The model theory of unitriangular groups

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Abstract

The model theory of groups of unitriangular matrices over rings is studied. An important tool in these studies is a new notion of a quasiunitriangular group. The models of the theory of all unitriangular groups (of fixed nilpotency class) are algebraically characterized; it turns out that all they are quasiunitriangular groups. It is proved that if \( R \) and \( S \) are domains or commutative associative rings then two quasiunitriangular groups over \( R \) and \( S \) are isomorphic only if \( R \) and \( S \) are isomorphic or antiisomorphic. This algebraic result is new even for ordinary unitriangular groups. The groups elementarily equivalent to a single unitriangular group \( UT_n(R) \) are studied. If \( R \) is a skew field, they are of the form \( UT_n(S) \), for some \( S = R \). In general, the situation is not so nice. Examples are constructed demonstrating that such a group need not be a unitriangular group over some ring; moreover, there are rings \( P \) and \( R \) such that \( UT_n(P) \equiv UT_n(R) \), but \( UT_n(P) \) cannot be represented in the form \( UT_n(S) \) for \( S \equiv R \). We also study the number of models in a power of the theory of a unitriangular group. In particular, we prove that, for any communicative associative ring \( R \) and any infinite power \( \lambda \), \( I(\lambda, R) = I(\lambda, UT_n(R)) \). We construct an associative ring such that \( I(\aleph_1, R) = 3 \) and \( I(\aleph_1, UT_n(R)) = 2 \). We also study models of the theory of \( UT_n(R) \) in the case of categorical \( R \).

For an associative ring with unit \( R \), let \( UT_n(R) \) be the group of all upper unitriangular matrices over \( R \), that is matrices with entries in \( R \) which have zeros below the main diagonal and units on it. For \( n = 1 \) the group is trivial and for \( n = 2 \) it is isomorphic to the additive group of \( R \); so the only interesting case is \( n \geq 3 \). For any \( n \), the group \( UT_n(R) \) is \((n - 1)\) step nilpotent.

The model theory of unitriangular groups began with Maltsev's paper [16]. He considered only the case \( n = 3 \). (Note that \( UT_3(R) \) is a group even if \( R \) is not associative.) He showed that the ring \( R \) can be interpreted in the group \( UT_3(R) \) with certain parameters and gave an algebraic characterization of groups of this form.

Rose [19] applied the idea of Maltsev's work to the ring \( NT_n(R) \) of all upper niltriangular \( n \times n \) matrices, \( n \geq 3 \). He showed that the ring \( R \) is interpretable in the ring \( NT_n(R) \) with parameters and gave a first order axiomatization of the class of all rings of the from \( NT_n(F) \), where \( F \) is a field. It follows that every ring elementarily

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equivalent to $\text{NT}_n(F)$ has a form $\text{NT}_n(K)$ for some field $K$. The problem whether $K \equiv F$ remained open. Wheeler [23] answered the question in the positive proving that $\text{NT}_n(F) \cong \text{NT}_n(K)$ iff $F \cong K$. The result obviously implies the $\aleph_1$-categoricity of the ring $\text{NT}_n(F)$ for an algebraically closed field $F$.

Videla [21] generalized Wheeler's result from fields to arbitrary associative rings with unit. He also found an explicit recursive axiom system for the first order theory of the class of rings of the form $\text{NT}_n(R)$ and gave an algebraic description of its models. He applied the latter result to prove that, for any infinite power $\lambda$, $I(\lambda, R) = I(\lambda, \text{UT}_n(R))$. (Here $I(\lambda, M)$ denotes the number of models of power $\lambda$ of the theory of a structure $M$.) Videla asked whether $I(\lambda, R) = I(\lambda, \text{UT}_n(R))$ for any associative ring with unit [21,22]. In [22] he generalized Maltsev's result to certain maximal unipotent subgroups of Chevalley groups and proved that, if $L$ is a root system and $F$ is a field of characteristic $\neq 2,3$, any group elementary equivalent to $U_L(F)$ is isomorphic to $U_L(K)$ for some $K \equiv F$. As a corollary he showed that, for any infinite field $F$ with char($F$) $\neq 2,3$, $I(\lambda, F) = I(\lambda, U_L(F))$. Note that $\text{UT}_n(F)$ is $U_L(F)$ for the root system $L$ of the type $A_{n-1}$. In the present paper we show that, in general, Videla's question has a negative answer, but in important special cases the answer is positive.

We extensively studied the model theory of groups of the form $\text{UT}_n(R)$ in the papers [2-7]. Here we extend the results to arbitrary $n \geq 3$.

Let $R$ be a ring with unit, associative if $n > 3$. For any $g_1, \ldots, g_{n-1}$, symmetric 2-cocycles from $R^*$ to itself, in Section 1 we define a group $\text{UT}_n(R, g_1, \ldots, g_{n-1})$ with the same underlying set as for the group $\text{UT}_n(R)$. If the cocycles are zeros, we get the ordinary $\text{UT}_n(R)$. The new group has the same commutation operation as the old one; hence it is $(n - 1)$ step nilpotent (Section 1.4). The groups of the form $\text{UT}_n(R, g_1, \ldots, g_{n-1})$ are called quasiunitriangular. We give an algebraic characterization of quasiunitriangular and unitriangular groups (Section 1.7). It turns out that the class of all quasiunitriangular groups of fixed nilpotency class is first order axiomatizable, but this fails for unitriangular groups (Section 2.2). Then we study the question to what extent a quasiunitriangular group over a ring determines the ring. As

$$\text{UT}_n(R, g_1, \ldots, g_{n-1}) \cong \text{UT}_n(R^{op}, g_{n-1}, \ldots, g_1),$$

one could only hope that the group determines the ring up to isomorphism or antiisomorphism (Section 1.10). Nevertheless, in general, it is not the case: there exist associative $R$ and $S$ such that $\text{UT}_n(R) \cong \text{UT}_n(S)$, but $R$ is not isomorphic to $S$ and $S^{op}$ (Proposition 1.9). We prove that if $R$ and $S$ are domains or commutative associative rings, then

$$\text{UT}_n(R, g_1, \ldots, g_{n-1}) \cong \text{UT}_n(S, q_1, \ldots, q_{n-1})$$

implies $R \cong S$ or $S^{op}$ (Sections 1.15, 1.17). The result seems new even for ordinary unitriangular groups.

In Section 2 we study models of the first order theory of the class of unitriangular groups. It turns out that they all are exactly the so called locally pure quasiunitriangular
groups (Section 2.2). We also study groups elementarily equivalent to a single unitriangular group $UT_n(R)$. We have a number of positive results here, under some restrictions on $R$. For example, if $R$ is a skew field, any group elementarily equivalent to $UT_n(R)$ is of the form $UT_n(S)$ for some $S \equiv R$ (Corollary 2.15). In general, the situation is not so nice. We construct examples demonstrating that such a group need not be a unitriangular group over some ring; moreover, there are rings $P$ and $R$ such that $UT_n(P) \equiv UT_n(R)$, but $UT_n(P)$ cannot be represented in the form $UT_n(S, q_1, \ldots, q_{n-1})$ for $S \equiv R$ (Proposition 2.20).

In Section 3 we study the number of models in a power of the theory of a unitriangular group. We prove that, for any commutative associative ring $R$ and any infinite power $\lambda$, $I(\lambda, R) = I(\lambda, UT_n(R))$ (Theorem 3.8). For domains $R$ we can prove the analogous result only for an uncountable $\lambda$; we know that $I(\aleph_0, R) \geq I(\aleph_0, UT_n(R))$ and $\lambda$ could be only if $R$ is a skew field with some pathological properties (Theorem 3.9); I conjecture that such skew fields do not exist. We construct an associative ring such that $I(\aleph_1, R) = 3$ and $I(\aleph_1, UT_n(R)) = 2$ (Theorem 3.20). We also study models of the theory of $UT_n(R)$ in the case of categorical $R$. We show that, for every infinite $\lambda$, a ring $R$ is $\lambda$-categorical iff the group $UT_n(R)$ is $\lambda$-categorical (Propositions 3.22, 3.23). If a ring $R$ is $\aleph_0$-categorical, all the models of $\text{Th}(UT_n(R))$ are of the form $UT_n(S)$, $S \equiv R$ (Proposition 3.22). I do not know whether an analogous result holds for an $\aleph_1$-categorical $R$. We discuss the question in detail and give some equivalent formulations of the problem (see 3.24–3.28, 3.30–3.32, 3.34).

1. Quasunitriangular groups

In this section we introduce a new notion of quasunitriangular group generalizing unitriangular groups. The special case $n = 3$ was considered in the author's papers [2–7].

1.1. Symmetric 2-cocycles and abelian group extensions

Recall that a symmetric 2-cocycle from an abelian group $B$ to an abelian group $A$ is defined to be a map $g: B \times B \rightarrow A$ satisfying the following conditions:

1. $g(x, y) + g(z, x + y) = g(z + x, y) + g(z, x),$
2. $g(x, 0) = g(0, x) = 0,$
3. $g(x, y) = g(y, x).$

A symmetric 2-cocycle $g$ is said to be a coboundary if, for some $q: B \rightarrow A,$

$g(x, y) = q(x) + q(y) - q(x + y).$

Cocycles $g_1$ and $g_2$ are called cohomologous if $g_1 - g_2$ is a coboundary.
For any symmetric 2-cocycle $g: B \times B \to A$, one can define an abelian group $[A, B, g]$ with the underlying set $A \times B$ and the group operation

$$(a, b) + (a', b') = (a + a' + g(b, b'), b + b').$$

For the homomorphisms $\mu(a) = (a, 0)$ and $\nu(a, b) = (0, b)$, the sequence

$$0 \to A \to [A, B, g] \to B \to 0$$

is exact, so it is an extension of $A$ by $B$; denote it by $E(g)$. The extension $E(g)$ splits iff $g$ is a coboundary; moreover, $g_1$ and $g_2$ are cohomologous iff $E(g_1)$ and $E(g_2)$ are equivalent. Every abelian extension of $A$ by $B$ is equivalent to $E(g)$ for some $g$.

### 1.2. Elementary matrices

Let $R$ be a ring with unit, $n \geq 3$. For $1 \leq i, j \leq n$, $e_{ij}$ denotes the matrix which has 1 as its $(i, j)$ entry and 0 elsewhere; $e$ denotes the identity $n \times n$ matrix. Clearly, $e_{ij} e_{ik} = 0$ for $j \neq l$ and $e_{ij} e_{jk} = e_{ik}$. For $i \neq j$ and $x \in R$, denote $t_{ij}(x) = e + x e_{ij}$ and $t_{ij} = t_{ij}(1)$. It is easy to verify that $t_{ij}(x)^{-1} = t_{ij}(-x)$; $[t_{ij}(x), t_{ik}(\beta)] = e$, for $i \neq k$, $j \neq l$; $[t_{ij}(x), t_{jk}(\beta)] = t_{ik}(x \beta)$, for $i \neq k$. Clearly, $t_{ij}(x) \in UT_n(R)$ for $i < j$.

### 1.3. New group operation

Let $g_1, \ldots, g_{n-1}$ be symmetric 2-cocycles from $R^+$, the additive group of $R$, to itself. Define a new binary operation $\odot$ on the set of all upper unitriangular $n \times n$ matrices over $R$. Let $\cdot$ denote the usual matrix multiplication. If $n > 3$, suppose that $R$ is associative; then, for $n \geq 3$, the operation $\cdot$ is associative. For $a = (a_{ij})$ and $b = (\beta_{ij})$, upper unitriangular $n \times n$ matrices over $R$, put

$$a \odot b = a \cdot b + \left( \sum_{i=1}^{n-1} g_i(\alpha_{i,i+1}, \beta_{i,i+1}) \right) e_{1n}.$$  

Note that

(i) $(\alpha_{ij})^\cdot (\beta_{ij}) = (\gamma_{ij})$ implies $\alpha_{i,i+1} + \beta_{i,i+1} = \gamma_{i,i+1},$

(ii) $(\alpha_{ij})^{-1} = (\delta_{ij})$ implies $\delta_{i,i+1} = -\alpha_{i,i+1}.$

Due to (2), $e$ is neutral with respect to $\odot$. A direct computation using (1) shows that $\odot$ is associative. Using (ii), (3) and the equality $a \cdot x e_{1n} = x e_{1n} \cdot a = x e_{1n}$, it is easy to see that

$$a^{-1} = a^{-1} - \left( \sum_{i=1}^{n-1} g_i(\alpha_{i,i+1}, -\alpha_{i,i+1}) \right) e_{1n}.$$  

is the inverse element for $a$ with respect to $\odot$.

So $\odot$ is a group operation. Denote the new group by $UT_n(R, g_1, \ldots, g_{n-1})$. We call a group of this form a quasunitriangular group over $R$. Clearly, $UT_n(R)$ is a special case of the construction, namely $UT_n(R, 0, \ldots, 0)$.
1.4. Commutation

A direct computation using (i), (ii), (1)–(3) shows that
\[ a^{(-1)} \odot b \odot a = a^{-1} \cdot b \cdot a, \]
\[ b^{(-1)} \odot a^{(-1)} \odot b \odot a = b^{-1} \cdot a^{-1} \cdot b \cdot a. \]

So the commutation operation in the new group coincides with the old one. It follows
that the new group has the same lower and upper central series, centralizers etc. as
the old one has. In particular, the new group is \((n - 1)\) nilpotent. Denote
\( \text{UT}_n(R, g_1, \ldots, g_{n-1}) \) by \( U \). Let \( U_k \) be the \( k \)th member of the central lower series of
\( U \) (that is, \( U_1 = U, U_{m+1} = [U_m, U] \)); then \( U_k \) consists of matrices \((\alpha_{ij})\) with \( \alpha_{ij} = 0, \)
for \( j - i < k \). The center of \( U \) consists of matrices of the form \( t_{1n}(\alpha) \).

1.5. Generators and defining relations

**Proposition 1.1.** The group \( U \) is generated by the set
\[ \{t_{ij}(\alpha): \alpha \in R, 1 \leq i < j \leq n\}. \]

The following relations \( \mathcal{R} \) in these generators define the group:

(i) \[ [t_{ij}(\alpha), t_{jk}(\beta)] = t_{ik}(\alpha \beta), \]
(ii) \[ [t_{ij}(\alpha), t_{ik}(\beta)] = e, \]
(iii) \[ t_{ij}(\alpha) \odot t_{ij}(\beta) = t_{ij}(\alpha + \beta), \]
(iv) \[ t_{i,i+1}(\alpha) \odot t_{i,i+1}(\beta) = t_{i,i+1}(\alpha + \beta) \odot t_{1n}(\alpha, \beta), \]

where \( i \neq k, j \neq l \) in (ii) and \( j > i + 1 \) in (iii).

**Proof.** Clearly, the elements \( t_{ij}(\alpha) \) satisfy \( \mathcal{R} \) in \( U \). It is easy to show that every element
in \( U \) can be uniquely represented as a product of elements \( t_{ij}(\alpha_{ij}) \), where \( t_{ij}(\alpha_{ij}) \) is on
the left of \( t_{ik}(\alpha_{ik}) \) iff \( j - i < k - l \) or \( j - i = k - l, i > l \). So the set of all the \( t_{ij}(\alpha) \)'s
generate \( U \). One can easily show that any group word in the \( t_{ij}(\alpha) \)'s is equal to
a product of the form above, modulo \( \mathcal{R} \). Suppose the word is equal to \( e \) in \( U \). Then
the product is equal to \( e \) in \( U \) too. Due to the uniqueness of the representation of an
element of \( U \) in the form above, the factors of the product are equal to \( e \) in \( U \). So the
word is equal to \( e \) in \( U \) iff it is in the normal closure of \( \mathcal{R} \) in the free group. \( \square \)

1.6. One-parameter subgroups

For \( i + 1 < j \), denote
\[ U_{ij} = \{t_{ij}(\alpha): \alpha \in R\}. \]
For \( i < n \), denote
\[ U_{i,i+1} = \{t_{i,i+1}(\alpha) + \beta e_{1n}: \alpha, \beta \in R\}. \]
Clearly, these are subgroups in $U$. The families

$$\{t_{ij}; 1 \leq i < j \leq n\} \quad \text{and} \quad \{U_{ij}; 1 \leq i < j \leq n\}$$

have the following properties.

