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# Moment Lyapunov exponents of the stochastic parametrical Hill's equation

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ABSTRACT

The Lyapunov exponent and moment Lyapunov exponents of Hill's equation with frequency and damping coefficient fluctuated by white noise stochastic process are investigated. A perturbation approach is used to obtain explicit expressions for these exponents in the presence of small intensity noises. The results are applied to the study of the almost-sure and the moment stability of the stationary solutions of the thin simply supported beam subjected to axial compressions and time-varying damping which are small intensity stochastic excitations.



## 1. Introduction

Dynamic stability of elastic systems under two random excitations has been investigated by many authors. There are numerous engineering structures that are subjected under the action of such loadings. The dynamical stability of these engineering structures are governed in general by the stability of the trivial solution of the stochastic differential equation of the form

$$\ddot{q}(t) + 2[\varsigma + g(t)]\dot{q}(t) + [1 + f(t)]q(t) = 0, \tag{1}$$

where $g(t)$ and $f(t)$ are stochastic processes and $\zeta$ is the damping constant.

Kapitaniak (1986) studied the non-Markovian process defined by Hill's Eq. (1) with frequency and damping coefficient fluctuated by non-white noise stochastic process. The stability of the first and second-order moments of the solution process is given by the well-known condition of stability of the differential equation with constant coefficients. Kozin and Wu (1973) obtained numerically sufficient almost-sure asymptotic stability boundaries when only one of stochastic process $f(t)$ and $g(t)$ is present. Ariaratnam and Xie (1988) have shown the method of obtaining a sufficient almost-sure asymptotic condition for linear systems (1) with ergodic damping coefficient. The probabilistic property of the derivative process of the damping coefficient is taken into account. A sufficient condition for almost- sure asymptotic stability is derived and numerical results are presented for the case of Gaussian noise coefficient. Ariaratnam and Ly (1989) obtained optimal results when both $f(t)$ and $g(t)$ are present, by solving for the envelope of the boundaries. Regions of the almost-sure asymptotic stability are obtained for arbitrary ergodic processes as well as ergodic Gaussian processes. Pavlović et al. (2005) determined the sufficient conditions for the almost-sure asymptotic stability of some continuous system, when the damping coefficient and axial compression are ergodic random processes. In that case, the probabilistic property of the derivative process of the damping coefficient is taken into account. The problem is solved by means of the Lyapunov direct method. In that way, they were able to obtain much sharper results than those obtained by earlier investigators, who considered only the influence of the mean

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and the variance of the coefficients of the stochastic processes. The sample or almost-sure stability of the trivial solution of the system (1) is determined by the Lyapunov exponent, which characterizes the average rate of growth of the solutions of system (1) for $t$ large, defined as

$$\lambda_q = \lim_{t\to\infty} \frac{1}{t} \log \|\boldsymbol{q}(t; q_0, \dot{q}_0)\|, \tag{2}$$

where $\|\boldsymbol{q}(t; q_0, \dot{q}_0)\| = (\boldsymbol{q}^{\boldsymbol{T}}(t; q_0, \dot{q}_0) \cdot \boldsymbol{q}(t; q_0, \dot{q}_0))^{1/2}$ is the Euclidean vector norm. If the largest Lyapunov exponent is negative, the trivial solution of system (1) is stable with probability 1; otherwise it is almost surely unstable. In the study of stability of the $p$th moment of solutions of random dynamical systems, the exponential growth rate of $E[\|\boldsymbol{q}(t; q_0, \dot{q}_0)\|^p]$ is provided by the $p$th moment Lyapunov exponent defined as

$$\Lambda_q(p) = \lim_{t\to\infty} \frac{1}{t} \log E[\|\boldsymbol{q}(t; q_0, \dot{q}_0)\|^p], \tag{3}$$

where $\boldsymbol{q}(t; q_0, \dot{q}_0) = \{q(t; q_0, \dot{q}_0), \dot{q}(t; q_0, \dot{q}_0)\}^T$ is the vector of states of the random dynamical system (1). If $\Lambda_q(p) < 0$, then by definition $E[\|\boldsymbol{q}(t; q_0, \dot{q}_0)\|^p] \to 0$ as $t \to \infty$ and this is referred to as $p$th moment stability. To have a complete picture of the dynamic stability of a stochastic system, it is important to study both the sample and moment stability and to determine both the Lyapunov exponents and the moment Lyapunov exponents. In despite of the importance of the moment Lyapunov exponents, publications are limited due to the difficulties in their actual determination. Moreover, almost all research of the moment Lyapunov exponents concerns the determination of approximate results of a single oscillator or two coupled oscillators under weak-noise excitations using perturbation methods. The connection between moment and almost sure stability of the trivial solution of a linear Itô equation was investigated by Kozin and Sugimoto (1977). They showed that the almost-sure stability region in some parameter space is the limit of the regions of the $p$th moment stability as $p\downarrow 0$. This implies that the sample stability criteria will include samples that are stable in some $p$th moment no matter how small $p$ may be. Arnold et al. (1977) constructed an approximation for the moment Lyapunov exponent, asymptotic growth rate of the moments of the response of a two-dimensional linear system under both real noise and white noise excitation. A perturbation approach is used to obtain explicit expressions for these exponents in the presence of small intensity noise. Khasminskii and Moshchuk (1998) obtained an asymptotic expansion of the moment Lyapunov exponent of a two-dimensional system under white noise parametric excitation in terms of the small fluctuation parameter. Sri Namachchivaya and Vedula (2000) obtained general asymptotic approximation for the moment Lyapunov exponent and the Lyapunov exponent for a four-dimensional system with one critical mode and another asymptotically stable mode driven by a small intensity stochastic process. These results, pertaining to $p$th moment stability and almost-sure stability, explain how the stochastic components that couple the stable and the critical modes play an important role in determining whether a noisy excitation can stabilize or destabilize the oscillatory critical mode. Xie (2001, 2003) applied a procedure similar to that employed in Khasminskii and Moshchuk (1998) to obtain weak noise expansions of the moment Lyapunov exponent, the Lyapunov exponent, and the stability index, in terms of the small fluctuation parameter, of two dimensional system under real noise excitation and under bounded noise excitation. In this paper, a procedure employed in Khasminskii and Moshchuk (1998) is applied to obtain an asymptotic expansion of the moment Lyapunov exponent and Lyapunov exponent of system (1) under two white-noise parametric excitation in terms of the small fluctuation parameter. These results are used to obtain explicit expressions an asymptotic expansion of the moment and almost sure stability boundaries of the simply supported beam which is subjected to the axial compressions and varying damping which are two random processes.

## 2. Discretisation of the equation of the motion

We present now an example which gives the best illustration of the theoretical results. In this sense, consider the elastic beam subjected to stochastically fluctuating axial compressions and damping force. It is assumed that the boundaries are simply supported. The motion of the beam governed by the partial differential equation, considered by Pavlović et al. (2005), is given by

$$L(w) = \frac{\partial^2 w}{\partial t^2} + 2[\varsigma + \sqrt{\varepsilon} g(t)]\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial z^4} + [f_0 + \sqrt{\varepsilon} f(t)]\frac{\partial^2 w}{\partial z^2} = 0, \tag{4}$$

with the following homogeneous boundary conditions

$$\left.\begin{matrix} z = 0 \\ z = 1 \end{matrix}\right\}, \quad w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0. \tag{5}$$

