal projection introduced in [Kac90, §6.2]. In particular, every nilpotent sequence of Δ is of length at most $|\overline{{}^{\mathrm{re}}\Delta}|$.

3. A bound on the order of torsion elements in Kac–Moody groups over finite fields

We briefly indicate how Corollary 1.3 can be deduced from Theorem 1.2.

Since an almost split Kac–Moody group over a finite field can be embedded in a split one [Rém02, Chapter 11], it suffices to consider the split case. Let thus G be a split Kac–Moody group over a finite field $\mathbb{F}_q$. Recall that any finite subgroup of G is contained in the intersection of two finite type parabolic subgroups of opposite signs [CM06, Corollary 3.8]. Now, the intersection of two such parabolic subgroups has a Levi decomposition [Rém02, §6.3.4] (see also [CM06, Proposition 3.6]): it can be written as a semi-direct product $L \ltimes U$ where L is a finite type Levi subgroup, namely the intersection of two opposite parabolic subgroups of finite type, and U is conjugate to a subgroup U_w for some w. Since G has finitely many orbits of pairs of opposite parabolic subgroups (for the action by conjugation), it follows that the order of the finite subgroup L is bounded above by a constant depending only on G. Thus the corollary will be proven if we show that the order of any element of U_w is bounded above by a constant which is independent of w. In view of [Tit87, Proposition 1], each factor group of the descending central series of U_w is isomorphic to a direct product of root subgroups. In particular, such a factor group is an elementary abelian p-group, where p is the characteristic of $\mathbb{F}_q$. In particular, the order of any element of U_w is bounded above by p^n, where n is the length of the descending central series of U_w. By Theorem 1.2 (see also the comment following its statement in the introduction), the number n is bounded above by a constant depending only on G, and the desired assertion follows.

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