1. $U_{1n} \leq U_{i,i+1}$.
2. $[U_{ij}, U_{ik}] = e$, for $i \neq k, j \neq l$ and $[U_{ij}, U_{jk}] = U_{ik}$.
3. $[U_{ij}, t_{jk}] = U_{ik}$ and $[t_{ki}, U_{ij}] = U_{kj}$.
4. The centralizers of $t_{jk}$ and $t_{ki}$ in $U_{ij}$ are trivial if $j - i > 1$ and equal to $U_{1n}$ if $j - i = 1$.

The properties (1)–(3) follow from (i) and (ii) of Proposition 1.1.

4. $[[x, y], z] = [x, [y, z]]$, for $x \in U_{ij}, y \in U_{jk}, z \in U_{kl}$.

The property (4) makes sense only if $n > 3$; it follows from (i) of Proposition 1.1 and the associativity of $R$.

To formulate the next property we need some notations. For $x \in U_{ij}$, define $\tau_{ij}(x)$ as follows:

$$\tau_{ij}(x) = \begin{cases} 
[t_{ij}, x], t_{jn} & \text{if } 1 < i < j < n, \\
[t_{ij}, x] & \text{if } 1 < i = j < n, \\
x & \text{if } 1 = i < j < n. 
\end{cases}$$

Clearly, $\tau_{ij}$ is an epimorphism from $U_{ij}$ onto $U_{1n}$ whose kernel is $U_{1n}$ if $j = i + 1$ and trivial if $j > i + 1$.

5. For $x \in U_{ij}, y \in U_{jk}, z \in U_{mp}, v \in U_{pq}$,

$$\tau_{ij}(x) = \tau_{mp}(z) \quad \text{and} \quad \tau_{jk}(y) = \tau_{pq}(v) \quad \text{implies} \quad \tau_{ik}([x, y]) = \tau_{mq}([z, v]).$$

The property (5) is a consequence of (i) of Proposition 1.1.

6. Every element in $U$ can be represented as a product of elements $u_{ij} \in U_{ij}$, where $u_{ij}$ is on the left of $u_{ik}$ iff $j - i < k - l$ or $j - i = k - l, i > l$. In such a representation $u_{ij}$ is uniquely determined if $i + 1 < j, (i, j) \neq (1, n)$ and uniquely determined modulo $U_{1n}$ otherwise.

The property (6) follows from the fact already mentioned in the proof of Proposition 1.1: every element in $U$ can be uniquely represented as a product of elements $t_{ij}(x_{ij})$, where $t_{ij}(x_{ij})$ is on the left of $t_{ik}(x_{ik})$ iff $j - i < k - l$ or $j - i = k - l, i > l$.

7. The extension $U_{1n} \leq U_{i,i+1}$ is equivalent to $E(g_l)$. In particular, $U_{1n}$ is a direct summand in $U_{i,i+1}$ iff $y_l$ is a coboundary.

1.7. Characterization theorem

It turns out that the properties above completely characterize quasiunitriangular groups.
Theorem 1.2. Let $H$ be a group, $n \geq 3$ and $h = \{h_{ij}: 1 \leq i < j \leq n\}$ a family in $H$. The following are equivalent:

(a) there exists a ring $R$ with unit (associative if $n > 3$) and symmetric 2-cocycles $g_1, \ldots, g_{n-1}$ from $R^+$ to itself such that

$$(H, h) \cong (UT_n(R, g_1, \ldots, g_{n-1}), t),$$

where $t = \{t_{ij}: 1 \leq i < j \leq n\};$

(b) $[h_{ij}, h_{jk}] = h_{ik}$ for $1 \leq i < j < k \leq n$, and there exists

$S = \{H_{ij}: 1 \leq i < j \leq n\},$

a family of subgroups in $H$ with $h_{ij} \in H_{ij}$, satisfying the conditions (0)–(6) from Section 1.6 in which $U$, $U_{ij}$ and $t_{ij}$ are replaced by $H$, $H_{ij}$ and $h_{ij}$, respectively.

Moreover, if (a) holds, we can choose $S$ in such a way that for all $i$ the extension $H_{1n} \leq H_{i.i+1}$ is equivalent to $E(g_i)$; if (b) holds, we can choose the $g_i$'s in such a way that for all $i$ the extension $H_{1n} \leq H_{i,i+1}$ is equivalent to $E(g_i)$.

Proof. As (a) $\Rightarrow$ (b) has been already proved in Section 1.6, we need to prove only (b) $\Rightarrow$ (a).

First of all, by the first part of (1), all the groups $H_{ij}$ are abelian, and, by (2), $\tau_{ij}$ maps $H_{ij}$ onto $H_{1n}$. Define a ring

$$\text{(Ring}(H, S, h), \boxplus, \boxtimes)$$

as follows. Its additive group is $H_{1n}$, so $u \boxplus v = uv$, for $u, v \in H_{1n}$. Put

$$\tau_{ij}(x) \boxtimes \tau_{jk}(y) = \tau_{ik}([x, y]),$$

for $x \in U_{ij}, y \in U_{jk}$. Due to (5), it is a well-defined operation on $H_{1n}$. Due to the first part of (1), the maps $\tau_{ij}$ are homomorphisms and $\boxtimes$ is distributive with respect to $\boxplus$; so $H_{1n}$ forms a ring with respect to $\boxplus$ and $\boxtimes$.

The element $h_{1n}$ is the unit of this ring. Indeed, let $u \in H_{1n}, u = \tau_{11}(x) = \tau_{nn}(y)$, where $1 \leq i < n, 1 \leq j < n, x \in H_{1i}, y \in H_{jn}$.

Then

$$u \boxplus h_{1n} = \tau_{11}(x) \boxplus \tau_{nn}(h_{1n}) = \tau_{1n}([x, h_{1n}]) = u,$$

$$h_{1n} \boxtimes u = \tau_{11}(h_{1n}) \boxtimes \tau_{nn}(y) = \tau_{1n}([h_{1n}, y]) = u.$$

We show that the operation $\boxtimes$ is associative if $n > 3$. Let $u, v, w \in H_{1n}$. Choose $1 \leq i < j < n, x \in H_{1i}, y \in H_{ij}$ and $z \in H_{j,n}$ such that $u = \tau_{1i}(x), v = \tau_{ij}(y) w = \tau_{jn}(z)$. Then, taking into account (4), we have

$$(u \boxplus v) \boxtimes w = \tau_{1j}([x, y]) \boxtimes \tau_{jn}(z)$$

$$= \tau_{1n}([x, y], z]) = \tau_{1n}([x, [y, z]])$$

$$= \tau_{1i}(x) \boxtimes \tau_{nn}([y, z]) = u \boxtimes (v \boxtimes w).$$
Remark 1.3. Due to (i) and (iii) of Proposition 1.1, the map \( \alpha \mapsto t_{1n}(\alpha) \) is an isomorphism from \( R \) onto

\[ \text{Ring}(UT_n(R, g_1, \ldots, g_{n-1}), U, t), \]

where \( U = \{ U_{ij} : 1 \leq i < j \leq n \} \). So \( \text{Ring}(H, \emptyset, \emptyset) \) can be non-associative for \( n = 3 \); in fact, \( \text{Ring}(UT_3(R, g_1, g_2), U, t) \) is associative iff \( R \) is.

We now show that, for \( R = (\text{Ring}(H, \emptyset, \emptyset) \) and for some \( g_1, \ldots, g_{n-1} \),

\[ (H, \emptyset) \simeq (UT_n(R, g_1, \ldots, g_{n-1}), t). \]

By (3), \( \tau_{ij} \) is an isomorphism if \( j > i + 1 \) and has a kernel \( H_{1n} \) if \( j = i + 1 \). For \( j > i + 1 \), put \( \rho_{ij} = \tau_{ij}^{-1} \). For any \( \alpha \in H_{1n} \), choose an element \( \rho_{i,i+1}(\alpha) \) in the set \( \tau_{i,i+1}^{-1}(\alpha) \). As \( \tau_{i,i+1}(e) = e \) and \( \tau_{i,i+1}(h_{i,i+1}) = h_{1n} \), we can choose \( \rho_{i,i+1}(e) = e \) and \( \rho_{i,i+1}(h_{1n}) = h_{i,i+1} \). By (6), the set

\[ \{ \rho_{ij}(\alpha) : \alpha \in H_{1n}, 1 \leq i < j \leq n \} \]
generates the group \( H \). For \( \alpha, \beta \in H_{1n} \), put

\[ g_i(\alpha, \beta) = \rho_{i,i+1}(\alpha) \cdot \rho_{i,i+1}(\beta) \cdot \rho_{i,i+1}(\alpha \beta)^{-1}. \]

Then \( g_i \) is a symmetric 2-cocycle from the additive group of \( R \) to itself, and the extension \( H_{1n} \leq H_{i,i+1} \) is equivalent to \( E(g_i) \). If this extension splits, one can choose \( \rho_{ij} \) to be a monomorphism; in this case \( g_i = 0 \). Indeed, let \( \pi_i \) be a homomorphism from \( H_{1n} \) to \( H_{i,i+1} \) such that \( \tau_{i,i+1} \circ \pi_i = \text{id} \). Then

\[ h_{i,i+1} \cdot \pi_i(h_{1n})^{-1} \in \text{Ker}(\tau_{i,i+1}) = H_{1n}. \]

For \( h \in H_{1n} \), put

\[ \rho_{i,i+1}(h) = \pi_i(h) \cdot (h \Box (h_{i,i+1} \cdot \pi_i(h_{1n})^{-1})). \]

Clearly, \( \rho_{i,i+1} \) is a homomorphism from \( H_{1n} \) to \( H_{i,i+1} \), \( \tau_{i,i+1} \circ \rho_{i,i+1} = \text{id} \) and \( \rho_{i,i+1}(h_{1n}) = h_{i,i+1} \).

We show that the mapping \( t_{ij}(\alpha) \mapsto \rho_{ij}(\alpha), 1 \leq i < j \leq n, \alpha \in R \), can be extended to a homomorphism \( f \) from \( UT_n(R, g_1, \ldots, g_{n-1}) \) onto \( H \). It suffices to note that the elements \( \rho_{ij}(\alpha) \) satisfy the defining relations from Proposition 1.1. The relation

\[ [\rho_{ij}(\alpha), \rho_{jk}(\beta)] - \rho_{ik}(\alpha \Box \beta) \]

is equivalent to \( \tau_{ik}([\rho_{ij}(\alpha), \rho_{jk}(\beta)]) = \alpha \Box \beta \); the latter holds by definition of \( \Box \). The relation

\[ [\rho_{ij}(\alpha), \rho_{ik}(\beta)] = \epsilon \]

holds for \( i \neq k, j \neq l \), due to the first part of (1). If \( i + 1 < j \), the relation

\[ \rho_{ij}(\alpha) \rho_{ij}(\beta) = \rho_{ij}(\alpha \ominus \beta) \]
is equivalent to \( x \oplus \beta = \gamma \beta \); the latter holds by definition of \( \oplus \). The relation

\[
\rho_{i,i+1}(\alpha)\rho_{i,i+1}(\beta) = \rho_{i,i+1}(\alpha \oplus \beta)\rho_{i,i+1}(\alpha, \beta)
\]

holds by definition of \( g_i \) because \( \rho_{1n} = \text{id} \) and \( \alpha \oplus \beta = \gamma \beta \), for \( \alpha, \beta \in H_{1n} \).

We show that \( f \) is an isomorphism. Suppose \( f \) sends the matrix \( a \) to \( e \). In the proof of Proposition 1.1 we showed how to represent the matrix \( a \) as a product of the \( t_{ij}(\alpha_{ij}) \)'s; then \( e \) is the product of the \( \rho_{ij}(\alpha_{ij}) \)'s in the same order. By (6), \( \rho_{ij}(\alpha_{ij}) = e \) if \( i + 1 < j \) and \( (i,j) \neq (1, n) \) and \( \rho_{ij}(\alpha_{ij}) \in U_{1n} \) otherwise. Hence in any case \( \alpha_{ij} = \tau_{ij}(\rho_{ij}(\alpha_{ij})) = e \); so every \( \alpha_{ij} \) is equal to the zero of the ring \( R \), and \( a \) is the identity of \( U \).

Note that \( f(t_{ij}) = \rho_{ij}(h_{1n}) = h_{ij} \) and \( f(U_{ij}) = H_{ij} \). The proof is completed. \( \square \)

**Corollary 1.4.** Let \( H \) be a group, \( n \geq 3 \) and \( b = \{ h_{ij}: 1 \leq i < j \leq n \} \) a family in \( H \). The following are equivalent:

(a) there exists a ring \( R \) with unit (associative if \( n > 3 \)) such that

\[
(H, b) \cong (U, U, t)
\]

where \( t = \{ t_{ij}: 1 \leq i < j \leq n \}; \)

(b) \[ h_{ij}, h_{jk} = h_{ik}, \text{for } 1 \leq i < j < k \leq n \text{ and there exists} \]

\( \mathcal{H} = \{ H_{ij}: 1 \leq i < j \leq n \}, \)

a family of subgroups in \( H \) with \( h_{ij} \in H_{ij} \), satisfying the conditions (0)–(6) from Section 1.6, in which \( U, U_{ij} \) and \( t_{ij} \) are replaced by \( H, H_{ij} \) and \( h_{ij} \), respectively and the condition ‘\( H_{1n} \) is a direct summand in every \( H_{i,i+1} \)’.

1.8. **Definability of one-parameter subgroups**

**Proposition 1.5.** Let \( H \) be a group and \( \mathcal{H}, b \) satisfy the conditions (b) of Theorem 1.2. Then every \( H_{ij} \) is first order definable in \( H \) with parameters \( b \) (by certain positive primitive formulas, uniformly with respect to \( H \) and \( b \)). In particular, in (b) of Theorem 1.2 the family \( \mathcal{H} \) is uniquely determined.

**Proof.** In the proof of Theorem 1.2 it was shown that

\[
(H, \mathcal{H}, b) \cong (U, U, t).
\]

Therefore it suffices to prove that \( U_{ij} \)'s are definable in \( U \) with parameters \( t \). (In fact, we shall define them using only the commutation operation.)

First of all, note that the centralizer of \( t_{ij} \) in \( U \) consists of all matrices \( u \) such that \( u - e \) has the zero \( i \)th column and the zero \( j \)th row.

For \( 1 \leq i < k < m \leq n \), let \( A^k_m \) be the set of all matrices \( u \in U \) such that \( u - e \) has the zero \( i \)th row, for \( i \neq 1, k \), and the zero \( j \)th column, for \( j \neq m, n \). Then \( A^k_m \) is the centralizer in \( U \) of the set \( \{ t_{1i}, t_{jm}: i \neq k, j \neq m \} \).
It is easy to show that, for $1 \leq i < j < n$,
\[
[e + ax_{ij} + \beta \epsilon_{in} + \gamma \epsilon_{jj} + \delta \epsilon_{1n}, t_{j,j+1}] = e + ax_{i,j+1} + \gamma \epsilon_{1,j+1},
\]
and, for $1 < i < j \leq n$,
\[
[t_{i-1,i}, e + ax_{ij} + \beta \epsilon_{in} + \gamma \epsilon_{jj} + \delta \epsilon_{1n}] = e + ax_{i-1,j} + \beta \epsilon_{i-1,n}.
\]
It follows that
\[
U_{1j} = \begin{cases} A_j^2 & \text{for } j = 2, \\ [A_{i-1}^1, t_{j-1,i}] & \text{for } j > 2, \end{cases}
\]
and
\[
U_{in} = \begin{cases} A_n^1 & \text{for } i = n - 1, \\ [t_{i+1,i}, A_{i+1}^n] & \text{for } i < n - 1. \end{cases}
\]
We have, for $1 < i < j < n, j \neq i + 1$,
\[
U_{ij} = [A_i^{-1}, t_{j-1,i}] \cap [t_{i,i+1}, A_{i+1}^j],
\]
and, for $1 < i < n - 1$,
\[
U_{i,i+1} = \{ u \in A_i^{i+1} : [u, t_{i+1,i+2}] \in [t_{i,i+1}, A_i^{i+2}] \text{ and } [t_{i-1,i}, u] \in [A_i^{i-1}, t_{i,i+1}] \}.
\]
Therefore every $U_{ij}$ can be defined in $U$ by a primitive positive formula with parameters $t$ and we are done.  

Due to Proposition 1.5, we can use the notation $\text{Ring}(H, h)$ instead of $\text{Ring}(H, \mathcal{S}, h)$.