The quantities $\varsigma$ and $f_0$ in Eq. (4), are positive constants, functions $g(t)$ and $f(t)$ are Gaussian white noise processes with zero mean and autocorrelation functions

$$\begin{aligned} R_{gg}(t_1, t_2) &= E[g(t_1)g(t_2)] = 2S_g\delta(t_2 - t_1), \\ R_{ff}(t_1, t_2) &= E[f(t_1)f(t_2)] = 2S_f\delta(t_2 - t_1), \end{aligned} \tag{6}$$

where $2S_g$ and $2S_f$ are spectral densities, $\delta()$ is the Dirac delta function and $E[]$ denotes expectation. In order to further simplify Eq. (4), a mode of the Galerkin method will be used for reducing Eq. (4) to a corresponding ordinary differential equation representing only the time varying part of the solution. Consider the shape function $\sin\pi z$, which satisfies the boundary conditions (5). For the first mode of the transverse motion of the beam can be described by

$$w(z,t) = q(t)\sin\pi z. \tag{7}$$

Furthermore, by Galerkin's method it is required that

$$\int_0^1 L(w)\delta w \mathrm{d}z = 0. \tag{8}$$

By substituting (7) into (8) and evaluating the integral as indicated, it follows that the given shape function will satisfy the following ordinary differential equation

$$\ddot{q}(t) + 2[\varsigma + \sqrt{\varepsilon}\, g(t)]\dot{q}(t) + [\pi^4 - \pi^2 f_0 - \pi^2\sqrt{\varepsilon}\, f(t)]q(t) = 0. \tag{9}$$

However, before formulating the eigenvalue problem, Eq. (9) can be simplified by removing the constant damping term, putting $q(t) = x(t)\exp(-\varsigma t)$ into Eq. (9), which results in

$$\ddot{x}(t) + 2\sqrt{\varepsilon}\, g(t)\dot{x}(t) + [\omega^2 - \pi^2\sqrt{\varepsilon}\, f(t) - 2\sqrt{\varepsilon}\varsigma g(t)]x(t) = 0. \tag{10}$$

where $\omega^2 = \pi^4 - \pi^2 f_0 - \varsigma^2$.

From the definitions of the Lyapunov exponent (2) and the moment Lyapunov exponent (3), it can be easily shown that the Lyapunov exponents and the moment Lyapunov exponents of the systems (9) and (10) are related as follows

$$\begin{aligned} \lambda_{q(t)} &= -\varsigma + \lambda_{x(t)}, \\ \Lambda_{q(t)} &= -\varsigma p + \Lambda_{x(t)}(p). \end{aligned} \tag{11}$$

## 3. Weak noise expansion of the moment Lyapunov exponent

Using the transformation $X_1 = \frac{1}{\omega}x(t), X_2 = \frac{1}{\omega^2}\dot{x}(t)$, the Eq. (10) may be written in the form of the state equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/\omega & 0 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} w_1(t) + \begin{bmatrix} 0 & 0 \\ \varsigma/\omega & -1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} w_2(t), \tag{12}$$

where are $w_1(t) = \pi^2\sqrt{\varepsilon}f(t)$ and $w_2(t) = 2\sqrt{\varepsilon}g(t)$.

The auto-correlation functions of the processes $w_1(t)$ and $w_2(t)$ are

$$\begin{aligned} R_{w_1 w_1}(t_1,t_2) &= E[w_1(t_1)w_1(t_2)] = \varepsilon\pi^4 S_f\delta(t_2 - t_1) = \varepsilon\beta\delta(t_2 - t_1) = \sigma_f^2\delta(t_2 - t_1), \\ R_{w_2 w_2}(t_1,t_2) &= E[w_2(t_1)w_2(t_2)] = 4\varepsilon S_g\delta(t_2 - t_1) = \varepsilon\gamma\delta(t_2 - t_1) = \sigma_g^2\delta(t_2 - t_1). \end{aligned} \tag{13}$$

Since the excitations are Wiener processes, Eq. (12) can be represented in the first-order form by a set of the Stratonovich differential equations

$$\mathrm{d}\mathbf{X} = \omega\mathbf{A}_0\mathbf{X}\mathrm{d}t + \sqrt{\varepsilon}\mathbf{B}_1\mathbf{X}\circ \mathrm{d}w_1(t) + \sqrt{\varepsilon}\mathbf{B}_2\mathbf{X}\circ \mathrm{d}w_2(t), \tag{14}$$

where $\mathbf{A}_0$,$\mathbf{B}_1$ and $\mathbf{B}_2$ are $2\times 2$ matrices $\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{B}_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\beta}/\omega & 0 \end{pmatrix}, \mathbf{B}_2 = \sqrt{\gamma}\begin{pmatrix} 0 & 0 \\ \varsigma/\omega & -1 \end{pmatrix}, \mathbf{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $w_1(t)$ and $w_2(t)$ are the standard Wiener processes.

Apply the Khasminskii transformation,

$$X_1 = a\cos\varphi, \quad X_2 = a\sin\varphi, \quad \sqrt{X_1^2 + X_2^2} = \|a\|, \quad P = \|a\|^p, \quad -\infty \leqslant p \leqslant \infty, \tag{15}$$

and denote

$$\mathbf{s}(\varphi) = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}, \quad \bar{\mathbf{s}}(\varphi) = \begin{Bmatrix} \sin\varphi \\ -\cos\varphi \end{Bmatrix}, \quad \mathbf{s}^T(\varphi) = (\cos\varphi, \quad \sin\varphi), \quad \bar{\mathbf{s}}^T(\varphi) = (\sin\varphi, \quad -\cos\varphi),$$

then angular component can be represented in the Khasminskii form

$$\mathrm{d}\varphi = -\omega\,\mathrm{d}t - \sqrt{\varepsilon}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\circ \mathrm{d}w_r(t), \tag{16}$$

and for the $p$th power of radial part $\|a\|^p$ we get the following linear equation

$$\mathrm{d}\|a\|^p = p\sqrt{\varepsilon}\|a\|^p\sum_{r=1}^{2}(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\circ \mathrm{d}w_r(t). \tag{17}$$

The Itô version of these equations have the following form

$$\begin{aligned}
\mathrm{d}\varphi &= \left[-\omega + \frac{1}{2}\varepsilon\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s}) - \frac{1}{2}\varepsilon\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\right]\mathrm{d}t - \sqrt{\varepsilon}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\cdot \mathrm{d}w_r(t),\\
\mathrm{d}\|a\|^p &= \varepsilon p\|a\|^p\left[\frac{1}{2}p\sum_{r=1}^{2}(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\right]\mathrm{d}t + p\sqrt{\varepsilon}\|a\|^p\sum_{r=1}^{2}(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\cdot \mathrm{d}w_r(t).
\end{aligned}\tag{18}$$

Applying a linear transformation

$$S = T(\varphi)P, \quad P = \frac{S}{T(\varphi)}, \tag{19}$$

introducing the new norm process $S$ by means of the scalar function $T(\varphi)$ which is defined on the stationary phase process $\varphi$ in the range $-\pi/2 \leqslant \varphi \leqslant \pi/2$, the Itô equation for the new $p$th norm process $S$ can be obtained from Itô‘s Lemma:

$$\begin{aligned}
\mathrm{d}S = P\Bigg\langle &-\omega\frac{\mathrm{d}T(\varphi)}{\mathrm{d}\varphi} + \varepsilon\Bigg\{\frac{1}{2}\frac{\mathrm{d}^2T(\varphi)}{\mathrm{d}\varphi^2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{\mathrm{d}T(\varphi)}{\mathrm{d}\varphi}\left[\frac{1-2p}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s}) - \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\right]\\
&+ pT(\varphi)\left[\frac{1}{2}p\sum_{r=1}^{2}(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\right]\Bigg\}\Bigg\rangle\mathrm{d}t\\
&+\sqrt{\varepsilon}P\sum_{r=1}^{2}\left[-\frac{\mathrm{d}T(\varphi)}{\mathrm{d}\varphi}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s}) + pT(\varphi)(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\right]\cdot \mathrm{d}w_r(t).
\end{aligned}\tag{20}$$

If the transformation function $T(\varphi)$ is bounded and non-singular, both processes P and S possess the same stability behaviour. Therefore, transformation function $T(\varphi)$ is chosen so that the drift term, of the Itô differential Eq. (20), does not depend on the phase process $\boldsymbol{\varphi}$, so that

$$\mathrm{d}S = \Lambda(p)S\mathrm{d}t + \sqrt{\varepsilon}ST^{-1}(\varphi)\sum_{r=1}^{2}\left[-\frac{\mathrm{d}T(\varphi)}{\mathrm{d}\varphi}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s}) + pT(\varphi)(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})\right]\cdot \mathrm{d}w_r(t). \tag{21}$$