1.9. Bases

Let $H$ be a group, $n \geq 3$. A family $\mathfrak{h}$ satisfying the conditions (b) of Theorem 1.2 is said to be a base in $H$. The base $t$ in $U$ is said to be a standard base in $U$. Due to Proposition 1.5 the set of all bases in $H$ is $0$-definable by a formula which is a conjunction of formulas of the form $\forall \vec{y} \phi(\vec{x}, \vec{y}) \Rightarrow \psi(\vec{x}, \vec{y})$, where $\phi$ and $\psi$ are positive primitive formulas and $\forall \vec{x} \phi(\vec{x}, \vec{e})$ holds in any group. A base $\mathfrak{h}$ in $H$ is said to be pure (splitting), if, for every $i$, the extension $H_{1n} \leq H_{i+1}$ is pure (splits). Clearly, every splitting base is pure. A group is said to be pure if it has a pure base. Corollary 1.4 says that a group is isomorphic to the group $U_{1n}(R)$ over some ring $R$ iff it has a splitting base.

Proposition 1.6. Let $\mathfrak{h}$ be a base in a group $H$. Each of the following conditions guarantees that $\mathfrak{h}$ splits:

1. $Z(H)$ is pure-injective and $\mathfrak{h}$ is pure,
2. $Z(H)$ is pure-injective and torsion-free,
3. $Z(H)$ is a direct sum of cyclic groups and $\mathfrak{h}$ is pure,
4. every $H_{1i,i+1}$ is an elementary abelian group.
Proof. (1) The subgroup $H_{1n}$ coincides with $Z(H)$ and so is pure-injective; therefore the extension $H_{1n} \leq H_{i,i+1}$ splits, provided it is pure.

(2) As $H_{i,i+1}/H_{1n} \cong H_{1n} \cong Z(H)$ is torsion-free, the extension $H_{1n} \leq H_{i,i+1}$ is pure and therefore splits, due to the pure-injectivity of $Z(H)$.

(3) As $H_{i,i+1}/H_{1n} \cong H_{1n} = Z(H)$ is a direct sum of cyclic groups and the extension $H_{1n} \leq H_{i,i+1}$ is pure, it splits by Kulikov’s theorem [13, 28.2].

(4) Every subgroup of an elementary abelian group is a direct summand. □

In the remaining part of this section we study the question to what extent a quasiunitriangular group over a ring determines the ring.

1.10. Quasiunitriangular groups over $R^{op}$

Proposition 1.7. $UT_n(R, g_1, \ldots, g_{n-1}) \cong UT_n(R^{op}, g_{n-1}, \ldots, g_1)$.

Proof. Denote the group operations in $UT_n(R)$, $UT_n(R^{op})$, $UT_n(R, g_1, \ldots, g_{n-1})$ and $UT_n(R^{op}, g_{n-1}, \ldots, g_1)$ by $\cdot$, $\ast$, $\odot$ and $\otimes$, respectively. Let $(\alpha_{ij})$ be the matrix $(\alpha_{n-j+1,n-i+1})$. One can verify that $(u \ast v) = v \ast u$. Moreover, for $u = (\alpha_{ij})$, $v = (\beta_{ij})$,

$$
' (u \odot v) = ' (u \cdot v + \sum_{i=1}^{n-1} g_i(\alpha_{i,i+1}, \beta_{i,i+1})) e_{1n}
$$

$$
= ' (v \cdot u + \sum_{i=1}^{n-1} g_{n-i}(\beta_{n-i,n-i+1}, g_{n-i,n-i+1})) e_{1n}
$$

Thus the map $u \mapsto ' u$ is an antiisomorphism from $UT_n(R, g_1, \ldots, g_{n-1})$ onto $UT_n(R^{op}, g_{n-1}, \ldots, g_1)$. Since every group is antiisomorphic to itself via the map $x \mapsto x^{-1}$ and the composition of two antiisomorphisms is an isomorphism, the result follows. □

Corollary 1.8. $UT_n(R) \cong UT_n(R^{op})$.

1.11. A unitriangular group need not determine a ring

Proposition 1.9. There exist associative rings $R$ and $S$ such that the groups $UT_n(R)$ and $UT_n(S)$ are isomorphic, for every $n \geq 3$, but $R$ and $S$ are not isomorphic nor anti-isomorphic.

Proof. Let $K$ be an indecomposable associative ring with unit which is not anti-isomorphic to itself. Put $R = K \times K$, $S = K \times K^{op}$. Then $R$ and $S$ are not isomorphic nor antiisomorphic; but

$$
UT_n(R) \cong UT_n(K) \times UT_n(K) \cong UT_n(K) \times UT_n(K^{op}) \cong UT_n(S). \quad \Box
$$
Below we shall show that in the class of commutative associative rings and in the class of associative rings without zero divisors such examples do not exist.

**Question.** Does $\text{UT}_n(R) \cong \text{UT}_n(S)$ imply $\text{UT}_m(R) \cong \text{UT}_m(S)$, for associative rings $R$, $S$ and $n, m \geq 3$?

1.12. **Bilinear maps in nilpotent groups**

Let $n \geq 3$, $G$ be a $(n - 1)$ step nilpotent group and

$$G = G_1 > G_2 > \ldots > G_{n-1} > G_n = 1$$

the lower central series of $G$. Consider the map

$$f_G : G_1/G_2 \times G_{n-2}/G_{n-1} \to G_{n-1}, \quad f_G(xG_2, yG_{n-1}) = [x, y],$$

for $x \in G_1$, $y \in G_{n-2}$. Using $[G_i, G_j] \leq G_{i+j}$ (see [1, Corollary 0.31]) one can show that $f_G$ is a well-defined bilinear map of abelian groups. Clearly, for isomorphic groups the corresponding bilinear maps are isomorphic.

1.13. **The bilinear map of a quasiunitriangular group**

**Proposition 1.10.** Let $n \geq 3$ and $R$ be a ring with unit, associative if $n > 3$. Let $U$ be the group $\text{UT}_n(R, g_1, \ldots, g_{n-1})$. Then $f_U$ is isomorphic to the bilinear map

$$f^R_n : R^{n-1} \times R^2 \to R, \quad f^R_n((\gamma_1, \ldots, \gamma_{n-1}), (\delta_1, \delta_2)) = \gamma_1 \delta_2 - \delta_1 \gamma_{n-1}. $$

**Proof.** For $(\gamma_1, \ldots, \gamma_{n-1}) \in R^{n-1}$, $(\delta_1, \delta_2) \in R^2$ and $\zeta \in R$ put

$$h_1(\gamma_1, \ldots, \gamma_{n-1}) = t_{n-1,n}(\gamma_{n-1}) \odot \ldots \odot t_{1,2}(\gamma_1) \odot U_2,$$

$$h_2(\delta_1, \delta_2) = t_{2,n}(\delta_2) \odot t_{1,n-1}(\delta_1) \odot U_{n-1},$$

$$h_0(\zeta) = t_{1,n}(\zeta).$$

It is easy to see that $h_1 : R^{n-1} \to U_1/U_2$, $h_2 : R^2 \to U_{n-2}/U_{n-1}$ and $h_0 : R \to U_{n-1}$ are isomorphisms of abelian groups and

$$f_U(h_1(\gamma_1, \ldots, \gamma_{n-1}), h_2(\delta_1, \delta_2)) = \left[t_{n-1,n}(\gamma_{n-1}) \odot \ldots \odot t_{1,2}(\gamma_1), t_{2,n}(\delta_2) \odot t_{1,n-1}(\delta_1) \right]$$

$$= t_{1,n}(\gamma_1 \delta_2 - \delta_1 \gamma_{n-1})$$

$$= h_0(f^R_n((\gamma_1, \ldots, \gamma_{n-1}), (\delta_1, \delta_2))).$$

So $f_U$ and $f^R_n$ are isomorphic. \(\square\)

Renaming variables we have

$$f^R_n((\alpha, \beta, \tilde{\gamma}), (\alpha', \beta')) = \alpha \beta' - \alpha' \beta,$$

where $\tilde{\gamma}$ is an $(n - 3)$-tuple in $R$.  

Corollary 1.11. If $UT_n(R, g_1, \ldots, g_{n-1}) \cong UT_n(S, q_1, \ldots, q_{n-1})$, then $f_n \cong f_n'$.

1.14. The ring of a bilinear map

Let $f : A_1 \times A_2 \to A_0$ be a bilinear map of abelian groups. Denote by $P(f)$ the set of all triples $(\varphi_0, \varphi_1, \varphi_2)$ such that $\varphi_i \in \text{End}(A_i)$ and for $x_1 \in A_1$, $x_2 \in A_2$,

$$f(x_1, x_2) = f(x_1, \varphi_2(x_2)) = \varphi_0(f(x_1, x_2)).$$

Clearly, $P(f)$ is a subring of the ring $\text{End}(A_0) \times \text{End}(A_1) \times \text{End}(A_2)$.

Proposition 1.12. Let $R$ be an associative ring with unit, $n \geq 3$,

$$f : R^{n-1} \times R^2 \to R, \quad f((x, \beta, \gamma), (x', \beta')) = x\beta' - \gamma x'.$$

Then $(\varphi_0, \varphi_1, \varphi_2) \in P(f)$ iff for some $\gamma \in Z(R)$ and some additive homomorphism $\pi : R^{n-3} \to R^{n-3}$

$$\varphi_0(x) = \gamma x, \quad \varphi_1(x, y, z) = (\gamma x, \gamma y, \pi(x, y, z)), \quad \varphi_2(x, y) = (y, y).$$

Proof. The sufficiency follows from the equations

$$(\gamma x)v - u(\gamma y) \equiv x(\gamma v) - (\gamma u)y \equiv \gamma(xv - uy).$$

Suppose a triple $(\varphi_0, \varphi_1, \varphi_2)$ is in $P(f)$. Let $\varphi_1 = (\tau_1, \sigma_1, \pi)$, $\varphi_2 = (\tau_2, \sigma_2)$, where $\tau_1, \sigma_1 \in \text{Hom}(R^{n-1}, R)$, $\pi \in \text{Hom}(R^{n-1}, R^{n-3})$ and $\tau_2, \sigma_2 \in \text{Hom}(R^2, R)$. The condition $(\varphi_0, \varphi_1, \varphi_2) \in P(f)$ is equivalent to the following:

$$\tau_1(x, y, z)y' - x'\sigma_1(x, y, z) \equiv x\sigma_2(x', y') - \tau_2(x', y')y \equiv \varphi_0(xy' - x'y).$$

Specializing $x, y, x', y'$, we have

$$x' = 0, y' = 1: \quad \tau_1(x, y, z) \equiv \varphi_0(x),$$
$$x' = 1, y' = 0: \quad \sigma_1(x, y, z) \equiv \varphi_0(y),$$
$$x = 0, y = 1: \quad \tau_2(x', y') \equiv \varphi_0(x'),$$
$$x = 1, y = 0: \quad \sigma_2(x', y') \equiv \varphi_0(y').$$

So the condition above is equivalent to

$$\varphi_0(x)y' - x'\varphi_0(y) \equiv x\varphi_0(y') - \varphi_0(x')y \equiv \varphi_0(xy' - x'y).$$

Putting $x = 1, x' = 0$, we have $\varphi_0(1)y' = \varphi_0(y')$. Putting $y = 1, y' = 0$, we have $x'\varphi_0(1) = \varphi_0(x')$. Thus

$$\varphi_0(x) \equiv \varphi_0(1)x \equiv x\varphi_0(1).$$
So $y = \varphi_0(1) \in Z(R)$ and

$$\varphi_0(x) \equiv y, \quad \varphi_1(x, y, z) \equiv (yx, yy, \pi(x, y, z)), \quad \varphi_2(x, y) \equiv (yx, yy),$$

and the proof is completed. \hfill \Box

1.15. Isomorphism theorem (the case of commutative rings)

**Theorem 1.13.** Let $R$ and $S$ be the associative rings with unit, $n \geq 3$. Suppose

$$UT_n(R, y_1, \ldots, y_{n-1}) \simeq UT_n(S, q_1, \ldots, q_{n-1}).$$

Then $Z(R) \simeq Z(S)$. If $R$ is commutative, $R \simeq S$.

**Proof.** By Corollary 1.11, $f_n^R \simeq f_n^S$. Clearly, we can assume that $R$ and $S$ have the same additive group, say, $A$. So there are mappings $\Phi_1 \in \text{Aut}(A^{n-1})$, $\Phi_2 \in \text{Aut}(A^2)$ and $\Psi \in \text{Aut}(A)$ such that

$$f_n^R(\Phi_1(x), \Phi_2(y)) = \Psi(f_n^S(x, y)), \quad \text{for } x \in A^{n-1}, \quad y \in A^2.$$  

Denote $P^R = P(f_n^R)$ and $P^S = P(f_n^S)$. It is easy to see that

$$P^S = \{ (\psi_0 \psi^{-1}, \Phi_1 \phi_1 \Phi^{-1}_1, \Phi_2 \phi_2 \Phi^{-1}_2) : (\phi_0, \phi_1, \phi_2) \in P^R \}.$$  

Let $\cdot$, $\star$ and $1^\ast$, $1^\ast$ be the multiplication operations and the units of $S$ and $R$, respectively. Fix $\delta \in Z(R)$. Define $\psi_0 \in \text{End}(A)$, $\psi_1 \in \text{End}(A^{n-1})$ and $\psi_2 \in \text{End}(A^2)$ as follows:

$$\psi_0(x) \equiv \delta \cdot x, \quad \psi_1(x, \beta, \gamma) \equiv (\delta \cdot x, \delta \cdot \beta, \delta \cdot \gamma), \quad \psi_2(x, \beta) \equiv (\delta \cdot x, \delta \cdot \beta).$$

Then $(\psi_0, \psi_1, \psi_2)$ are $P^R$ and therefore $(\Psi \psi_0 \psi^{-1}, \Phi_1 \phi_1 \Phi^{-1}_1, \Phi_2 \phi_2 \Phi^{-1}_2) \in P^S$. By Proposition 1.12, for some $\chi(\delta) \in Z(S)$, $\psi_0 \psi^{-1}(x) \equiv \chi(\delta) \cdot x$. So, for $\alpha \in A$, $\delta \in Z(R)$, $\psi(\delta \cdot \alpha) = \chi(\delta) \cdot \psi(x)$. Analogously, there is a map $\theta : Z(S) \to Z(R)$ such that, for $\alpha \in A$, $\zeta \in Z(S)$, $\psi^{-1}(\zeta \cdot x) = \theta(\zeta) \cdot \psi^{-1}(x)$. If follows that $\chi \circ \theta = \theta \circ \chi = id$; so $\chi$ is a bijection from $Z(R)$ onto $Z(S)$.

We show that in fact $\chi$ is a ring isomorphism. Clearly, $\Psi(\delta) = \chi(\delta) \cdot \Psi(1^\ast)$, for $\delta \in Z(R)$. Therefore, for $\zeta$, $\zeta' \in Z(S)$, $\zeta \cdot \Psi(1^\ast) = \zeta' \cdot \Psi(1^\ast)$ implies $\zeta = \zeta'$. For $\delta$, $\delta' \in Z(R)$,

$$\chi(\delta + \delta') \cdot \Psi(1^\ast) = \Psi(\delta + \delta') = \Psi(\delta) + \Psi(\delta') = (\chi(\delta) + \chi(\delta')) \cdot \Psi(1^\ast),$$

$$\chi(\delta \cdot \delta') \cdot \Psi(1^\ast) = \Psi(\delta \cdot \delta') = \chi(\delta) \cdot \Psi(\delta') = \chi(\delta) \cdot \chi(\delta') \cdot \Psi(1^\ast).$$

Hence, for $\delta, \delta' \in Z(R)$, $\chi(\delta + \delta') = \chi(\delta) + \chi(\delta')$, $\chi(\delta \cdot \delta') = \chi(\delta) \cdot \chi(\delta')$. So $Z(R) \simeq Z(S)$.

Note that if $Z(R) = R$ then $\chi(R) = S$. Indeed, in this case, for $\delta \in A$, we have $\Psi(\delta) = \chi(\delta) \cdot \Psi(1^\ast)$. Then $\Psi(1^\ast)$ is invertible in $S$: its inverse element is $\chi(\Psi^{-1}(1^\ast))$. Therefore $\chi$ maps $R$ onto $S$. So if $R$ is commutative, $R \simeq S$. \hfill \Box
1.16. Comments

For the idea to recover a ring via bilinear mappings we were inspired by the work of Myasnikov [17], who showed that every non-degenerate bilinear map of abelian groups \( f : A_1 \times A_2 \to A_0 \) such that \( f(A_1, A_2) \) generates \( A_0 \) is \( \mathscr{P}_f \)-bilinear, for some commutative ring \( \mathscr{P}_f \) and structures of \( \mathscr{P}_f \)-modules on \( A_0, A_1 \) and \( A_2 \), which are the greatest with respect to this property. In [3] we proved the second part of Theorem 1.13 for \( n = 3 \) by another method. For a commutative associative ring \( R \), we have found all the bases of the group \( \text{UT}_3(R, g_1, g_2) \) and have shown that the ring corresponding to each of them is isomorphic to \( R \).

**Question.** Are there a non-associative ring \( R \) and an associative ring \( S \) with

\[
\text{UT}_3(R, g_1, g_2) \cong \text{UT}_3(S, q_1, q_2)?
\]

The question is open even for ordinary unitriangular groups.

1.17. Isomorphism theorem (the case of domains)

**Theorem 1.14.** Let \( R \) and \( S \) be associative rings with unit, \( n \geq 3 \). Suppose \( S \) is a non-commutative domain and

\[
\text{UT}_n(R, g_1, \ldots, g_{n-1}) \cong \text{UT}_n(S, q_1, \ldots, q_{n-1}).
\]

Then \( R \) is isomorphic or antiisomorphic to \( S \).

**Proof.** By Corollary 1.11, \( f^R_n \cong f^S_n \). Clearly, we can assume that \( R \) and \( S \) have the same additive group, say, \( A \). So there exist \( \Phi_1 \circ \text{Aut}(A^{n-1}), \Phi_2 \circ \text{Aut}(A^2) \) and \( \Psi \circ \text{Aut}(A) \) such that

\[
f^S_n(\Phi_1(x), \Phi_2(y)) = \Psi(f^R_n(x, y)), \quad \text{for } x \in A^{n-1}, \ y \in A^2.
\]

For a bilinear map \( f : A^{n-1} \times A^2 \to A \) let

\[
P[f] = \{(\phi_1, \phi_2) \in \text{End}(A^{n-1}) \times \text{End}(A^2) : f(x, \phi_2(y)) \equiv f(x, \phi_2(y))\}.
\]

Denote \( P_R = P[f^R] \) and \( P_S = P[f^S] \). It is easy to see that they are subgroups of the additive group of the ring \( \text{End}(A^{n-1}) \times \text{End}(A^2) \) and

\[
P_S = \{(\Phi_1 \Phi_1^{-1}, \Phi_2 \Phi_2^{-1}) : (\phi_1, \phi_2) \in P_R\}.
\]

Let \( \star \) and \( 1^* \) be the multiplication operations and the units of \( S \) and \( R \), respectively. Fix \( \rho, \lambda \in A \). Define \( \psi_1 \in \text{End}(A^{n-1}) \) and \( \psi_2 \in \text{End}(A^2) \) as follows:

\[
\psi_1(x, \beta, \gamma) \equiv (x \star \rho, \lambda \star \beta, \delta), \quad \psi_2(x, \beta) \equiv (x \star \lambda, \rho \star \beta).
\]

Then \( (\psi_1, \psi_2) \in P_R \) and therefore \( (\Phi_1 \psi_1 \Phi_1^{-1}, \Phi_2 \psi_2 \Phi_2^{-1}) \in P_S \).
Lemma 1.15. \((\varphi_1, \varphi_2) \in P_S\) iff, for some (uniquely determined) \(\rho', \lambda' \in A\),
\[ \varphi_1(\alpha, \beta, \bar{\gamma}) \equiv (\alpha \cdot \rho', \lambda' \cdot \beta, \pi(\alpha, \beta, \bar{\gamma})), \quad \varphi_2(\alpha, \beta) \equiv (\alpha \cdot \lambda', \rho' \cdot \beta). \]

Proof of Lemma 1.15. Since the sufficiency can be verified as above, we need to prove only the necessity. Let \(\varphi_1 = (\tau_1, \sigma_1, \pi), \varphi_2 = (\tau_2, \sigma_2)\). Then \((\varphi_1, \varphi_2) \in P_S\) means that
\[ \tau_1(\alpha, \beta, \bar{\gamma}) \cdot \beta' - \alpha' \cdot \sigma_1(\alpha, \beta, \bar{\gamma}) \equiv \alpha \cdot \sigma_2(\alpha', \beta') - \tau_2(\alpha', \beta') \cdot \beta. \]

Specializing \(\alpha, \beta, \alpha', \beta', \) we have
\[ \begin{align*}
\alpha' &= 0, \beta' = 1^\bullet: & \tau_1(\alpha, \beta, \bar{\gamma}) &\equiv \alpha \cdot \sigma_2(0, 1^\bullet) - \tau_2(0, 1^\bullet) \cdot \beta, \\
\alpha' &= 1^\bullet, \beta' = 0: & \sigma_1(\alpha, \beta, \bar{\gamma}) &\equiv - \alpha \cdot \sigma_2(1^\bullet, 0) + \tau_2(1^\bullet, 0) \cdot \beta, \\
\alpha &= 0, \beta = 1^\bullet: & \tau_2(\alpha', \beta') &\equiv - \tau_1(0, 1^\bullet, \bar{\gamma}) \cdot \beta' + \alpha' \cdot \sigma_1(0, 1^\bullet, \bar{\gamma}), \\
\alpha = 1^\bullet, \beta = 0: & \sigma_2(\alpha', \beta') &\equiv \tau_1(1^\bullet, 0, \bar{\gamma}) \cdot \beta' - \alpha' \cdot \sigma_1(0, 1^\bullet, \bar{\gamma}).
\end{align*} \]

It follows from the first two identities that \(\tau_1\) and \(\sigma_1\) do not depend on \(\bar{\gamma}\). So
\[ \begin{align*}
\tau_1(\alpha, \beta, \bar{\gamma}) &\equiv \alpha \cdot \rho_1 + v_1 \cdot \beta, & \tau_2(\alpha, \beta) &\equiv \alpha \cdot \rho_2 + v_2 \cdot \beta, \\
\sigma_1(\alpha, \beta, \bar{\gamma}) &\equiv \alpha \cdot \kappa_1 + \lambda_1 \cdot \beta, & \sigma_2(\alpha, \beta) &\equiv \alpha \cdot \kappa_2 + \lambda_2 \cdot \beta.
\end{align*} \]

Therefore
\[ \begin{align*}
(\alpha \cdot \rho_1 + v_1 \cdot \beta) \cdot \beta' - \alpha' \cdot (\alpha \cdot \kappa_1 + \lambda_1 \cdot \beta) &\equiv \alpha \cdot (\alpha' \cdot \kappa_2 + \lambda_2 \cdot \beta') - (\alpha' \cdot \rho_2 + v_2 \cdot \beta') \cdot \beta.
\end{align*} \]

Specializing \(\alpha, \beta, \alpha', \beta', \) we have
\[ \begin{align*}
\alpha &= \alpha' = 1^\bullet, \beta = \beta' = 0: & \kappa_1 &= - \kappa_2, \\
\alpha &= \alpha' = 0, \beta = \beta' = 1^\bullet: & v_1 &= - v_2, \\
\alpha &= \beta' = 1^\bullet, \alpha' = \beta = 0: & \rho_1 &= \lambda_2, \\
\alpha &= \beta' = 0, \alpha' = \beta = 1^\bullet: & \lambda_1 &= \rho_2.
\end{align*} \]

So
\[ \begin{align*}
\tau_1(\alpha, \beta, \bar{\gamma}) &\equiv \alpha \cdot \rho_1 + v_1 \cdot \beta, & \tau_2(\alpha, \beta) &\equiv \alpha \cdot \lambda_1 - v_1 \cdot \beta, \\
\sigma_1(\alpha, \beta, \bar{\gamma}) &\equiv \alpha \cdot \kappa_1 + \lambda_1 \cdot \beta, & \sigma_2(\alpha, \beta) &\equiv - \alpha \cdot \kappa_1 + \rho_1 \cdot \beta.
\end{align*} \]

Therefore
\[ \begin{align*}
(\alpha \cdot \rho_1 + v_1 \cdot \beta) \cdot \beta' - \alpha' \cdot (\alpha \cdot \kappa_1 + \lambda_1 \cdot \beta) &\equiv \alpha \cdot (- \alpha' \cdot \kappa_1 + \rho_1 \cdot \beta') - (\alpha' \cdot \lambda_1 - v_1 \cdot \beta') \cdot \beta, \\
\end{align*} \]
that is
\[ v_1 \cdot (\beta \cdot \beta' - \beta' \cdot \beta) \equiv (\alpha' \cdot \alpha - \alpha \cdot \alpha') \kappa_1. \]
As $S$ is a non-commutative domain, it follows that $\nu_1 = \kappa_1 = 0$. Thus

$$\tau_1(\alpha, \beta, \gamma) = \alpha \cdot \rho_1, \quad \tau_2(\alpha, \beta) = \alpha \cdot \lambda_1,$$

$$\sigma_1(\alpha, \beta, \gamma) \equiv \lambda_1 \cdot \beta, \quad \sigma_2(\alpha, \beta) \equiv \rho_1 \cdot \beta.$$

So we can put $\rho' = \rho_1$, $\lambda' = \lambda_1$. To see the uniqueness of $\rho'$ and $\lambda'$ put $\alpha = \beta = 1^*$. The lemma is proved. \(\square\)

**Proof of Theorem 1.14 (continued).** By the lemma,

$$\Phi_2 \psi_2 \Phi_2^{-1}(\alpha, \beta) \equiv (\alpha \cdot \lambda', \rho' \cdot \beta).$$

So for every $\lambda, \rho \in A$ there are uniquely determined $\lambda', \rho' \in A$ such that

$$(\theta_1(\alpha \star \lambda, \rho \star \beta), \theta_2(\alpha \star \lambda, \rho \star \beta)) = (\theta_1(\alpha, \beta) \cdot \lambda', \rho' \cdot \theta_2(\alpha, \beta)),$$

where $\Phi_2(\alpha, \beta) \equiv (\theta_1(\alpha, \beta), \theta_2(\alpha, \beta))$. Putting $\alpha = \beta = 1^*$, we have, for every $\lambda, \rho \in A$,

$$\theta_1(\lambda, \rho) = \theta_1(1^*, 1^*) \cdot \lambda', \quad \theta_2(\lambda, \rho) = \rho' \cdot \theta_2(1^*, 1^*).$$

Denote $\theta_1(1^*, 1^*)$, $\theta_2(1^*, 1^*)$ by $\xi_1$, $\xi_2$, respectively. Since $\Phi_2$ is a permutation of $A^2$, $\theta_1(\lambda_0, \rho_0) = \theta_2(\lambda_0, \rho_0) = 1^*$, for some $\lambda_0, \rho_0 \in A$. Then $1^* = \xi_1 \cdot \lambda_0 = \rho_0 \cdot \xi_2$. As in domains one-side inverses are inverses, $\xi_1$ and $\xi_2$ are invertible in $S$; let $\eta_1$ and $\eta_2$ be their inverses in $S$. So

$$\lambda' = \eta_1 \cdot \theta_1(\lambda, \rho), \quad \rho' = \theta_2(\lambda, \rho) \cdot \eta_2,$$

and, for every $\alpha, \beta, \lambda, \rho \in A$,

$$\theta_1(\alpha \star \lambda, \rho \star \beta) = \theta_1(\alpha, \beta) \cdot \eta_1 \cdot \theta_1(\lambda, \rho),$$

$$\theta_2(\alpha \star \lambda, \rho \star \beta) = \theta_2(\lambda, \rho) \cdot \eta_2 \cdot \theta_2(\alpha, \beta),$$

that is,

$$\eta_1 \cdot \theta_1(\alpha \star \lambda, \beta \star \rho) \equiv (\eta_1 \cdot \theta_1(\alpha, \beta)) \cdot (\eta_1 \cdot \theta_1(\lambda, \rho)),$$

$$\theta_2(\alpha \star \lambda, \beta \star \rho) \cdot \eta_2 \equiv (\theta_2(\alpha, \beta) \cdot \eta_2) \cdot \theta_2(\lambda, \rho) \cdot \eta_2.$$

The latter identities mean that the automorphism of the abelian group $A^2$

$$h(\alpha, \beta) = (\eta_1 \cdot \theta_1(\alpha, \beta), \theta_2(\alpha, \beta) \cdot \eta_2)$$

is an isomorphism from the ring $R \times R^{op}$ onto $S \times S^{op}$. As $S$ is a domain, it is indecomposable into a direct product; therefore $K \simeq S$ or $S^{op}$. \(\square\)

**Remark 1.16.** In [4] we have proved this theorem by another method in a special case, namely, for $n = 3$ and a non-commutative skew field $S$. We have found all the bases of the group $UT_3(S, q_1, q_2)$ and have shown that the rings corresponding to them are isomorphic to $S$ or $S^{op}$. 
2. Groups elementary equivalent to a unitriangular group

Denote the expanded group \((UT_n(R, g_1, \ldots, g_{n-1}), !)\) by \(UT_n^*(R, g_1, \ldots, g_{n-1})\). By Section 1.7, such expanded groups can be characterized as groups with a distinguished base; the expanded groups of the form \(UT_n^*(R)\) can be characterized as groups with a distinguished splitting base.

Clearly,

\[ UT_n^*(R, g_1, \ldots, g_{n-1}) \approx (UT_n^*(S, q_1, \ldots, q_{n-1}) \]

implies \( R \cong S \) because \( R \cong \text{Ring}(UT_n^*(R, g_1, \ldots, g_{n-1})) \).

For a class of rings \( R \), denote by \( UT_n(R) \), \( UT_n^*(R) \), \( QU_n(R) \) and \( QU_n^*(R) \) the classes of groups of the forms \( UT_n(R), UT_n^*(R), UT_n(R, g_1, \ldots, g_{n-1}) \) and \( UT_n^*(R, g_1, \ldots, g_{n-1}) \), respectively, where \( R \in R \). The classes of groups or expanded groups of the forms \( UT_n(R, g_1, \ldots, g_{n-1}), UT_n^*(R, g_1, \ldots, g_{n-1}) \), where \( R \in R \) and the standard base is pure, are denoted by \( PQUT_n(R) \) and \( PQUT_n^*(R) \), respectively. We shall omit \( R \) if it is the class of all rings with unit, for \( n = 3 \), or the class of all associative rings with unit, for \( n > 3 \).

We denote the axiomatizable closure of a class \( X \) of structures by \( \bar{X} \).

2.1. The axiomatizable closure of \( UT_n^* \)

**Proposition 2.1.** Let \( R \) be a class of rings. \( 1 \) If \( R \) is (finitely) axiomatizable, \( QU_n^*(R) \) is also (finitely) axiomatizable. \( 2 \) \( UT_n^*(R) = PQUT_n^*(R) \).

In particular, \( QU_n^*(R) \) is finitely axiomatizable and \( UT_n^* = PQUT_n^* \).

**Proof.** \( 1 \) is a consequence of Theorem 1.2 and Proposition 1.5.

\( 2 \) Using Theorem 1.2 and Proposition 1.5, it is easy to show that \( PQUT_n^*(R) \) is axiomatizable. As it contains \( UT_n^*(R) \), it contains also \( UT_n^*(R) \). To prove the converse inclusion it suffices to show that every \( N_1 \)-saturated member \( G \) of \( PQUT_n^*(R) \) is in \( UT_n^*(R) \). The center of \( G \) is \( N_1 \)-saturated and so pure-injective (see \([11, 1.11]\)). By \( 1 \) of Proposition 1.6, \( G \in UT_n^*(R) \). As \( UT_n^*(R) \subseteq UT_n^*(R) \), we are done. \( \square \)

**Remark 2.2.** The class \( UT_n^* \) is a proper subclass of the class \( PQUT_n^* \). For example, let \( R = \mathbb{Z} \times \mathbb{Q} \). Then \( \text{Ext}(R^+, R^+) \cong \text{Ext}(Q, Z) \neq 0 \). If \( g_1, \ldots, g_{n-1} \) are symmetric 2-cocycles from the group \( R^+ \) to itself and not all of them are coboundaries, \( UT_n^*(R, g_1, \ldots, g_{n-1}) \) is in \( PQUT_n^* \) but not in \( UT_n^* \). So the class \( UT_n^* \) is not axiomatizable.

2.2. The axiomatizable closure of \( UT_n \)

For \( I \subseteq \omega \), a base \( \mathfrak{h} \) in a group \( H \) is said to be \( I \)-pure if, for every \( m \in I \), \( H_{1,m} \) is an \( m \)-pure subgroup of every \( H_{i,i+1} \). (A subgroup \( A \) of abelian group \( B \) is said to be
A group \( H \) is said to be **locally pure** if it has an \( I \)-pure base, for each finite \( I \subset \omega \).