Comparing Eqs. (20) and (21), it is seen that such transformation function $T(\varphi)$ is given by the following equation:

$$[L_1 + \varepsilon L_2]T(\varphi) = \Lambda(p)T(\varphi). \tag{22}$$

Here $L_1$ and $L_2$ are the following first and second-order differential operators

$$L_1 = -\omega\frac{\mathrm{d}}{\mathrm{d}\varphi}, \quad L_2 = a(\varphi)\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + b(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi} + c(\varphi),$$

and $a(\varphi)$, $b(\varphi)$ and $c(\varphi)$ can be evaluated as follows:

$$a(\varphi) = \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 = \frac{1}{2}\left[\frac{\beta+\varsigma^2\gamma}{\omega^2}\cos^2\varphi - \frac{\varsigma\gamma}{\omega}\sin 2\varphi + \gamma\sin^2\varphi\right]\cos^2\varphi, \tag{23}$$

$$\begin{aligned}
b(\varphi) &= \frac{1-2p}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s}) - \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\\
&= (p-1)\left(\frac{\beta+\varsigma^2\gamma}{\omega^2}\cos^2\varphi - \frac{\varsigma\gamma}{\omega}\sin 2\varphi + \gamma\sin^2\varphi\right)\sin\varphi\cos\varphi - \frac{\varsigma\gamma}{2\omega}\cos^2\varphi + \frac{\gamma}{2}\sin\varphi\cos\varphi,
\end{aligned}\tag{24}$$

$$\begin{aligned}
c(\varphi) &= p\left[\frac{1}{2}p\sum_{r=1}^{2}(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})^2 + \frac{1}{2}\sum_{r=1}^{2}(\bar{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_r\mathbf{s})(\mathbf{s}^{\mathrm{T}}\mathbf{B}_r\bar{\mathbf{s}})\right]\\
&= p\left[\left(\frac{\beta+\varsigma^2\gamma}{\omega^2}\cos^2\varphi - \frac{\varsigma\gamma}{\omega}\sin 2\varphi + \gamma\sin^2\varphi\right)\left(\cos^2\varphi + \frac{p}{2}\sin^2\varphi\right) - \frac{\beta+\varsigma^2\gamma}{2\omega^2}\cos^2\varphi + \frac{\varsigma\gamma}{2\omega}\sin\varphi\cos\varphi\right].
\end{aligned}\tag{25}$$

Eq. (22) defines an eigenvalue problem in which $\Lambda(p)$ is the eigenvalue and $T(\varphi)$ is the associated eigenfunction. From (21), we see that the eigenvalue $\Lambda(p)$ is the Lyapunov exponent of the $p$th moment system (12). This approach was first applied by Wedig (1988) to derive the eigenvalue problem for the moment Lyapunov exponent of a two-dimensional linear Itô stochastic system.

Applying the method of regular perturbation, both the moment Lyapunov exponent $\Lambda(p)$ and the eigenfunction $T(\varphi)$ are expanded in power series of $\varepsilon$ as

$$\begin{aligned}
\Lambda(p) &= \Lambda_0(p) + \varepsilon\Lambda_1(p) + \varepsilon^2\Lambda_2(p) + \ldots + \varepsilon^n\Lambda_n(p) + \ldots,\\
T(\varphi) &= T_0(\varphi) + \varepsilon T_1(\varphi) + \varepsilon^2 T_2(\varphi) + \ldots + \varepsilon^n T_n(\varphi) + \ldots.
\end{aligned}\tag{26}$$

Substituting the perturbation series (26) into the eigenvalue problem (22) and equating terms of the equal powers of $\varepsilon$ leads to the following equations:

$$\begin{aligned}
&\varepsilon^0 : L_1T_0(\varphi) = \Lambda_0(p)T_0(\varphi),\\
&\varepsilon^1 : L_1T_1(\varphi) + L_2T_0(\varphi) = \Lambda_0(p)T_1(\varphi) + \Lambda_1(p)T_0(\varphi),\\
&\varepsilon^2 : L_1T_2(\varphi) + L_2T_1(\varphi) = \Lambda_0(p)T_2(\varphi) + \Lambda_1(p)T_1(\varphi) + \Lambda_2(p)T_0(\varphi),\\
&\varepsilon^3 : L_1T_3(\varphi) + L_2T_2(\varphi) = \Lambda_0(p)T_3(\varphi) + \Lambda_1(p)T_2(\varphi) + \Lambda_2(p)T_1(\varphi) + \Lambda_3(p)T_0(\varphi),\\
&\ldots\ \ldots\ \ldots\\
&\varepsilon^n : L_1T_n(\varphi) + L_2T_{n-1}(\varphi) = \Lambda_0(p)T_n(\varphi) + \Lambda_1(p)T_{n-1}(\varphi) + \Lambda_2(p)T_{n-2}(\varphi) + \ldots + \Lambda_n(p)T_0(\varphi),
\end{aligned} \tag{27}$$

where each function $T_i(\varphi), i = 0, 1, 2, \ldots$ must be positive and periodic in $\varphi$ of period $\pi$. The first equation in (27) has a periodic solution if and only if $\Lambda_0(p) = 0$ and $T_0(\varphi) = 1$.

### 3.1. First-order perturbation

The first-order perturbation equation is

$$-\omega\frac{dT_1(\varphi)}{d\varphi} + c(\varphi) = \Lambda_1(p). \tag{28}$$

The Eq. (28) can be easily solved to yield

$$T_1(\varphi) = \frac{1}{\omega}\int_0^{\varphi}[c(s) - \Lambda_1(p)]ds. \tag{29}$$

It is required that function $T_1(\varphi)$ is a periodic in $\varphi$, so we have

$$\Lambda_1(p) = \frac{1}{\pi}\int_0^{\pi} c(s)ds = p\left(\frac{\gamma}{8} + \frac{\beta + \varsigma^2\gamma}{8\omega^2}\right) + p^2\left(\frac{3\gamma}{16} + \frac{\beta + \varsigma^2\gamma}{16\omega^2}\right). \tag{30}$$

Substituting $\Lambda_1(p)$ into Eq. (29) we obtain function $T_1\ (\varphi)$

$$T_1(\varphi) = A_1 + B_1\sin 2\varphi + C_1\sin 4\varphi + D_1\cos 2\varphi + E_1\cos 4\varphi, \tag{31}$$

where

$$\begin{aligned}
&A_1 = -\frac{3\varsigma\gamma}{16\omega^2}p - \frac{3\varsigma\gamma}{32\omega^2}p^2, \quad B_1 = \frac{\beta + \varsigma^2\gamma}{8\omega^3}p - \frac{\gamma}{8\omega}p^2,\\
&C_1 = \left(-\frac{\gamma}{32\omega} + \frac{\beta + \varsigma^2\gamma}{32\omega^3}\right)p + \left(\frac{\gamma}{64\omega} - \frac{\beta + \varsigma^2\gamma}{64\omega^3}\right)p^2, \quad D_1 = \frac{\varsigma\gamma}{8\omega^2}p + \frac{\varsigma\gamma}{8\omega^2}p^2, \quad E_1 = \frac{\varsigma\gamma}{16\omega^2}p - \frac{\varsigma\gamma}{32\omega^2}p^2.
\end{aligned}$$

### 3.2. Second-order perturbation

The equation for the second-order perturbation is

$$-\omega\frac{dT_2(\varphi)}{d\varphi} + a(\varphi)\frac{d^2T_1(\varphi)}{d\varphi^2} + b(\varphi)\frac{dT_1(\varphi)}{d\varphi} + c(\varphi)T_1(\varphi) = \Lambda_1(p)T_1(\varphi) + \Lambda_2(p). \tag{32}$$

The Eq. (32) can be easily solved to yield

$$T_2(\varphi) = \frac{1}{\omega}\int_0^{\varphi}\left[a(s)\frac{d^2T_1(s)}{ds^2} + b(s)\frac{dT_1(s)}{ds} + c(s)T_1(s) - \Lambda_1(p)T_1(s) - \Lambda_2(p)\right]ds. \tag{33}$$