**Proposition 2.3.** Let \( \mathcal{R} \) be a class of rings.

1. If \( \mathcal{R} \) is (finitely) axiomatizable, \( \text{QUT}_n(\mathcal{R}) \) is also (finitely) axiomatizable.
2. The axiomatizable closure of \( \text{UT}_n(\mathcal{R}) \) consists of all groups \( H \) such that, for every finite \( I \subset \omega \) and \( \phi \in \text{Th}(\mathcal{R}) \), there is an \( I \)-pure base \( \mathfrak{h} \) with \( \text{Ring}(H, \mathfrak{h}) \) a model of \( \phi \). In particular, \( \text{UT}_n \) is the class of all locally pure groups in \( \text{QUT}_n \).

**Proof.** (1) is a consequence of Theorem 1.2 and Proposition 1.5.

(2) For a finite \( I \subset \omega \) and \( \phi \in \text{Th}(\mathcal{R}) \), let \( \Phi(\phi, I) \) be the group sentence expressing the existence of an \( I \)-pure base \( \mathfrak{h} \) such that \( \phi \) holds in \( \text{Ring}(H, \mathfrak{h}) \). Every \( \Phi(\phi, I) \) holds in \( \text{UT}_n(\mathcal{R}) \) and therefore in \( \text{UT}_n(\mathcal{R}) \). If \( H \) is a model of all the \( \Phi(\phi, I) \)'s, in an \( \aleph_1 \)-saturated \( H' \equiv H \) there is a pure base \( \mathfrak{h} \) with \( \text{Ring}(H', \mathfrak{h}) \in \mathcal{R} \). As in the proof of Proposition 2.1, \( H' \in \text{UT}_n(\mathcal{R}) \equiv \text{UT}_n(\mathcal{R}) \). Thus \( H \in \text{UT}_n(\mathcal{R}) \).

**Remark 2.4.** The result contrasts with Proposition 2.1: the axiomatizable closure of \( \text{UT}_n^* \) consists of all expanded **pure** groups, but the axiomatizable closure of \( \text{UT}_n \) consists of all **locally pure** groups. In [5] we have constructed a rather subtle example of a locally pure group (over a commutative ring, for \( n = 3 \)), which is not pure. (The construction depends on \( n = 3 \), but I believe that the notions **pure** and **locally pure** differ for every \( n \geq 3 \).) As a byproduct, we have that the class \( \text{UT}_3 \) is not elementarily closed, even though closed under ultraproducts. The non-axiomatizability of \( \text{UT}_n \) is a more non-trivial fact than that of \( \text{UT}_n^* \). The reason is that a group of the form \( \text{UT}_n(R, g_1, \ldots, g_{n-1}) \) can be isomorphic to an ordinary unitriangular group, even if not all the \( g_i \)s are coboundaries. In [4] we have constructed a group elementarily equivalent to \( \text{UT}_3^{*}(\mathbb{Z}) \), which is not in \( \text{UT}_3^{*} \).

Below we shall study what are the groups elementarily equivalent to the group \( \text{UT}_n(R) \). Of course, they are all of the form \( \text{UT}_n(S, q_1, \ldots, q_{n-1}) \). The problem is what one can say about \( S \) and \( q_1, \ldots, q_{n-1} \).

**2.3. Positive results**

**Proposition 2.5.** Suppose \( \text{UT}_n(S, q_1, \ldots, q_{n-1}) \equiv \text{UT}_n(R, g_1, \ldots, g_{n-1}) \). Then \( R^+ \equiv S^+ \).

**Proof.** The groups \( S^+ \) and \( R^+ \) are isomorphic to the centers of the quasiunitriangular groups considered.

**Proposition 2.6.** If \( \text{UT}_n(S, q_1, \ldots, q_{n-1}) \equiv \text{UT}_n(R, g_1, \ldots, g_{n-1}) \) and \( R \) is torsion-free, then the cocycles \( q_1, \ldots, q_{n-1} \) are pure.

**Proof.** By Proposition 2.5, \( S^+ \) is torsion-free. But any abelian extension of any group by a torsion-free group is pure.
Proposition 2.7. If $G \cong \text{UT}_n(R, g_1, \ldots, g_{n-1})$, where $R$ is a domain or a commutative associative ring, then $G$ has a form $\text{UT}_n(S, q_1, \ldots, q_{n-1})$, for some $S \equiv R$.

Proof. It follows from (1) of Proposition 2.3 that, for every $\phi \in \text{Th}(R)$, the group $G$ has a form $\text{UT}_n(S^\phi, q_1^\phi, \ldots, q_{n-1}^\phi)$, where $S^\phi$ is a model of $\phi$, $S^\phi$ is a domain if $R$ is, and $S^\phi$ is a commutative associative ring if $R$ is.

If $R$ is a commutative associative ring, all the $S^\phi$s are isomorphic by Theorem 1.13 and are therefore elementarily equivalent to $R$.

Suppose $R$ is a domain. Let $\text{Th}(R) = \{\psi_m: m < \omega\}$, $\phi_m = \psi_0 \wedge \cdots \wedge \psi_m$ and $S_m = S^\phi_m$. By Theorem 1.14, $S_m \simeq S_0$ or $S_0^{op}$, for any $m$. Then $\{m: S_m \simeq S_0\}$ or $\{m: S_m \simeq S_0^{op}\}$ is infinite. If the first set is infinite, $S_0 \equiv R$; otherwise the second set is infinite and $S_0^{op} \equiv R$. Since, by Proposition 1.7,

$$G \cong \text{UT}_n(S_0^{op}, q_{n-1}^{\phi_0}, \ldots, q_1^\phi),$$

in both the cases the result follows. \qed

Proposition 2.8. If a group $H$ is $\mathcal{N}_1$-saturated, $H \cong \text{UT}_n(R, g_1, \ldots, g_{n-1})$ and $g_1, \ldots, g_{n-1}$ are pure, then $H \cong \text{UT}_n(R')$, for some $R' \equiv R$.

Proof. Since $H$ is $\mathcal{N}_1$-saturated there is a pure base $h$ in $H$ such that $\text{Ring}(H, h) \equiv R$. The centre of $H$ is $\mathcal{N}_1$-saturated and therefore pure-injective; so $h$ is a splitting base. Therefore, by Corollary 1.4, $H \cong \text{UT}_n(R')$, where $R' = \text{Ring}(H, h)$. \qed

Proposition 2.9. Suppose $S \equiv R$. Then $\text{UT}_n(S) \equiv \text{UT}_n(R)$. Moreover,

$$\text{UT}_n(S, q_1, \ldots, q_{n-1}) \equiv \text{UT}_n(R, g_1, \ldots, g_{n-1}),$$

provided the cocycles $g_i$'s and $q_i$'s are pure.

Proof. The first claim follows from the fact that the group $\text{UT}_n(R)$ can be interpreted in the ring $R$ without parameters uniformly in $R$.

To prove the second claim consider $\mathcal{N}_1$-saturated groups elementarily equivalent to $\text{UT}_n(S, q_1, \ldots, q_{n-1})$ and $\text{UT}_n(R, g_1, \ldots, g_{n-1})$; by Proposition 2.8 they are of the forms $\text{UT}_n(S')$, $\text{UT}_n(R')$, where $S' \equiv S$, $R' \equiv R$, respectively. Now the result follows from the first claim. \qed

Proposition 2.10. If $R$ is a domain or a commutative associative ring and $R$ is torsion-free, then $G \cong \text{UT}_n(R, g_1, \ldots, g_{n-1})$ iff $G$ has a form $\text{UT}_n(S, q_1, \ldots, q_{n-1})$, for some $S \equiv R$.

Proof. The if part follows from Propositions 2.6 and 2.9 and the only if part is a special case of Proposition 2.7. \qed
Proposition 2.11. Suppose $R^+$ is the direct sum of a bounded group and a divisible group. If $G \equiv \text{UT}_n(R)$, then $G \simeq \text{UT}_n(S)$, for some $S$. For every $\phi \in \text{Th}(R)$, the ring $S$ can be chosen to be a model of $\phi$.

For the proof we need the following lemma.

Lemma 2.12. Let $A$ be an abelian group, $A = B \oplus D$, where $B$ is bounded and $D$ is divisible. Let $I$ be the set of all prime power divisors of $\exp(B)$. Then an abelian extension $A < C$ is pure iff it is I-pure.

Proof of Lemma 2.12. As the only if part is trivial, we need to prove only the if part. It suffices to show that $B$ is a pure subgroup of $C$, that is, $B \cap p^k C \leq p^k B$, for every prime $p$ and natural number $k$. Suppose $p^m$ divides $\exp(B)$, but $p^{m+1}$ does not. Then $p^m B$ is $p$-torsion free; hence $p^k B = p^m B$, for $k \geq m$. As $B$ is a $p^m$-pure subgroup of $C$, we have, for every $k \geq m$,

$$B \cap p^k C \leq B \cap p^m C = p^m B = p^k B.$$ 

If $k < m$, we have $B \cap p^k C \leq p^k B$ because $B$ is an I-pure subgroup of $C$. □

Proof of Proposition 2.11. Let $m$ be the exponent of the reduced part of $R^+$. As $G \equiv \text{UT}_n(R)$, we have $Z(G) \equiv R^+$, hence $Z(G) = B \oplus D$, where $B$ is of exponent $m$ and $D$ is divisible. Let $I$ be the set of all prime power divisors of $m$; it is a finite set. Let $\phi \in \text{Th}(R)$. Due to $G \equiv \text{UT}_n(R)$, there is an I-pure base $g$ in $G$ such that $\text{Ring}(G, g)$ is a model of $\phi$. By Lemma 2.12 this base is pure. By [13, 21.2, 27.5, 38.3], $Z(G)$ is pure-injective. So $g$ is a splitting base and the result follows from Corollary 1.4. □

Proposition 2.13. Suppose $R^+$ is the direct sum of a bounded group and a divisible group and $R$ is a domain or a commutative associative ring. Then $G \equiv \text{UT}_n(R)$ iff $G \simeq \text{UT}_n(S)$, for some $S \equiv R$.

We need the following lemma.

Lemma 2.14. Suppose $\text{UT}_n(R, g_1, \ldots, g_{n-1}) \equiv \text{UT}_n(S, q_1, \ldots, q_{n-1})$. If $R$ and $S$ are commutative associative rings, $R = S$. If $R$ and $S$ are domains, $R = S$ or $R = S^{op}$.

Proof of Lemma 2.14. Let $H$ be a model of the complete theory of the two groups considered such that the types of their standard bases are realized in $H$ by some tuples $r$ and $s$. Then

$$\text{Ring}(H, r) \equiv \text{Ring}(\text{UT}_n^*(R, g_1, \ldots, g_{n-1})) \simeq R,$$

$$\text{Ring}(H, s) \equiv \text{Ring}(\text{UT}_n^*(S, q_1, \ldots, q_{n-1})) \simeq S.$$
By Theorem 1.2,

\[ H \simeq \text{UT}_n(R', q'_1, \ldots, q'_{n-1}) \simeq \text{UT}_n(S', q'_1, \ldots, q'_{n-1}) \]

where \( R' = \text{Ring}(H, r) \equiv R \) and \( S' = \text{Ring}(H, s) \equiv S \).

If \( R \) and \( S \) are commutative and associative, \( R' \) and \( S' \) are also commutative and associative; then, by Theorem 1.13, \( R' \approx S' \); therefore \( R \equiv S \).

If \( R \) and \( S \) are domains, \( R' \) and \( S' \) are also domains; then, by Theorem 1.14, \( R' \) is isomorphic or antiisomorphic to \( S' \); therefore \( R \equiv S \) or \( R \equiv S^{op} \).

**Proof of Proposition 2.13.** The if part holds for an arbitrary \( R \), by Proposition 2.9. We prove the only if part. If \( G = \text{UT}_n(R) \), then by Proposition 2.11, \( G \approx \text{UT}_n(S) \), where \( S \) is commutative and associative if \( R \) is, and \( S \) is a domain if \( R \) is. By Lemma 2.14, \( S \equiv R \) in the first case and \( S \equiv R \) or \( S^{op} \equiv R \) in the second case. Since \( \text{UT}_n(S) \approx \text{UT}_n(S^{op}) \), by Corollary 1.8, the result follows. \( \square \)

**Corollary 2.15.** If \( R \) is a skew field then \( G \equiv \text{UT}_n(R) \) iff \( G \approx \text{UT}_n(S) \), for some \( S \equiv R \).

We shall show that the conditions in the results above are essential. To construct examples we need the following observations.

### 2.4. Quasiunitriangular groups and Cartesian products

**Proposition 2.16.** If \( R = \prod_{i \in I} R_i \), then \( \text{UT}_n(R) = \prod_{i \in I} \text{UT}_n(R_i) \).

**Proof.** Easy. \( \square \)

**Proposition 2.17.** Let \( \{H_i : i \in I\} \) be a family of groups, \( H = \prod_{i \in I} H_i \). Let \( h = \{h_k : 1 \leq k < l \leq n\} \) be a family in \( H \), \( n \geq 3 \). Then

1. \( h \) is a base in the group \( H \) iff, for every \( i \in I \), \( h_i = \{h_k(i) : 1 \leq k < l \leq n\} \) is a base in the group \( H_i \);
2. if \( h \) is a base in \( H \), then \( \text{Ring}(H, h) = \prod_{i \in I} \text{Ring}(H_i, h_i) \).

**Proof.** To prove (1) we need the following

**Claim 2.18.** Let \( \theta(x) \) be a formula of the form \( \forall y \phi(x, y) \rightarrow \psi(x, y) \), where \( \phi \) and \( \psi \) are positive primitive formulas. Let \( \{\mathcal{A}_i : i \in I\} \) be a family of structures of the language of \( \theta \) such that \( \forall x \exists y \phi(x, y) \) holds in every \( \mathcal{A}_i \). Then \( \theta(\vec{f}) \) holds in the direct product of the family \( \mathcal{A}_i \), for all \( i \in I \).

**Proof of Claim 2.18.** The if part holds as \( \theta \) is a Horn formula; the only if part can be easily verified using the condition \( \forall x \exists y \phi(x, y) \) holds in every \( \mathcal{A}_i \). \( \square \)
Now (1) follows from the O-definability of the notion of base by a formula of the form $\forall \bar{y} (\phi(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$ where $\phi$ and $\psi$ are positive primitive formulas and $\forall x \phi(x, e)$ holds in any group (see Section 1.9).

(2) It can be easily seen that, for every $n \geq 3$, there is a positive primitive formula $\rho(u, v, w, \bar{x})$ such that, for any group $G$ with a base $g$, the formula $\rho(u, v, w, g)$ defines the graph of the multiplication operation in $\text{Ring}(G, g)$.

Clearly, $Z(H) = \bigcap_{i \in I} Z(H_i)$. Let $a, b, c \in Z(H)$. Then

$\text{Ring}(H, h) = a \Box b = c$

if $H \models \rho(a, b, c, h)$

if $H_i \models \rho(a_i, b_i, c_i, h_i)$ for all $i \in I$

if $\text{Ring}(H_i, h_i) = a_i \Box b_i = c_i$ for all $i \in I$.

So (2) holds. $\square$

**Corollary 2.19.** If a ring $R$ is indecomposable into a Cartesian product, the group $UT_n(R, q_1, \ldots, q_{n-1})$ is also indecomposable. The group $UT_n(R)$ is decomposable into a Cartesian product iff the ring $R$ is.

2.5. Negative results

**Proposition 2.20.** There exist associative rings $P$ and $R$ of prime characteristic such that, for every $n \geq 3$, $UT_n(P) = UT_n(R)$, but, for every $S \equiv R$, $UT_n(P)$ cannot be represented in the form $UT_n(S, q_1, \ldots, q_{n-1})$.

To prove the proposition we need the following lemma.

**Lemma 2.21.** Let $\{\mathcal{U}_i : i \in I\}$ be an infinite family of $L$-structures, $T$ the set of all sentences $\phi$ such that $\{i \in I : \mathcal{U}_i \models \neg \phi\}$ is finite, and $\mathcal{A}$ a model of $T$. Then

$\mathcal{A} \times \prod_{i \in I} \mathcal{U}_i \equiv \prod_{i \in I} \mathcal{U}_i$.

In particular, this holds for the ultraproduct $\mathcal{A} = \prod_{i \in I} \mathcal{U}_i / D$ modulo a non-principal ultrafilter $D$ on $I$.

We first need the following fact.

**Fact 2.22 [12].** For every $L$-sentence $\psi$, there are a positive integer $m$ and $L$-sentences $\theta_0, \ldots, \theta_{m-1}$ such that for any non-empty set $I$ there exist a positive integer $n$, subsets $S_0, \ldots, S_{n-1}$ of $m$ and functions $g_0, \ldots, g_{n-1} \in \omega^m$ such that, for every family $\{\mathcal{U}_i : i \in I\}$ of $L$-structures, $\psi$ holds in $\prod_{i \in I} \mathcal{U}_i$ iff, for some $k < n$, $|\{i : \mathcal{U}_i \models \theta_j\}|$ is equal to $g_k(j)$ if $j \in S_k$ and greater than $g_k(j)$ otherwise.
Proof of Lemma 2.21. W.l.o.g. we can assume that $0 \notin I$; denote $\mathcal{U}_0 = \mathcal{U}_I, I_0 = I \cup \{0\}$.

For an $L$-sentence $\theta$, denote $J = \{i \in I: \mathcal{U}_i \models \theta\}$ and $J_0 = \{i \in I_0: \mathcal{U}_i \models \theta\}$. If $J$ is finite, $J_0 = J$; if $J$ is infinite, $J_0$ is infinite, too.

For an $L$-sentence $\psi$, consider the corresponding number $m$ and the $L$-sentences $\theta_0, \ldots, \theta_{m-1}$ from the fact above. Since $|I| = |I_0|$, for $I$ and $I_0$ the parameters $n, S_k$'s and $g_k$'s are the same. Therefore $\psi$ holds in $\prod_{i \in I} \mathcal{U}_i$ iff $\psi$ holds in $\prod_{i \in I_0} \mathcal{U}_i$. Then Lemma 2.21 is proved. \(\square\)

Proof of Proposition 2.20. Let $p$ be a prime. For any field $F$ of characteristic $p$, $F \neq F_p$, the ring

$$K(F) = \begin{pmatrix} F_p & F \\ 0 & F \end{pmatrix}$$

satisfies the following property: every non-zero right ideal has a cardinality $\geq |F|$.

Indeed, it is easy to see that the proper right ideals of $K(F)$ are exactly

$$\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}, \quad \left\{ \begin{pmatrix} 0 & ab \\ 0 & b \end{pmatrix} : b \in F \right\}, \quad a \neq 0.$$

For every additive subgroup $A$ of $F$ the set

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

is a left ideal in $K(F)$, so there are left ideals of cardinality $p < |F|$.

Let $D$ be a non-principal ultrafilter on the set $\{2, 3, 4, \ldots\}$, $F_\omega = \prod_{m \geq 1} F_{p^m}/D$.

Denote $K_m = K(F_{p^m}), K_\omega = K(F_\omega)$. It is easy to see that $K_\omega = \prod_{m \geq 1} K_m/D$.

Put $P = \prod_{m > 1} K_m, R = P \times K_\omega$. Since, by Corollary 1.8, $\mathsf{UT}_n(K_\omega) \simeq \mathsf{UT}_n(K_\omega)$ and, by Lemma 2.21, $P \times K_\omega \equiv P$, we have

$$\mathsf{UT}_n(R) \simeq \mathsf{UT}_n(P) \times \mathsf{UT}_n(K_\omega)$$

$$\sim \mathsf{UT}_n(P) \times \mathsf{UT}_n(K_\omega) \sim \mathsf{UT}_n(P \times K_\omega) = \mathsf{UT}_n(P).$$

We show that $\mathsf{UT}_n(P)$ cannot be represented in the form $\mathsf{UT}_n(S, q_1, \ldots, q_{n-1})$, where $S \equiv R$. Suppose the contrary. Then, for some base $h$ in $\mathsf{UT}_n(P),$

$$S \simeq \mathsf{Ring}(\mathsf{UT}_n(P), h) \equiv R.$$

For every $m > 1$ the ring $R$ satisfies the following first order property:

There is a minimal central idempotent $e$ such that (1) every non-zero left ideal of the ring $Re$ has a cardinality $\geq p^m$; (2) for any minimal central idempotent $e'$ different from $e$ there is a left ideal of cardinality $p$ in $Re'$.

Indeed, in a ring of the form $K(F)$ the only non-zero central idempotent is the unit. Therefore in $R$ the minimal central idempotents are exactly the units of the factors
Since, by the remark above, every non-zero left ideal of $K^p_m$ is infinite and there is a left ideal in $K^p_m$ of cardinality $p$, one can take the unit of $K^p_m$ as $e$.

So this property holds in $S$. Then there is a minimal central idempotent $e$ in $S$ such that every non-zero left ideal in $Se$ is infinite; in particular, $Se$ is infinite.

By (2) of Proposition 2.17, $S \simeq \prod_{m > 1} \text{Ring}(UT_a(K_m), b_m)$. Since every minimal central idempotent of a direct product belongs to one of the factors, $\text{Ring}(UT_a(K_m), b_m)$ is infinite for some $m$. But it is of the same cardinality as $K_m$, a contradiction $\square$.

Proposition 2.20 demonstrates that in Propositions 2.7 and 2.13 the condition $R$ is a domain or a commutative associative ring is essential.

In Proposition 2.6 the condition $R$ is torsion-free is essential because there are a commutative ring $R$ and a group $H \equiv UT_3(R)$ such that $H$ is not pure [5]. The latter fact also shows that in Proposition 2.8 the condition $H$ is $\mathbb{N}_1$-saturated is essential.

In Proposition 2.9 the condition of purity of the cocycles is essential: we have shown in [5] that, for even $k$, the groups $UT_3(Z_k)$ and $UT_3(Z_k, pr, pr)$ are not isomorphic, where $pr$ is the cocycle $(x, y) \mapsto xy$.

By the same reason, in Proposition 2.10 the condition $R$ is torsion-free is essential.

In [3] we have shown that there is a group $G \equiv UT_3(Z), G \not\equiv UT_3$. By this reason, the condition on $R^+$ in Proposition 2.11 is essential.

3. The number of models in a power of the theory of a unitriangular group

In this section we study connections between the number of models in a power of the theory of a ring and the one of the unitriangular group over the ring.

For a structure $\mathfrak{U}$ and an infinite power $\lambda$, denote by $I(\lambda, \mathfrak{U})$ the number of models of $Th(\mathfrak{U})$ in $\lambda$.

3.1. Equi-stability and equi-smallness of $UT_\Lambda(R)$ and $R$

**Proposition 3.1.** For any cardinal $\lambda$, $UT_\Lambda(R)$ is $\lambda$-stable iff $R$ is $\lambda$-stable. The theory of $UT_\Lambda(R)$ is small iff the theory of $R$ is small.

**Proof.** The ring $R$ and the group $UT_\Lambda(R)$ interpret each other, by Section 1.7. $\square$

**Corollary 3.2.** If $R$ is unsuperstable, then, for every uncountable $\lambda$, $I(\lambda, UT_\Lambda(R)) = I(\lambda, R) = 2^\lambda$. If $R$ is not small, $I(\mathbb{N}_0, UT_\Lambda(R)) = I(\mathbb{N}_0, R) = 2^{\mathbb{N}_0}$.

**Proof.** The first part is a consequence of [20, VIII 2.1]. The second part follows from the fact that a theory having uncountably many types has $2^{\mathbb{N}_0}$ types. $\square$
3.2. DCC for stable or small rings

We need the following well-known observation.

**Fact 3.3.** Any stable ring \( R \) satisfies the descending chain condition for principal one-sided ideals.

**Proof.** If DCC fails for, say, principal right ideals, the formula \( (\exists z)x = yz \) has the order property. \( \Box \)

We also need the following fact which is a modification of Cherlin's lemma [25, Lemma 10.4].

**Proposition 3.4.** Let \( R \) be a ring. If \( a_0 R \triangleright a_1 R \triangleright \cdots \) or \( Ra_0 \triangleright Ra_1 \triangleright \cdots \), then there are \( 2^{\aleph_0} \) 2-types over \( \{ a_i: i < \omega \} \).

**Proof.** Suppose, say, \( a_0 R \triangleright a_1 R \triangleright \cdots \). We can assume that \( |a_i R: a_{i+1} R| \geq 2^{i+1} \), for every \( i \). We show that the set of formulas \( p(v) = \{ (\forall y) a_i v \neq a_{i+1} y: i < \omega \} \) is finitely satisfiable in \( R \). Suppose the contrary. Then \( R = A_0 \cup \cdots \cup A_k \), for some \( k < \omega \), where \( A_i = \{ x \in R: a_i x \in a_{i+1} R \} \). Clearly, \( |R: A_i| \geq |a_i R: a_{i+1} R| \geq 2^{i+1} \). By Neumann's lemma (see [18], Theorem 2.12), one can assume that all the \( A_i \)'s are of finite index in \( R \); then \( A = A_0 \cap \cdots \cap A_k \) is of a finite index \( m \) in \( R \). Thus \( m \leq m/2 + \cdots + m/2^k < m \), a contradiction.

Let \( b \) be a realization of the type \( p \) in \( R' \), an elementary extension of \( R \); then \( a_i b \notin a_{i+1} R' \). By a well-known observation of Macintyre [15], there are \( 2^{\aleph_0} \) 1-types over \( \{ b, a_0, a_1, \ldots \} \); therefore there are \( 2^{\aleph_0} \) 2-types over \( \{ a_0, a_1, \ldots \} \). \( \Box \)

**Proposition 3.5** If \( R \) is an associative ring without zero divisors, which is stable or small, then \( R \) is a skew field.

**Proof.** Let \( a \in R, a \neq 0 \). Then, by Fact 3.3 and Proposition 3.4, for some \( m \), \( a^m R = a^{m+1} R \). As \( R \) has no zero divisors, for every \( b \in R \), there is \( c \in R \) with \( b = ac \). Analogously, for every \( b \in R \), there is \( c \in R \) with \( b = ca \). Hence \( R \) is a skew field. \( \Box \)

**Proposition 3.6.** If \( R \) is a stable or small ring with unit, then \( R^+ \) is the direct sum of a bounded subgroup and a divisible subgroup.

**Proof.** By Fact 3.3 and Proposition 3.4, \( m! R = (m+1)! R = \ldots \), for some \( m \). Then \( m! R^+ \) is a divisible subgroup of \( R^+ \); hence \( R^+ = B \oplus m! R^+ \) and \( m! B = 0 \). \( \Box \)
3.3. The number of models of $\text{Th}(\text{UT}_n(R))$: positive results

**Proposition 3.7.** If $R$ is stable or small, every group elementarily equivalent to $\text{UT}_n(R)$ has a form $\text{UT}_n(S)$, for some ring $S$. If, in addition, $R$ is a commutative associative ring or a domain $G \equiv \text{UT}_n(R)$ iff $G \simeq \text{UT}_n(S)$, for some $S \simeq R$.

**Proof.** Apply Proposition 2.11, 2.13 and 3.6. □

**Theorem 3.8.** Let $R$ be a commutative associative ring with unit. Then, for every infinite cardinal $\beta$, $I(\lambda, R) = I(\lambda, \text{UT}_n(R))$.

**Proof.** If $R$ is stable or small then, by Theorem 1.13 and Proposition 3.7, $S \mapsto \text{UT}_n(S)$ is a one-to-one correspondence between the models of $\text{Th}(R)$ and the models of $\text{Th}(\text{UT}_n(R))$, and the result follows. If $R$ is unstable and non-small, then, by Corollary 3.2, $I(\lambda, R) = I(\lambda, \text{UT}_n(R)) = 2^\beta$. □

**Theorem 3.9** Let $R$ be a domain. Then

1. for every uncountable $\lambda$, $I(\lambda, R) = I(\lambda, \text{UT}_n(R))$;
2. $I(\aleph_0, R) \equiv I(\aleph_0, \text{UT}_n(R))$, and $\beta$ holds iff $R$ is a skew field, $I(\aleph_0, R) < \aleph_0$ and there is a countable $S$ such that $S \equiv S^{op} \equiv R$, but $S$ is not isomorphic to $S^{op}$.

**Proof.** (1) If $R$ is unsuperstable, then by Corollary 3.2, $I(\lambda, \text{UT}_n(R)) = I(\lambda, R) = 2^\beta$, for every uncountable $\lambda$. If $R$ is superstable, then, by Proposition 3.5, $R$ is a skew field and, in fact, a commutative algebraically closed field, by [10]. Then, by Theorem 3.8, $I(\lambda, \text{UT}_n(R)) = I(\lambda, R)$. (This number is equal to 1, for $\lambda > \aleph_0$, or $\aleph_0$, for $\lambda = \aleph_0$.)

(2) If $R$ is not small, $I(\aleph_0, \text{UT}_n(R)) = I(\aleph_0, R) = 2^{\aleph_0}$. If $R$ is small then, by Proposition 3.5, $R$ is a skew field and, by Theorem 1.14 and Proposition 3.7, $S \mapsto \text{UT}_n(S)$ is an (at most two)-to-one correspondence between the countable models of $\text{Th}(R)$ and the countable models of $\text{Th}(\text{UT}_n(R))$, so the result follows. □

**Proposition 3.10.** If $R$ is a stable domain, $I(\lambda, \text{UT}_n(R)) = I(\lambda, R)$, for every $\lambda$.

**Proof.** Due to (1) of Theorem 3.9 we need to consider only $\lambda = \aleph_0$. Suppose $I(\aleph_0, \text{UT}_n(R)) \not= I(\aleph_0, R)$. Then, by Section 3.1 and Proposition 3.5, $R$ is a stable small skew field and hence a commutative algebraically closed field, by [24], a contradiction to Theorem 3.8. □

**Conjecture 3.11.** For every domain $R$ and infinite $\lambda$, $I(\lambda, \text{UT}_n(R)) = I(\lambda, R)$.

This conjecture is equivalent to the following:

**Conjecture 3.12.** There is no skew field $R$ such that
(a) $I(\aleph_0, R) < \aleph_0$.
(b) $K$ is antiisomorphic to itself, but $S$ is not, for some countable $S$, $K \equiv R$. 
Indeed, (a) and (b) obviously imply the condition for \( > \) in (2) of Theorem 3.9; if the condition holds, one can take the countable saturated model of \( \text{Th}(R) \) as \( K \) in (a).

I conjecture that there is actually no skew field \( R \) with \( I(\mathbb{N}_0, R) < \mathbb{N}_0 \). By [24], such an \( R \) must be unstable. The question is open even for commutative fields \( R \).

3.4. The number of models of \( \text{Th}(\text{UT}_a(R)) \): a negative result

Videla [21, 22] asked whether \( I(\lambda, R) = I(\lambda, \text{UT}_a(R)) \), for every \( R \) and \( \lambda \). We answer the question in the negative. To do this we need the following construction.

3.4.1. Idealization of a bimodule

Let \( S \) be an associative ring with unit and \( M \) an \((S, S)\)-bimodule. Define operations + and \( \cdot \) on the set \( M + S \) of formal sums \( m + s \) \((m \in M, s \in S)\) as follows:

\[
(m + s) + (m' + s') = (m + m') + (s + s')
\]

\[
(m + s) \cdot (m' + s') = (sm' + s'm) + ss'.
\]

It is easy to see that \( M + S \) is an associative ring with unit. Moreover, \( M \) is its ideal with trivial multiplication, \( S \) is its subring and \( M + S \) is a semidirect product of them.

3.4.2. Indecomposability of idealizations

**Proposition 3.13.** If \( S \) is indecomposable into a direct product, \( M + S \) also is.

**Proof.** Let \( m + s \) be a central idempotent in \( M + S \). Then \( ms + sm = m \) and \( s \) is a central idempotent in \( S \). As \( S \) is indecomposable, \( s \) is 0 or 1. In both the cases \( ms + sm = m \) implies \( m = 0 \); so \( m + s \) is 0 or 1.

3.4.3. Rings elementarily equivalent to idealizations

**Proposition 3.14.** Let \( S \) be finite and assume

(i) there is \( m_0 \in M \) such that \( sm_0 = m_0 s = 0 \) implies \( s = 0 \), for every \( s \in S \),

(ii) for every non-zero \( s \in S \), \( \text{Ann}_M(s) \) is of infinite index in \( M \).

Then the rings elementarily equivalent to \( M + S \) are exactly rings of the form \( M' + S \), where \( M' \equiv M \).

**Example.** An example of an \((S, S)\)-bimodule satisfying (i) and (ii) is \( M = S^{(\lambda)} \), for any infinite \( \lambda \). Indeed, one can take as \( m_0 \) in (i) any function \( f \in S^{(\lambda)} \) with \( f(\alpha) = 1 \), for some \( \alpha < \lambda \). The condition (ii) holds, due to \( 1 \notin \text{Ann}_S(s) \), for \( s \neq 0 \), and \( \text{Ann}_M(s) = \text{Ann}_S(s)^{(\lambda)} \).