Analogously from the periodicity of $T_2(\varphi)$, we obtain

$$\begin{aligned}
\Lambda_2(p) &= \frac{1}{\pi}\int_0^{\pi}\left[a(s)\frac{d^2T_1(s)}{ds^2} + b(s)\frac{dT_1(s)}{ds} + c(s)T_1(s) - \Lambda_1(p)T_1(s)\right]ds\\
&= -p^2\left(\frac{\beta\gamma\varsigma}{16\omega^4} + \frac{\gamma^2\varsigma^3}{16\omega^4} + \frac{\gamma^2\varsigma}{16\omega^2}\right) - p^3\left(\frac{\beta\gamma\varsigma}{32\omega^4} + \frac{\gamma^2\varsigma^3}{32\omega^4} + \frac{\gamma^2\varsigma}{32\omega^2}\right).
\end{aligned} \tag{34}$$

Now, when we substitute value $\Lambda_2(p)$ into Eq. (33) we obtain function $T_2(\varphi)$ in the form of a partial sum of a Fourier series in $\varphi$

$$T_2(\varphi) = A_2 + B_2\sin 2\varphi + C_2\sin 4\varphi + D_2\sin 6\varphi + E_2\sin 8\varphi + F_2\cos 2\varphi + G_2\cos 4\varphi + H_2\cos 6\varphi + I_2\cos 8\varphi, \tag{35}$$

in which the values $A_2$, $B_2$, $C_2$, $D_2$, $E_2$, $F_2$, $G_2$, $H_2$ and $I_2$ are given in Appendix A.

### 3.3. Third-order perturbation

From Eq. (27), the equation for third-order perturbation is

$$-\omega\frac{\mathrm{d}T_3(\varphi)}{\mathrm{d}\varphi}+a(\varphi)\frac{\mathrm{d}^2T_2(\varphi)}{\mathrm{d}\varphi^2}+b(\varphi)\frac{\mathrm{d}T_2(\varphi)}{\mathrm{d}\varphi}+c(\varphi)T_2(\varphi)=\Lambda_1(p)T_2(\varphi)+\Lambda_2(p)T_1(\varphi)+\Lambda_3(p). \tag{36}$$

The solution of Eq. (36) is

$$T_3(\varphi)=\frac{1}{\omega}\int_0^{\varphi}\left[a(s)\frac{\mathrm{d}^2T_2(s)}{\mathrm{d}s^2}+b(s)\frac{\mathrm{d}T_2(s)}{\mathrm{d}s}+c(s)T_2(s)-\Lambda_1(p)T_2(s)-\Lambda_2(p)T_1(s)-\Lambda_3(p)\right]\mathrm{d}s. \tag{37}$$

Analogously from the periodicity of $T_3(\varphi)$, we obtain

$$\begin{aligned}\Lambda_3(p)&=\frac{1}{\pi}\int_0^{\pi}\left[a(s)\frac{\mathrm{d}^2T_2(s)}{\mathrm{d}s^2}+b(s)\frac{\mathrm{d}T_2(s)}{\mathrm{d}s}+c(s)T_2(s)-\Lambda_1(p)T_2(s)-\Lambda_2(p)T_1(s)\right]\mathrm{d}s\\
&=p\left(-\frac{15\beta^3}{512\omega^8}-\frac{45\beta^2\varsigma^2\gamma}{512\omega^8}-\frac{45\beta\varsigma^4\gamma^2}{512\omega^8}-\frac{15\varsigma^6\gamma^3}{512\omega^8}+\frac{9\beta^2\gamma}{512\omega^6}-\frac{9\beta\gamma^2\varsigma^2}{256\omega^6}-\frac{27\gamma^3\varsigma^4}{512\omega^6}-\frac{\beta\gamma^2}{512\omega^4}-\frac{13\gamma^3\varsigma^2}{512\omega^4}-\frac{\gamma^3}{512\omega^2}\right)\\
&\quad+p^2\left(-\frac{19\beta^3}{2048\omega^8}-\frac{57\beta^2\varsigma^2\gamma}{2048\omega^8}-\frac{57\beta\varsigma^4\gamma^2}{2048\omega^8}-\frac{19\varsigma^6\gamma^3}{2048\omega^8}+\frac{5\beta^2\gamma}{2048\omega^6}-\frac{13\beta\gamma^2\varsigma^2}{1024\omega^6}-\frac{31\gamma^3\varsigma^4}{2048\omega^6}-\frac{\beta\gamma^2}{2048\omega^4}-\frac{13\gamma^3\varsigma^2}{2048\omega^4}-\frac{\gamma^3}{2048\omega^2}\right)\\
&\quad+p^3\left(\frac{11\beta^3}{2048\omega^8}+\frac{33\beta^2\varsigma^2\gamma}{2048\omega^8}+\frac{33\beta\varsigma^4\gamma^2}{2048\omega^8}+\frac{11\varsigma^6\gamma^3}{2048\omega^8}-\frac{13\beta^2\gamma}{2048\omega^6}+\frac{37\beta\gamma^2\varsigma^2}{1024\omega^6}+\frac{87\gamma^3\varsigma^4}{2048\omega^6}+\frac{17\beta\gamma^2}{2048\omega^4}+\frac{93\gamma^3\varsigma^2}{2048\omega^4}+\frac{17\gamma^3}{2048\omega^2}\right)\\
&\quad+p^4\left(\frac{11\beta^3}{8192\omega^8}+\frac{33\beta^2\varsigma^2\gamma}{8192\omega^8}+\frac{33\beta\varsigma^4\gamma^2}{8192\omega^8}+\frac{11\varsigma^6\gamma^3}{8192\omega^8}-\frac{13\beta^2\gamma}{8192\omega^6}+\frac{69\beta\gamma^2\varsigma^2}{4096\omega^6}+\frac{151\gamma^3\varsigma^4}{8192\omega^6}+\frac{33\beta\gamma^2}{8192\omega^4}+\frac{173\gamma^3\varsigma^2}{8192\omega^4}+\frac{33\gamma^3}{8192\omega^2}\right).\end{aligned} \tag{38}$$

Now, when we substitute $\Lambda_3(p)$ into Eq. (37) we obtain function $T_3(\varphi)$ in the form of a partial sum of a Fourier series in $\varphi$

$$\begin{aligned}T_3(\varphi)&=A_3+B_3\sin 2\varphi+C_3\sin 4\varphi+D_3\sin 6\varphi+E_3\sin 8\varphi+F_3\sin 10\varphi+G_3\sin 12\varphi+H_3\cos 2\varphi+I_3\cos 4\varphi\\
&\quad+J_3\cos 6\varphi+K_3\cos 8\varphi+L_3\cos 10\varphi+M_3\cos 12\varphi,\end{aligned} \tag{39}$$

in which the $A_3$, $B_3$, $C_3$, $D_3$, $E_3$, $F_3$, $G_3$, $H_3$, $I_3$, $J_3$, $K_3$ and $L_3$ are also obtained, but not presented here due to the limitation of space.