**Proof.** It is easy to see that, for a finite \( S \), the ring \( M + S \) can be interpreted in \( M \) without parameters (uniformly in \( M \)). Therefore, \( M \equiv M' \) implies \( M + S \equiv M' + S \).

Let \( |S| = n \). Then \( M \) is an ideal of index \( n \) in \( M + S \). By (i), \( \text{Ann}_{M + S}(m_0) = M \).
We show that, for every \( a \in M + S \), the following three conditions are equivalent:

1. \( a \in M \),
2. \( M \subseteq \text{Ann}_{M + S}(a) \),
3. \( \text{Ann}_{M + S}(a) \) is of finite index in \( M + S \).

The implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are obvious; we show (3) \( \Rightarrow \) (1). Let \( a = m + s \). We have

\[
|M : \text{Ann}_M(s)| = |M : M \cap \text{Ann}_{M + S}(a)| \leq |M + S : \text{Ann}_{M + S}(a)| < \infty.
\]

Then, by (ii), \( s = 0 \), that is, \( a \in M \).

It follows that, for \( a \in M + S \), \( M = \text{Ann}_{M + S}(a) \) iff \( \text{Ann}_{M + S}(a) \) is of index \( n \) in \( M + S \). So \( M \) is 0-definable in \( M + S \) by a ring formula \( \rho(v) \) saying that there is \( u \) such that \( \text{Ann}(u) \) has the index \( n \) and \( v \in \text{Ann}(u) \).

Clearly, \( M + S \) satisfies the first order sentence \( \chi \) saying that, for every \( a \) and \( b \), if \( \text{Ann}(a) \) and \( \text{Ann}(b) \) are of index \( n \), then \( \text{Ann}(a) = \text{Ann}(b) \) and \( \text{Ann}(a)^2 = 0 \).

Note that, if \( S' \) is a complement of \( M \) in \( M + S \), then there is an isomorphism \( \pi : S \to S' \) such that \( sm = \pi(s)m \) and \( ms = m\pi(s) \), for every \( s \in S \). Indeed, every \( s \in S \) can be uniquely represented as \( m + s' \), for some \( m \in M \), \( s' \in S' \); put \( \pi(s) = s' \). It is easy to verify that \( \pi \) satisfies the conditions above.

Therefore \( M + S \) satisfies the first order sentence \( \psi \) saying that, for every \( a, r_1, r'_1, \ldots, r_n, r'_n \), if \( \text{Ann}(a) \) is an ideal of index \( n \) and \( \{r_1, \ldots, r_n\}, \{r'_1, \ldots, r'_n\} \) are subrings being complements of \( \text{Ann}(a) \), then, for some permutation \( \sigma \), the map \( r_i \mapsto r'_{\sigma(i)} \) is an isomorphism of these subrings and \( r_1x = r'_{\sigma(1)}x, xr_1 = xr'_{\sigma(1)} \), for every \( x \in \text{Ann}(a) \).

Let \( S = \{s_1, \ldots, s_n\} \). For every sentence \( \phi \) of the language of \((S, S)\)-bimodules such that \( M \) satisfies \( \phi \), the ring \( M + S \) satisfies the sentence \( \phi^* \) saying that there are \( a, r_1, \ldots, r_n \) such that (i) \( \text{Ann}(a) \) is an ideal of index \( n \), (ii) \( \{r_1, \ldots, r_n\} \) forms a subring \( R \) isomorphic to \( S \) under \( r_i \mapsto s_i \), (iii) the whole ring is a semidirect product of \( \text{Ann}(a) \) and \( R \), (iv) \( \text{Ann}(a) \) considered as an \((R, R)\)-bimodule satisfies \( \phi \) if \( s_i \) is interpreted as \( r_i \).

Suppose \( K = M + S \) and \( \phi \in \text{Th}(M) \). As \( \phi^* \) holds in \( K \), the ring \( K \) is a semidirect product of an ideal \( I \) of the form \( \text{Ann}(a) \) and a subring \( S' \) isomorphic to \( S \). Since \( \chi \) holds in \( K \) such an \( I \) is unique and \( I^2 = 0 \). As \( \phi^* \) holds in \( K \), there is a monomorphism \( \tau_\phi : S \to K \) such that \( \tau_\phi(S) \) is a complement of \( I \) in \( K \) and \( I \) satisfies \( \phi \) if we consider it as \((S, S)\)-bimodule interpreting \( s \) as \( \tau_\phi(s) \). Since \( K \) satisfies \( \psi \), one can choose \( \tau_\phi \) in such a way that \( \tau_\phi(S) = S' \).

Let \( \text{Th}(M) = \{\theta_0, \theta_1, \ldots\} \), \( \phi^i = \theta_0 \land \cdots \land \theta_i \). Since there are only finitely many isomorphisms from \( S \) onto \( S' \), there is a monomorphism \( \tau \) such that \( \tau_\phi = \tau \), for infinitely many \( i \). We can consider \( I \) as \((S, S)\)-bimodule, interpreting \( s \) as \( \tau(s) \). Denote this bimodule by \( M' \); clearly \( M' \equiv M \). Obviously, \( M' + S \simeq K \) under the isomorphism \( u + s \mapsto u + \tau(s) \), for \( u \in I \), \( s \in S \).

3.4.4 Categoricity of a bimodule

Proposition 3.15. Let \( S \) be a finite indecomposable ring with unit. Then, for every infinite \( \lambda \), the \((S, S)\)-bimodule \( S^{(2)} \) is totally categorical.

Proof. It is known [25] that, for any finite indecomposable right module \( M \), the module \( M^{(2)} \) is totally categorical. It is well known that \((R, S)\)-bimodules
can be naturally considered as right \((R^{\text{op}} \otimes S)\)-modules and vice versa. The indecomposability of the ring \(S\) means that \(S\) is indecomposable as an \((S, S)\)-bimodule or, equivalently, as a right \((S^{\text{op}} \otimes S)\)-module. So \(S^{(1)}\) is totally categorical as a right \((S^{\text{op}} \otimes S)\)-module, hence as an \((S, S)\)-bimodule. □

3.4.5. Categoricity of idealizations

**Proposition 3.16.** Let \(S\) be a finite indecomposable ring with unit, \(\lambda \geq \aleph_0\), \(K_{\lambda} = S^{(\lambda)} + S\). Then the ring \(K_{\lambda}\) is totally categorical and indecomposable; the rings elementarily equivalent to it are exactly the rings of the form \(K_{\mu}, \mu \geq \aleph_0\).

**Proof.** By 3.4.2, 3.4.3 and 3.4.4. □

**Proposition 3.17.** Let \(S\) and \(R\) be finite associative rings with unit, \(M\) an \((S, S)\)-bimodule, \(N\) an \((R, R)\)-bimodule and \(M + S = N + R\). Then \(S \simeq R\). In particular, \(K_{\lambda} \equiv K_{\mu}^{\text{op}}\) implies \(S \simeq S^{\text{op}}\).

**Proof.** The formula \(\rho\) from the proof of Proposition 3.14 defines the ideals \(M\) and \(N\) in the rings \(M + S\) and \(N + R\), respectively. Then \(S \simeq (M + S)/M \equiv (N + R)/N \simeq R\). As \(S\) and \(R\) are finite, \(R \simeq S\). □

3.4.6. A counterexample for Videla’s question

**Proposition 3.18.** Let \(S\) be a finite indecomposable ring with unit, which is not isomorphic to \(S^{\text{op}}\). Let \(K_{\lambda} = S^{(\lambda)} + S, R^{\lambda \mu} = K_{\lambda} \times K_{\mu}^{\text{op}}, R = R^{\aleph_0 \aleph_0}\). Then

1. the models of \(\text{Th}(R)\) are exactly \(R^{\lambda \mu}\), for \(\lambda, \mu \geq \aleph_0\);
2. \(R^{\lambda \mu} \simeq R^{\nu \rho}\) iff \(\lambda = \nu, \mu = \rho\);
3. \(\text{Th}(R)\) is \(\aleph_0\)-categorical and \(\aleph_0\)-stable;
4. \(I(\aleph_1, R) = 3\);
5. \(I(\aleph_1, \text{UT}_n(R)) = 2, \text{for every } n \geq 3\).

**Proof.** (1) follows from Proposition 3.16 and the following claim.

**Claim 3.19.** If \(A\) and \(B\) are indecomposable rings with unit, the models of \(\text{Th}(A \times B)\) are exactly the rings of the form \(A' \times B'\), where \(A' \equiv A\) and \(B' \equiv B\).

**Proof of Claim 3.19.** By the Feferman–Vaught theorem, if \(A' \equiv A\) and \(B' \equiv B\), then \(A' \times B' \equiv A \times B\). Suppose \(C \equiv A \times B\). Let \(\text{Th}(A) = \{\phi_0, \phi_1, \ldots\}\), \(\text{Th}(B) = \{\psi_0, \psi_1, \ldots\}\). For every \(n\) the ring \(A \times B\) satisfies the first order sentence saying that there is a non-trivial central idempotent \(e_n\) such that \(e_n\) generates an ideal satisfying \(\phi_0 \wedge \cdots \wedge \phi_n\) and \(1 - e_n\) generates an ideal satisfying \(\psi_0 \wedge \cdots \wedge \psi_n\). As \(A\) and \(B\) are indecomposable rings, \(A \times B\) has only two non-trivial central idempotents; so the same is true for \(C\). Therefore there is a non-trivial central idempotent
Proof of Proposition 3.18 (continued). (2) As $K_\lambda$ and $K_\mu^{\text{op}}$ are indecomposable, an isomorphism $R^{K_\lambda} \simeq R^{K_\mu}$ induces isomorphisms $K_\lambda \simeq K_\mu$, $K_\mu^{\text{op}} \simeq K_\lambda^{\text{op}}$ or $K_\lambda \simeq K_\lambda^{\text{op}}$, $K_\mu^{\text{op}} \simeq K_\mu$. Since the latter is impossible, by Proposition 3.17, the result follows.

(3) holds because, as is well-known, the product of two totally categorical structures is always $\aleph_0$-categorical and $\aleph_0$-stable.

(4) follows from (1) and (2): the models of $\text{Th}(R)$ in $\mathcal{N}_1$ are exactly $R^{\mathcal{N},\mathcal{N}_1}$, $R^{\mathcal{N},\mathcal{N}_0}$, $R^{\mathcal{N},\mathcal{N}_1}$, and they are pairwise non-isomorphic.

(5) Since $R$ is $\aleph_0$-categorical, the models of $\text{Th}(\text{UT}_n(R))$ are exactly groups of the form $\text{UT}_n(R')$, $R' \equiv R$ (see Proposition 3.22), that is, by (1), $\text{UT}_n(R^{K_\lambda})$, $\lambda, \mu \geq \aleph_0$. In particular, the models in $\mathcal{N}_1$ are exactly $\text{UT}_n(R^{\mathcal{N},\mathcal{N}_1})$, $\text{UT}_n(R^{\mathcal{N},\mathcal{N}_0})$, $\text{UT}_n(R^{\mathcal{N},\mathcal{N}_1})$. The latter two groups are isomorphic because, due to Corollary 1.8,

$$\text{UT}_n(R^{\mathcal{N},\mathcal{N}_0}) \simeq \text{UT}_n(K_{\mathcal{N}_1}) \times \text{UT}_n(K_{\mathcal{N}_0}^{\text{op}}) \simeq \text{UT}_n(K_{\mathcal{N}_0}^{\text{op}}) \times \text{UT}_n(K_{\mathcal{N}_0}) \simeq \text{UT}_n(R^{\mathcal{N},\mathcal{N}_1}).$$

The first two groups are not isomorphic. Indeed, the rings $K_{\mathcal{N}_1}$ and $K_{\mathcal{N}_0}^{\text{op}}$ are saturated as uncountable models of an uncountably categorical theory. Therefore $R^{\mathcal{N},\mathcal{N}_1}$ and $\text{UT}_n(R^{\mathcal{N},\mathcal{N}_1})$ are also saturated. The ring $R^{\mathcal{N},\mathcal{N}_0}$ is of power $\aleph_1$ and has a countable definable ideal $K_{\mathcal{N}_0}^{\text{op}}$, so it is not saturated. As this ring is interpretable in the group $\text{UT}_n(R^{\mathcal{N},\mathcal{N}_1})$, the group is not saturated. So the theory of $\text{UT}_n(R)$ has exactly two models in $\mathcal{N}_1$. □

So we have the negative answer to Videla’s question:

**Theorem 3.20.** There exists an associative ring $R$ such that, for every $n \geq 3$, $I(\mathcal{N}_1, R) = 3$, but $I(\mathcal{N}_1, \text{UT}_n(R)) = 2$.

3.4.7. Open questions

**Problem.** Is there a ring $R$ with $I(\mathcal{N}_0, \text{UT}_n(R)) \neq I(\mathcal{N}_0, R)$?

$I(\mathcal{N}_0, \text{UT}_n(R)) > I(\mathcal{N}_0, R)$? $I(\mathcal{N}_0, \text{UT}_n(R)) < I(\mathcal{N}_0, R)$?

**Problem.** Is there a ring $R$ with $I(\lambda, R) < I(\lambda, \text{UT}_n(R))$, for some uncountable $\lambda$?

By Corollary 3.2, such a ring $R$ must be small, for the first of the problems and superstable, for the second one. Therefore, due to Proposition 3.7 the models of $\text{Th}(\text{UT}_n(R))$ are of the form $\text{UT}_n(S)$. In general, $S$ cannot be chosen elementarily equivalent to $R$, as Proposition 2.20 shows. This is why the existence of such rings does not seem improbable.

Note that if, for an uncountable $\lambda$, one of the cardinals $I(\lambda, R)$ and $I(\lambda, \text{UT}_n(R))$ is finite, then they are both finite and $I(\lambda, R) \geq I(\lambda, \text{UT}_n(R))$. Indeed, by Proposition
3.23, \( R \) is \( N_1 \)-categorical iff \( UT_n(R) \) is. Suppose they are not \( N_1 \)-categorical, but \( I(\lambda, R) \) or \( I(\lambda, UT_n(R)) \) is finite, for some uncountable \( \lambda \). By [14], \( R \) or \( UT_n(R) \) is \( N_0 \)-categorical and \( N_0 \)-stable. So, by Propositions 3.1 and 3.22, they are both \( N_0 \)-categorical and \( N_0 \)-stable. (So the construction of Proposition 3.18 is not accidental!) Then \( R^+ \) is bounded and so pure-injective; taking into account Proposition 2.1(2) and Corollary 1.4, we see that the models of \( Th(UT_n^*(R)) \) are exactly the expanded groups of the form \( UT_n^*(S) \), where \( S \equiv R \). Since \( S_1 \simeq S_2 \) iff \( UT_n^*(S_1) \simeq UT_n^*(S_2) \), for every infinite \( \lambda \) we have \( I(\lambda, R) = I(\lambda, UT_n^*(R)) \). It can be shown that, if \( T' \) is an inessential extension of a complete countable theory \( T \) by a finite number of constants, then, for any uncountable \( \lambda \), \( I(\lambda, T') \) is finite iff \( I(\lambda, T) \) is; moreover, \( I(\lambda, T) \leq I(\lambda, \lambda', T') \). So, for an uncountable \( \lambda \), \( I(\lambda, UT_n(R)) \leq I(\lambda, UT_n^*(R)) = I(\lambda, R) < N_0 \).

**Problem.** Is there a ring \( R \) such that, for some uncountable \( \lambda \), \( I(\lambda, UT_n(R)) \) and \( I(\lambda, R) \) are different and infinite?

**Conjecture 3.21.** For fixed \( \lambda \) and \( R \), \( I(\lambda, UT_n(R)) \) does not depend on \( n \).

3.5. Unitriangular groups over \( N_0 \)-categorical rings

**Proposition 3.22.** Let \( n > 3 \) and \( R \) be a ring with unit, associative if \( n > 3 \). Then \( R \) is \( N_0 \)-categorical iff \( UT_n(R) \) is. In this case the groups elementarily equivalent to \( UT_n(R) \) are exactly the groups of the form \( UT_n(S) \), where \( S \equiv R \).