### 3.4. Moment Lyapunov exponent, Lyapunov exponent and stability conditions

The weak noise expansion of the moment Lyapunov exponent in third-order perturbation for system (10) is obtained as

$$\Lambda_{x(t)}(p)=\varepsilon\Lambda_1(p)+\varepsilon^2\Lambda_2(p)+\varepsilon^3\Lambda_3(p)+O(\varepsilon^4), \tag{40}$$

where $\Lambda_1(p)$,$\Lambda_2(p)$ and $\Lambda_3(p)$ are given by Eqs. (30), (34) and (38), respectively. The procedure of the regular perturbation presented in Sections 3.1–3.3 can be extended easily to higher-order terms and carried out using software *Mathematica 5.* However, the number of terms involved in higher-order expansions increases drastically, and the higher-order terms obtainable are limited by the computer systems. The Lyapunov exponent for system (10) can be obtained from Eq. (40) by using a property of the moment Lyapunov exponent

$$\lambda_{x(t)}=\frac{\mathrm{d}\Lambda_{x(t)}(p)}{\mathrm{d}p}\bigg|_{p=0}=\varepsilon\lambda_1+\varepsilon^2\lambda_2+\varepsilon^3\lambda_3+O(\varepsilon^4), \tag{41}$$

where

$$\lambda_1=\frac{\gamma}{8}+\frac{\beta+\varsigma^2\gamma}{8\omega^2},\quad \lambda_2=0,$$
$$\lambda_3=-\frac{15\beta^3}{512\omega^8}-\frac{45\beta^2\varsigma^2\gamma}{512\omega^8}-\frac{45\beta\varsigma^4\gamma^2}{512\omega^8}-\frac{15\varsigma^6\gamma^3}{512\omega^8}+\frac{9\beta^2\gamma}{512\omega^6}-\frac{9\beta\gamma^2\varsigma^2}{256\omega^6}-\frac{27\gamma^3\varsigma^4}{512\omega^6}-\frac{\beta\gamma^2}{512\omega^4}-\frac{13\gamma^3\varsigma^2}{512\omega^4}-\frac{\gamma^3}{512\omega^2}.$$

Having obtained an approximation of the first-order perturbation $\Lambda_1(p)$, the moment Lyapunov exponent of the system (9) is given by

$$\Lambda_{q(t)}=-\varsigma p+\Lambda_{x(t)}(p)=-\varsigma p+\varepsilon\Lambda_1(p)+O(\varepsilon^2)=-\varsigma p+\varepsilon p\left(\frac{\gamma}{8}+\frac{\beta+\varsigma^2\gamma}{8\omega^2}\right)+\varepsilon p^2\left(\frac{3\gamma}{16}+\frac{\beta+\varsigma^2\gamma}{16\omega^2}\right)+O(\varepsilon^2), \tag{42}$$

and the Lyapunov exponent is

$$\lambda_{q(t)} = \left.\frac{\mathrm{d}\Lambda_{q(t)}(p)}{\mathrm{d}p}\right|_{p=0} = -\varsigma + \lambda_{x(t)} = -\varsigma + \varepsilon\lambda_1 + O(\varepsilon^2) = -\varsigma + \varepsilon\left(\frac{\gamma}{8} + \frac{\beta + \varsigma^2\gamma}{8\omega^2}\right) + O(\varepsilon^2). \tag{43}$$

Using the above result for the moment Lyapunov exponent, relation (13), with the definition of the moment stability $\Lambda_{q(t)} < 0$, we determine analytically the $p$th moment stability boundary in the first-order perturbation for various values of, $p = 1, 2, 4$, respectively,

$$\begin{aligned}
\sigma_f &< \sqrt{\frac{16}{3}\omega^2\varsigma - \frac{1}{3}(5\omega^2 + 3\varsigma^2)\sigma_g^2},\\
\sigma_f &< \sqrt{4\omega^2\varsigma - (2\omega^2 + \varsigma^2)\sigma_g^2},\\
\sigma_f &< \sqrt{\frac{8}{3}\omega^2\varsigma - \frac{1}{3}(7\omega^2 + 3\varsigma^2)\sigma_g^2}.
\end{aligned} \tag{44}$$

It is known that the system is asymptotically stable only if the Lyapunov exponent $\lambda_{q(t)}$ is negative. Then, expression (43) is employed to determine almost-sure stability boundary of the system (9)

$$\sigma_f < \sqrt{8\omega^2\varsigma - (\omega^2 + \varsigma^2)\sigma_g^2}. \tag{45}$$

The second-order perturbation of the moment Lyapunov exponent and Lyapunov exponent are given by the following expressions,

$$\begin{aligned}
\Lambda_{q(t)} &= -\varsigma p + \Lambda_{x(t)}(p) = -\varsigma p + \varepsilon\Lambda_1(p) + \varepsilon^2\Lambda_2(p) + O(\varepsilon^3) = -\varsigma p + \varepsilon p\left(\frac{\gamma}{8} + \frac{\beta + \varsigma^2\gamma}{8\omega^2}\right) + \varepsilon p^2\left(\frac{3\gamma}{16} + \frac{\beta + \varsigma^2\gamma}{16\omega^2}\right)\\
&\quad - \varepsilon^2 p^2\left(\frac{\beta\gamma\varsigma}{16\omega^4} + \frac{\gamma^2\varsigma^3}{16\omega^4} + \frac{\gamma^2\varsigma}{16\omega^2}\right) - \varepsilon^2 p^3\left(\frac{\beta\gamma\varsigma}{16\omega^4} + \frac{\gamma^2\varsigma^3}{16\omega^4} + \frac{\gamma^2\varsigma}{16\omega^2}\right) + O(\varepsilon^3),
\end{aligned} \tag{46}$$

$$\lambda_{q(t)} = \left.\frac{\mathrm{d}\Lambda_{q(t)}(p)}{\mathrm{d}p}\right|_{p=0} = -\varsigma + \lambda_{x(t)} = -\varsigma + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + O(\varepsilon^3) = -\varsigma + \varepsilon\left(\frac{\gamma}{8} + \frac{\beta + \varsigma^2\gamma}{8\omega^2}\right) + O(\varepsilon^3). \tag{47}$$

By the same procedure applied to Eqs. (46), (47) we determine the moment stability boundary in the second-order perturbation for various values of $p = 1, 2, 4$, respectively,

$$\begin{aligned}
\sigma_f &< \sqrt{\omega^2\frac{[2\omega^2(16\varsigma - 5\sigma_g^2) + 3\sigma_g^4\varsigma]}{3(2\omega^2 - \sigma_g^2\varsigma)} - \sigma_g^2\varsigma},\\
\sigma_f &< \sqrt{\omega^2\frac{[2\omega^2(2\varsigma - \sigma_g^2) + \sigma_g^4\varsigma]}{(\omega^2 - \sigma_g^2\varsigma)} - \sigma_g^2\varsigma},\\
\sigma_f &< \sqrt{\omega^2\frac{[\omega^2(8\varsigma - 7\sigma_g^2) + 6\sigma_g^4\varsigma]}{3(\omega^2 - 2\sigma_g^2\varsigma)} - \sigma_g^2\varsigma}.
\end{aligned} \tag{48}$$

Note that in the first and second-order perturbation of the Lyapunov exponent we obtain the same analytical expression for the almost-sure stability boundary. The same procedure is applied to Eqs. (40), (41) to determine the moment stability boundary for various values of $p = 1, 2, 4$ and almost-sure stability boundary in the third-order perturbation by the solution of the following equation

$$A\sigma_f^6 + B(\sigma_g)\sigma_f^4 + C(\sigma_g)\sigma_f^2 + D(\sigma_g) = 0, \tag{49}$$

in which the values $A$, $B(\sigma_g)$, $C(\sigma_g)$ and $D(\sigma_g)$ are given in Appendix B.

## 4. Numerical results and conclusions

In this paper, the moment Lyapunov exponents of the elastic simply supported beam under the both white noises parametric excitation are studied. The method of regular perturbation is applied to obtain a weak noise expansion of the moment Lyapunov exponent in terms of the small fluctuation parameter. The weak noise expansion of the Lyapunov exponent is also obtained. The slope of the moment Lyapunov exponent curve at $p = 0$ is the Lyapunov exponent.

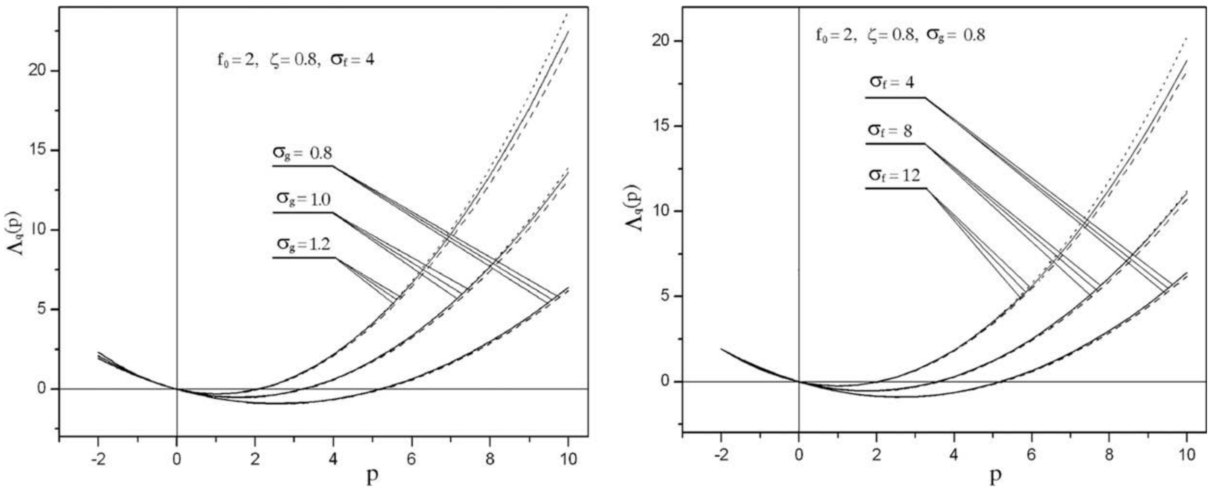


**Fig. 1.** Variation of the moment Lyapunov exponent, $\Lambda(p)$ with $p$. Solid line in the first, dashed line in the second and dot line in the third perturbation.

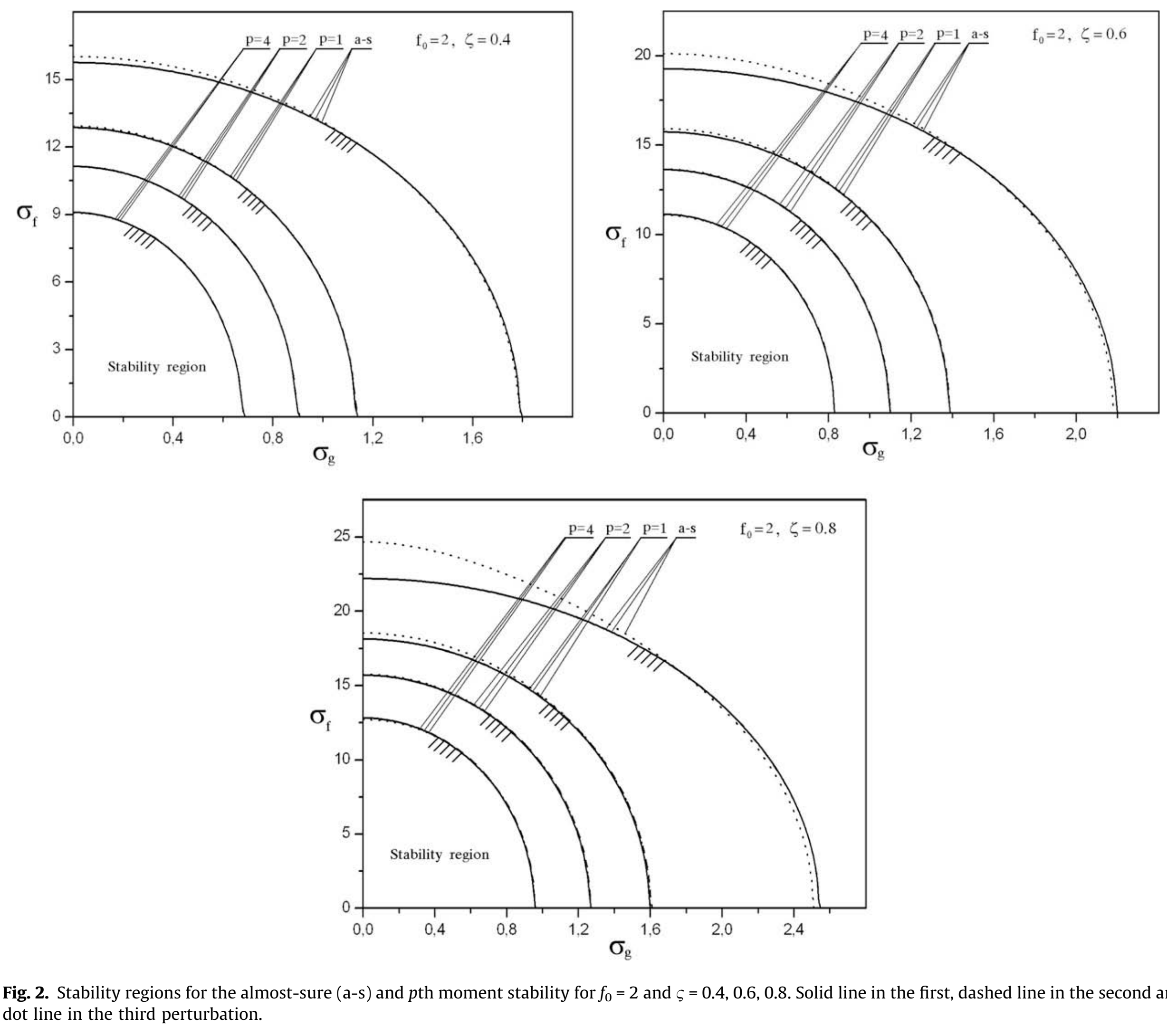


**Fig. 2.** Stability regions for the almost-sure (a-s) and $p$th moment stability for $f_0 = 2$ and $\varsigma = 0.4, 0.6, 0.8$. Solid line in the first, dashed line in the second and dot line in the third perturbation.