**Proof.** The ring \( R \) and the group \( UT_n(R) \) are interpretable each in other, so the first statement holds by Ryll–Nardzewski. If \( UT_n(R) \) is \( N_0 \)-categorical, every \( H \equiv UT_n(R) \) is \( N_0 \)-saturated. Therefore, there is a pure base \( h \) in \( H \) such that \( S = Ring(H, h) \equiv R \). Due to the \( N_0 \)-categoricity, the group \( H \) is bounded, so its centre is pure-injective [13, 27.5]. Therefore \( h \) splits, and, by Section 1.7, \( H \simeq UT_n(S) \).

3.6. Unitriangular groups over \( N_1 \)-categorical rings

**Proposition 3.23.** Let \( n > 3 \) and \( R \) be a ring with unit, associative if \( n > 3 \). Then \( R \) is \( N_1 \)-categorical iff \( UT_n(R) \) is.

**Proof.** Suppose \( R \) is not \( N_1 \)-categorical. Then there is an uncountable non-saturated \( R' \equiv R \). As \( R' \) is interpretable in \( UT_n(R') \), by Section 1.7, this group is an uncountable non-saturated model of the theory of \( UT_n(R) \); hence this theory is not \( N_1 \)-categorical.

Suppose \( R \) is \( N_1 \)-categorical. To prove the \( N_1 \)-categoricity of \( UT_n(R) \) it suffices to show that the expanded group \( UT_n^*(R) \) is \( N_1 \)-categorical because, as is well known, an inessential extension of a theory is \( N_1 \)-categorical iff the theory itself is. Suppose \( (H, h) \equiv UT_n^*(R) \). Then \( h \) is a pure base in \( H \) and

\[ R' = Ring(H, h) \equiv Ring(UT_n^*(R)) \equiv R. \]
As \( R \) is \( \aleph_1 \)-categorical, \( R^+ \) is the direct sum of a bounded subgroup and a divisible subgroup, by Proposition 3.6. Therefore the same holds for \( Z(\text{UT}_n(R)) \) and hence for \( Z(H) \). So \( Z(H) \) is pure-injective and, by Proposition 1.6(1), the base \( \mathfrak{h} \) splits. Thus \( (H, \mathfrak{h}) \cong \text{UT}_n^*(R') \), by Corollary 1.4. Clearly, \( |H| = |R'| \). If \( |H| \) is uncountable, there is a unique \( R' \equiv R \) in the power \( |H| \); so \( (H, \mathfrak{h}) \) is uniquely determined by its cardinality, up to isomorphism. So \( \text{UT}_n^*(R) \) is \( \aleph_1 \)-categorical and we are done. \( \square \)

**Conjecture 3.24.** For an \( \aleph_1 \)-categorical \( R \), the groups elementarily equivalent to \( \text{UT}_n(R) \) are exactly the groups of the form \( \text{UT}_n(S) \), where \( S \equiv R \).

Since \( \text{UT}_n(S) \equiv \text{UT}_n(R) \), for \( S \equiv R \) and \( \text{Th}(\text{UT}_n(R)) \) is \( \aleph_1 \)-categorical, every uncountable model of this theory is isomorphic to \( \text{UT}_n(S) \), for some \( S \equiv R \). So the question is what are the countable models of the theory. By Proposition 3.7 they all are of the form \( \text{UT}_n(S) \); the problem is whether one can choose \( S \equiv R \). The question is non-trivial, as Proposition 2.20 shows. The following result sheds some light on the question.

**Proposition 3.25.** Suppose \( R^+ \) is the direct sum of a bounded subgroup and a divisible subgroup, and the theory of the ring \( R \) has a prime model \( R_0 \). Then the following are equivalent:

(i) the groups elementarily equivalent to \( \text{UT}_n(R) \) are exactly the groups of the form \( \text{UT}_n(S) \), where \( S \equiv R \);

(ii) \( \text{UT}_n(R_0) \) is a prime model;

(iii) the standard base \( t \) realizes a principal type in \( \text{UT}_n(R_0) \).

**Proof.** (i) \( \Rightarrow \) (ii). Since \( R_0 \) is elementarily embeddable into any \( S \equiv R \), the group \( \text{UT}_n(R_0) \) is elementarily embeddable into any \( \text{UT}_n(S) \), \( S \equiv R \).

(ii) \( \Rightarrow \) (iii). By Vaught's theorem, prime models are exactly countable atomic models.

(iii) \( \Rightarrow \) (i). Suppose \( H \equiv \text{UT}_n(R) \). As the type of \( t \) is principal, it has a realization \( \mathfrak{b} \) in \( H \). Then \( \mathfrak{b} \) is a pure base and \( S = \text{Ring}(H, \mathfrak{b}) \equiv \text{Ring}(\text{UT}_n^*(R_0)) \cong R_0 \equiv R \). As \( Z(H) \equiv R^+ \), the reduced part of \( Z(H) \) is bounded; therefore \( Z(H) \) is pure-injective. Hence the base \( \mathfrak{h} \) splits, by Proposition 1.6(1), and \( H \cong \text{UT}_n(S) \), by Corollary 1.4. \( \square \)

As an \( \aleph_1 \)-categorical ring \( R \) satisfies the conditions of Proposition 3.25, Conjecture 3.24 is equivalent to the following.

**Conjecture 3.26.** If a ring \( R_0 \) is a prime model of an \( \aleph_1 \)-categorical theory, the group \( \text{UT}_n(R_0) \) is also a prime model.

3.7. Morley towers

For any \( \aleph_1 \)-categorical but not \( \aleph_0 \)-categorical theory \( T \), there is an elementary chain of its countable models \( \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \) such that \( \mathfrak{A}_0 \) is a prime minimal model
and, for \( i < \omega \), \( \mathfrak{U}_{i+1} \) is a prime minimal extension of \( \mathfrak{U}_i \). This chain is said to be a Morley tower of \( T \). It is known (see [20, IX. 2.21]) that every countable non-saturated model of \( T \) is isomorphic to a unique \( \mathfrak{U}_i \).

**Proposition 3.27.** Suppose \( R_0 \) is an \( \aleph_1 \)-categorical but not \( \aleph_0 \)-categorical ring, and \( R_0 \) is a prime model. Then there is an \( \aleph_1 \)-categorical but not \( \aleph_0 \)-categorical theory of rings such that, for its Morley tower \( S_0 < S_1 < \cdots \), the chain of groups \( UT_n(S_0) < UT_n(S_1) < \cdots \) is a Morley tower of the corresponding theory of groups and \( UT_n(R_0) \cong UT_n(S_i) \), for some \( i < \omega \).

(Conjecture 3.26 just claims that one can take \( R_0 \) as \( S_0 \).

**Proof.** By Propositions 3.22 and 3.23, \( \text{Th}(UT_n(R_0)) \) is \( \aleph_1 \)-categorical but not \( \aleph_0 \)-categorical. Let \( H \) be its prime minimal model. By Proposition 3.7, \( H \cong UT_n(S_0) \) for some \( S_0 \). This \( S_0 \) is a minimal model, due to Proposition 3.28 below.

By Propositions 3.22 and 3.23, the ring \( S_0 \) is \( \aleph_1 \)-categorical but not \( \aleph_0 \)-categorical. Let \( S_0 < S_1 < \cdots \) be a Morley tower. Then, by Proposition 3.28, the chain of groups \( UT_n(S_0) < UT_n(S_1) < \cdots \) is a Morley tower of \( \text{Th}(UT_n(R_0)) \). The ring \( R_0 \) is not saturated being prime and non-\( \aleph_0 \)-categorical. So \( UT_n(R_0) \cong UT_n(S_i) \), for some \( i < \omega \). \( \square \)

### 3.8. Minimal extensions

In the proof of Proposition 3.27 we used the following claim, which is of some independent interest.

**Proposition 3.28.** Let \( R \) be a subring of \( S \). Then

1. \( UT_n(R) \) is a subgroup of \( UT_n(S) \); it is a proper subgroup iff \( R \) is a proper subring;
2. \( UT_n(R) \leq UT_n(S) \) iff \( R \leq S \);
3. the extension \( UT_n(R) < UT_n(S) \) is minimal iff the extension \( R < S \) is.

**Proof.** (1) is easy. In (2) the if part holds as \( UT_n(R) \) is uniformly interpretable in \( R \) without parameters. To prove the only if part suppose \( UT_n(R) < UT_n(S) \). Then \( UT_n(R) < UT_n(S) \) and therefore

\[
R' = \text{Ring}(UT_n(S)) \leq \text{Ring}(UT_n(S)) = S'.
\]

The isomorphism \( \omega \rightarrow t_\omega(\alpha) \) from \( S \) onto \( S' \) maps \( R \) onto \( R' \), so \( R \leq S \).

(3) The only if part is obvious: \( UT_n(R) < UT_n(P) < UT_n(S) \) if \( R < P < S \). To prove the if part suppose that \( UT_n(R) \leq H \leq UT_n(S) \) and the extension \( R < S \) is minimal. Then the standard base \( t \) in \( UT_n(R) \) is a base in \( H \) and

\[
R' = \text{Ring}(UT_n(R)) \leq \text{Ring}(H, t) \leq \text{Ring}(UT_n(S)) = S'.
\]
As the extension $R' < S'$ is isomorphic to $R < S$, it is minimal; so $\text{Ring}(H, t)$ is equal to $R'$ or $S'$, that is, $Z(H)$ is equal to $Z(\text{UT}_n(R))$ or $Z(\text{UT}_n(S))$. Then $H$ is equal to $\text{UT}_n(R)$ or $\text{UT}_n(S)$, by the following fact.

**Lemma 3.29.** Let $H$ be a subgroup of $G$ and $h$ a base in both $H$ and $G$. Then $Z(G) = Z(H)$ iff $G = H$.

**Proof of Lemma 3.29.** The group $H$ is generated by subgroups $H_{kl}$ and the group $G$ is generated by subgroups $G_{kl}$, $1 < k < l < n$. Suppose $Z(G) = Z(H)$. To prove $G = H$ it suffices to show that $H_{kl} = G_{kl}$, for all $k, l$. By Proposition 1.5, $H_{kl}$ is definable in $H$ and $G_{kl}$ is definable in $G$ by the same existential formula with parameters $h$. Therefore $H_{kl} \leq G_{kl}$. For every $k, l$ there is an epimorphism $\tau_{kl}: G_{kl} \to G_{1n}$ such that $\text{Ker}(\tau_{kl})$ is $G_{1n}$ for $l = k + 1$, trivial for $l \neq k + 1$, and $\tau_{kl}(H_{kl}) = H_{1n}$ (see Section 1.6). Let $g \in G_{kl}$. Then $\tau_{kl}(g) \in G_{1n} = Z(G) = Z(H) = H_{1n}$. Therefore $\tau_{kl}(g) = \tau_{kl}(h)$, for some $h \in H_{kl}$. If $l \neq k + 1$ then $g = h$. If $l = k + 1$ then $gh^{-1} \in \text{Ker}(\tau_{kl}) \leq G_{1n} = H_{1n} \leq H_{kl}$, so $g \in H_{kl}$. Thus $G_{kl} \leq H_{kl}$ and we are done. \(\square\)

As a corollary of the lemma we also have the following result.

**Proposition 3.30.** Let $R$ be a domain or a commutative associative ring with unit. If $R$ is a minimal model, $\text{UT}_n(R, q_1, \ldots, q_{n-1})$ is also a minimal model. In particular, $R$ is minimal iff $\text{UT}_n(R)$ is minimal.

**Proof.** Denote $\text{UT}_n(R, q_1, \ldots, q_{n-1})$ by $G$. Suppose $H \leq G$. Then there is a base $h$ in $H$ such that $P = \text{Ring}(H, h)$ is a commutative associative ring or a domain. Clearly, $P \leq \text{Ring}(G, h)$ and $G \simeq \text{UT}_n(S, q_1, \ldots, q_{n-1})$, for $S = \text{Ring}(G, h)$ and some $q_1, \ldots, q_{n-1}$. As $S$ and $R$ are commutative associative rings or domains, $S$ is isomorphic to $R$ or $R^{op}$, by Theorems 1.13 and 1.14. Since $R$ is minimal, $S$ is also minimal; so $P = S$. Therefore $Z(G) = Z(H)$, by Lemma 3.29, $G = H$. \(\square\)

**Question.** Is there $R$ such that $R$ is minimal but $\text{UT}_n(R)$ is not minimal?

**Corollary 3.31.** For commutative associative rings and domains Conjecture 3.26 is valid.

**Proof.** For $\aleph_1$-categorical but not $\aleph_0$-categorical theories the notion of minimal model and the one of prime model coincide. Another proof: in the case considered Conjecture 3.24 is valid, by Proposition 3.7. \(\square\)

3.9. When is the type of the standard base in $\text{UT}_n(R)$ principal?

This question is natural in connection with Proposition 3.25.

**Proposition 3.32.** The type of the standard base in $\text{UT}_n(R)$ is principal iff the groups elementarily equivalent to $\text{UT}_n(R)$ are exactly the groups of the form $\text{UT}_n(S, q_1, \ldots, q_{n-1})$, where $S \equiv R$ and the cocycles $q_1, \ldots, q_{n-1}$ are pure.
Proof. To prove the if part it suffices to show, by the Omitting Types Theorem, that the type of the standard base \( t \) can be realized in \( \text{UT}_n(S, q_1, \ldots, q_{n-1}) \), if \( S \equiv R \) and \( q_1, \ldots, q_{n-1} \) are pure. We show that \( \text{UT}_n^*(R) \equiv \text{UT}_n^*(S, q_1, \ldots, q_{n-1}) \). The argument in Proposition 2.8 shows that an \( \mathbb{N}_1 \)-saturated model of the theory of the latter expanded group has a form \( \text{UT}_n^*(S') \), for some \( S' \equiv S \). As \( \text{UT}_n^*(R) \equiv \text{UT}_n^*(S') \), the result follows.

We prove the only if part. Let the type of \( t \) in \( \text{UT}_n(R) \) is principal. The type contains an infinite collection of formulas saying that the base is pure and the corresponding ring satisfies \( \text{Th}(R) \). Since the type is principal, it can be realized in every \( H \equiv \text{UT}_n(R) \); so, by Theorem 1.2, \( H \) has a form \( \text{UT}_n(S, q_1, \ldots, q_{n-1}) \), where \( S \equiv R \) and \( q_1, \ldots, q_{n-1} \) are pure. \( \square \)

Remark 3.33. The existence of locally pure non-pure groups [5] and Proposition 2.20 show that the standard base can realize a non-principal type. On the other hand, by Propositions 2.10 and 2.13, if \( R \) is a commutative associative ring or a domain and \( R^+ \) is the direct sum of a bounded group and a divisible group or torsion-free, then the standard base realizes a principal type.

In Conjectures 3.24 and 3.26 the assumption of \( \mathbb{N}_1 \)-categoricity is essential, as the following result shows.

Proposition 3.34. Let \( R \) be the ring \( \mathbb{Z} \times \mathbb{Q} \). Then

1. \( R \) is a prime minimal model,
2. \( \text{UT}_3(R) \) is a minimal but not prime model,
3. the type of the standard base in \( \text{UT}_3(R) \) is principal.

Proof. By [13, Section 5.1, Ex. 7, 52.2], there is a symmetric 2-cocycle \( f \) from \( R^+ \) to itself, which is not a coboundary. As \( R^+ \) is torsion-free, \( f \) is pure. Hence by Proposition 2.9, \( \text{UT}_3(R) \equiv \text{UT}_3(R, f, 0) \).

It is easy to see that every element in \( R \) is definable; hence \( R \) is a prime minimal model. By Proposition 3.30 \( \text{UT}_3(R) \) and \( \text{UT}_3(R, f, 0) \) are minimal models.

It follows that both of them are not prime. Indeed, otherwise \( \text{UT}_3(R) \approx \text{UT}_3(R, f, 0) \). The cocycle \( (x, y) \mapsto xy \) from \( R^+ \) to itself is a coboundary, as \( xy = q(x + y) - q(x) - q(y) \) in \( R \), for \( q(x) = x(x - 1)/2 \). Then, by [3, 5.9], the groups \( \text{UT}_3(R) \) and \( \text{UT}_3(R, f, 0) \) are naturally isomorphic, that is, by [3, 4.9], \( \text{UT}_3^*(R) \) and \( \text{UT}_3^*(R, f, 0) \) are isomorphic. Then, by [3, 4.10], \( f \) is a coboundary, a contradiction.

By Proposition 2.10, the groups elementarily equivalent to \( \text{UT}_3(R) \) are exactly the groups of the form \( \text{UT}_3(S, q_1, q_2) \), where \( S \equiv R \) and \( q_1, q_2 \) are pure. Then, by Proposition 3.32, the type of the standard base in \( \text{UT}_3(R) \) is principal. \( \square \)

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