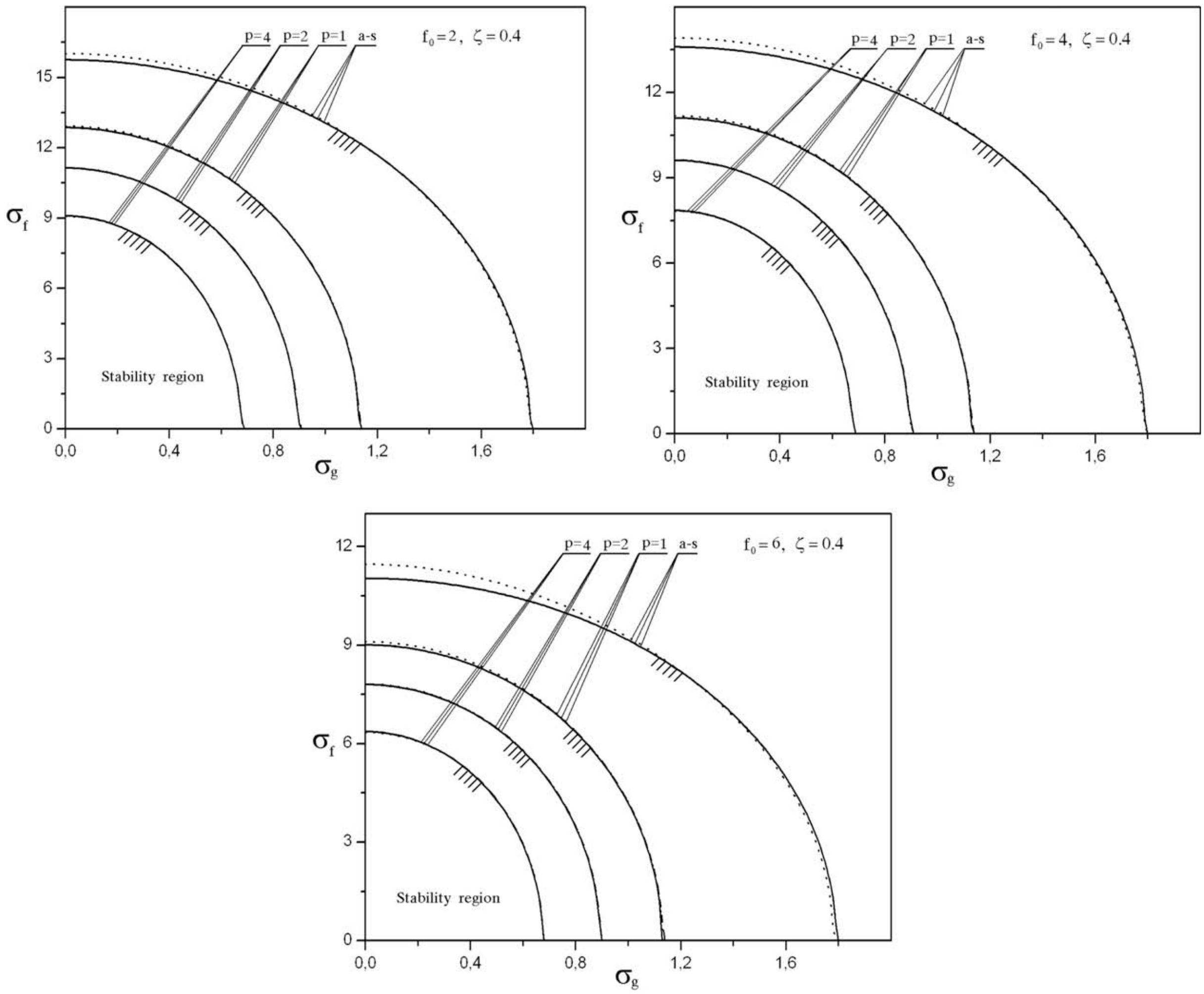


**Fig. 3.** Stability regions for almost-sure (a-s) and $p$th moment stability for $f_0 = 2, 4, 6$ and $\varsigma = 0.4$. Solid line in the first, dashed line in the second and dot line in the third perturbation.

When the Lyapunov exponent is negative, system (9) is stable with probability 1, otherwise it is unstable. For the purpose of illustration, in the numerical study we considered set system parameters $f_0 = 2, 4, 6$ and $\varsigma = 0.4, 0.6, 0.8$. Typical results of the moment Lyapunov exponent $\Lambda_{q(t)}(p)$ for system (9) are shown in Fig. 1, for $f_0 = 2$, $\varsigma = 0.8$ and various values of the noise intensity $\sigma_g = 0.8, 1.0, 1.2$, $\sigma_f = 4, 8, 12$. It is seen that, when the noise intensity $\sigma_g$ and $\sigma_f$ increases, the slope of the moment Lyapunov exponent curve of the origin decreases from positive to negative values. In addition we have shown some numerical results to illustrate the applicability of the analytical results of the stability boundaries for system (9). A comparison of these stability boundaries is presented in Figs. 2 and 3. The almost-sure and $p$th moment stability boundaries in the first, second and third-order perturbation for different values system parameters $f_0 = 2, 4, 6$ and $\varsigma = 0.4, 0.6, 0.8$ are shown in Figs. 2 and 3. By comparing these regions it is seen from numerical simulation that there are very small differences between stability boundaries in the first, second and third-order perturbation for different values of the system parameters. Also, note that the moment stability boundaries are more conservative than the almost-sure boundary. These boundaries become increasingly more conservative as $p$ increases. As expected, the increase of the value of $\varsigma$ has a stabilizing effect in the sense that the stability regions are increased, as shown in Fig. 2. On the contrary, the increase of the values of $f_0$ has a destabilizing effect in the sense that the stability regions are reduced as shown in Fig. 3.

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## Appendix A. Values of $A_2$, $B_2$, $C_2$, $D_2$, $E_2$, $F_2$, $G_2$, $H_2$ and $I_2$

$$\mathbf{A}_2 = p\left(-\frac{425\beta^2}{3072\omega^6}-\frac{425\beta\gamma\varsigma^2}{1536\omega^6}-\frac{425\gamma^2\varsigma^4}{3072\omega^6}+\frac{73\beta\gamma}{1536\omega^4}+\frac{37\gamma^2\varsigma^2}{512\omega^4}+\frac{5\gamma^2}{1024\omega^2}\right)+p^2\left(\frac{203\beta^2}{4096\omega^6}+\frac{203\beta\gamma\varsigma^2}{2048\omega^6}+\frac{203\gamma^2\varsigma^4}{3072\omega^6}+\frac{85\beta\gamma}{2048\omega^4}+\frac{319\gamma^2\varsigma^2}{2048\omega^4}+\frac{59\gamma^2}{4096\omega^2}\right)$$
$$+p^3\left(\frac{23\beta^2}{12288\omega^6}+\frac{23\beta\gamma\varsigma^2}{12288\omega^6}+\frac{23\gamma^2\varsigma^4}{12288\text{ømega}^6}-\frac{151\beta\gamma}{6144\omega^4}+\frac{73\gamma^2\varsigma^2}{2048\omega^4}-\frac{3\gamma^2}{4096\omega^2}\right)+p^4\left(\frac{\beta^2}{16384\omega^6}+\frac{\beta\gamma\varsigma^2}{8192\omega^6}+\frac{\gamma^2\varsigma^4}{16384\omega^6}-\frac{\beta\gamma}{8192\omega^4}+\frac{69\gamma^2\varsigma^2}{8192\omega^4}+\frac{65\gamma^2}{16384\omega^2}\right).$$

$$\mathbf{B}_2 = p\left(-\frac{9\beta\gamma\varsigma}{128\omega^5}-\frac{9\gamma^2\varsigma^3}{128\omega^5}-\frac{5\gamma^2\varsigma}{128\omega^3}\right)+p^2\left(-\frac{51\beta\gamma\varsigma}{512\omega^5}-\frac{51\gamma^2\varsigma^3}{512\omega^5}-\frac{7\gamma^2\varsigma}{512\omega^3}\right)+p^3\left(-\frac{9\beta\gamma\varsigma}{1024\omega^5}-\frac{9\gamma^2\varsigma^3}{1024\omega^5}+\frac{59\gamma^2\varsigma}{1024\omega^3}\right)+p^4\left(-\frac{\beta\gamma\varsigma}{1024\omega^5}-\frac{\gamma^2\varsigma^3}{1024\omega^5}+\frac{11\gamma^2\varsigma}{1024\omega^3}\right).$$

$$\mathbf{C}_2 = p\left(-\frac{3\beta\gamma\varsigma}{32\omega^5}-\frac{3\gamma^2\varsigma^3}{32\omega^5}-\frac{\gamma^2\varsigma}{64\omega^3}\right)+p^2\left(\frac{\beta\gamma\varsigma}{512\omega^5}+\frac{\gamma^2\varsigma^3}{512\omega^5}+\frac{27\gamma^2\varsigma}{512\omega^3}\right)+p^3\left(\frac{5\beta\gamma\varsigma}{256\omega^5}+\frac{5\gamma^2\varsigma^3}{256\omega^5}-\frac{\gamma^2\varsigma}{256\omega^3}\right)+p^4\left(\frac{3\beta\gamma\varsigma}{2048\omega^5}+\frac{3\gamma^2\varsigma^3}{2048\omega^5}-\frac{19\gamma^2\varsigma}{2048\omega^3}\right).$$

$$\mathbf{D}_2 = p\left(-\frac{7\beta\gamma\varsigma}{128\omega^5}-\frac{7\gamma^2\varsigma^3}{128\omega^5}+\frac{7\gamma^2\varsigma}{384\omega^3}\right)+p^2\left(\frac{17\beta\gamma\varsigma}{512\omega^5}+\frac{17\gamma^2\varsigma^3}{512\omega^5}+\frac{5\gamma^2\varsigma}{512\omega^3}\right)+p^3\left(-\frac{\beta\gamma\varsigma}{1024\omega^5}-\frac{\gamma^2\varsigma^3}{1024\omega^5}-\frac{47\gamma^2\varsigma}{3072\omega^3}\right)+p^4\left(-\frac{\beta\gamma\varsigma}{1024\omega^5}-\frac{\gamma^2\varsigma^3}{1024\omega^5}+\frac{3\gamma^2\varsigma}{1024\omega^3}\right).$$

$$\mathbf{E}_2 = p\left(-\frac{3\beta\gamma\varsigma}{256\omega^5}-\frac{3\gamma^2\varsigma^3}{256\omega^5}+\frac{3\gamma^2\varsigma}{256\omega^3}\right)+p^2\left(\frac{11\beta\gamma\varsigma}{1024\omega^5}+\frac{11\gamma^2\varsigma^3}{1024\omega^5}-\frac{11\gamma^2\varsigma}{1024\omega^3}\right)+p^3\left(-\frac{3\beta\gamma\varsigma}{1024\omega^5}-\frac{3\gamma^2\varsigma^3}{1024\omega^5}+\frac{3\gamma^2\varsigma}{3072\omega^3}\right)+p^4\left(\frac{\beta\gamma\varsigma}{4096\omega^5}+\frac{\gamma^2\varsigma^3}{4096\omega^5}-\frac{\gamma^2\varsigma}{4096\omega^3}\right).$$

$$\mathbf{F}_2 = p\left(\frac{9\beta^2}{128\omega^6}+\frac{9\beta\gamma\varsigma^2}{64\omega^6}+\frac{9\gamma^2\varsigma^4}{128\omega^6}-\frac{\beta\gamma}{128\omega^4}+\frac{5\gamma^2\varsigma^2}{128\omega^4}\right)+p^2\left(-\frac{5\beta^2}{512\omega^6}-\frac{5\beta\gamma\varsigma^2}{256\omega^6}-\frac{5\gamma^2\varsigma^4}{512\omega^6}-\frac{15\beta\gamma}{512\omega^4}-\frac{61\gamma^2\varsigma^2}{512\omega^4}-\frac{3\gamma^2}{128\omega^2}\right)$$
$$+p^3\left(-\frac{5\beta^2}{1024\omega^6}-\frac{5\beta\gamma\varsigma^2}{512\omega^6}-\frac{5\gamma^2\varsigma^4}{1024\omega^6}+\frac{7\beta\gamma}{1024\omega^4}-\frac{75\gamma^2\varsigma^2}{1024\omega^4}-\frac{\gamma^2}{512\omega^2}\right)+p^4\left(\frac{\beta\gamma}{1024\omega^4}-\frac{13\gamma^2\varsigma^2}{1024\omega^4}-\frac{\gamma^2}{1024\omega^2}\right).$$

$$\mathbf{G}_2 = p\left(\frac{3\beta^2}{64\omega^6}+\frac{3\beta\gamma\varsigma^2}{32\omega^6}+\frac{3\gamma^2\varsigma^4}{64\omega^6}-\frac{\beta\gamma}{64\omega^4}-\frac{5\gamma^2\varsigma^2}{128\omega^4}-\frac{\gamma^2}{128\omega^2}\right)+p^2\left(-\frac{3\beta^2}{128\omega^6}-\frac{3\beta\gamma\varsigma^2}{64\omega^6}-\frac{3\gamma^2\varsigma^4}{128\omega^6}-\frac{3\beta\gamma}{128\omega^4}-\frac{9\gamma^2\varsigma^2}{128\omega^4}+\frac{\gamma^2}{256\omega^2}\right)$$
$$+p^3\left(\frac{\beta\gamma}{64\omega^4}+\frac{\gamma^2\varsigma^2}{32\omega^4}+\frac{\gamma^2}{128\omega^2}\right)+p^4\left(\frac{7\gamma^2\varsigma^2}{1024\omega^4}-\frac{\gamma^2}{256\omega^2}\right).$$

$$\mathbf{H}_2 = p\left(\frac{7\beta^2}{384\omega^6}+\frac{7\beta\gamma\varsigma^2}{192\omega^6}+\frac{7\gamma^2\varsigma^4}{384\omega^6}-\frac{7\beta\gamma}{384\omega^4}-\frac{7\gamma^2\varsigma^2}{128\omega^4}\right)+p^2\left(-\frac{7\beta^2}{512\omega^6}-\frac{7\beta\gamma\varsigma^2}{256\omega^6}-\frac{7\gamma^2\varsigma^4}{512\omega^6}+\frac{3\beta\gamma}{512\omega^4}+\frac{9\gamma^2\varsigma^2}{512\omega^4}+\frac{\gamma^2}{128\omega^2}\right)$$
$$+p^3\left(\frac{7\beta^2}{3072\omega^6}+\frac{7\beta\gamma\varsigma^2}{1536\omega^6}+\frac{7\gamma^2\varsigma^4}{3072\omega^6}+\frac{11\beta\gamma}{3072\omega^4}+\frac{11\gamma^2\varsigma^2}{1024\omega^4}-\frac{3\gamma^2}{512\omega^2}\right)+p^4\left(-\frac{\beta\gamma}{1024\omega^4}-\frac{3\gamma^2\varsigma^2}{1024\omega^4}+\frac{\gamma^2}{1024\omega^2}\right).$$

$$\mathbf{I}_2 = p\left(\frac{3\beta^2}{1024\omega^6}+\frac{3\beta\gamma\varsigma^2}{512\omega^6}+\frac{3\gamma^2\varsigma^4}{1024\omega^6}-\frac{3\beta\gamma}{512\omega^4}-\frac{9\gamma^2\varsigma^2}{512\omega^4}+\frac{3\gamma^2}{128\omega^2}\right)+p^2\left(-\frac{11\beta^2}{4096\omega^6}-\frac{11\beta\gamma\varsigma^2}{2048\omega^6}-\frac{11\gamma^2\varsigma^4}{4096\omega^6}+\frac{11\beta\gamma}{2048\omega^4}+\frac{33\gamma^2\varsigma^2}{2048\omega^4}-\frac{11\gamma^2}{4096\omega^2}\right)$$
$$+p^3\left(\frac{3\beta^2}{4096\omega^6}+\frac{3\beta\gamma\varsigma^2}{2048\omega^6}+\frac{3\gamma^2\varsigma^4}{4096\omega^6}-\frac{3\beta\gamma}{2048\omega^4}-\frac{9\gamma^2\varsigma^2}{2048\omega^4}+\frac{3\gamma^2}{4096\omega^2}\right)+p^4\left(-\frac{\beta^2}{16384\omega^6}-\frac{\beta\gamma\varsigma^2}{8192\omega^6}-\frac{\gamma^2\varsigma^4}{16384\omega^6}+\frac{\beta\gamma}{8192\omega^4}+\frac{3\gamma^2\varsigma^2}{8192\omega^4}-\frac{\gamma^2}{16384\omega^2}\right).$$

## Appendix B. Values of $A$, $B$, $C$ and $D$

In Eq. (49) are used to determine moment stability boundary

$$p=1,\ A=-261,\quad B(\sigma_g)=9(11\omega^2-87\varsigma^2)\sigma_g^2,\quad C(\sigma_g)=(-783\varsigma^4+42\omega^2\varsigma^2+81\omega^4)\sigma_g^4-768\omega^4\varsigma\sigma_g^2+1536\omega^6,$$
$$D(\sigma_g)=(-261\varsigma^6-57\omega^2\varsigma^4+285\omega^4\varsigma^2+81\omega^6)\sigma_g^6-768\omega^4\varsigma(\omega^2+\varsigma^2)\sigma_g^4+512\omega^6(5\omega^2+3\varsigma^2)\sigma_g^2-8192\omega^8\varsigma,$$
$$p=2,\quad A=-1,\quad B(\sigma_g)=-(\omega^2+3\varsigma^2)\sigma_g^2,\quad C(\sigma_g)=(-3\varsigma^4+14\omega^2\varsigma^2+4\omega^4)\sigma_g^4-16\omega^4\varsigma\sigma_g^2+16\omega^6,$$
$$D(\sigma_g)=(-\varsigma^6+15\omega^2\varsigma^4+20\omega^4\varsigma^2+4\omega^6)\sigma_g^6-16\omega^4\varsigma(\omega^2+\varsigma^2)\sigma_g^4+16\omega^6(2\omega^2+\varsigma^2)\sigma_g^2-64\omega^8\varsigma,$$
$$p=4,\quad A=27,\quad B(\sigma_g)=-9(5\omega^2-9\varsigma^2)\sigma_g^2,\quad C(\sigma_g)=(81\varsigma^4+402\omega^2\varsigma^2+99\omega^4)\sigma_g^4-192\omega^4\varsigma\sigma_g^2+96\omega^6,$$
$$D(\sigma_g)=(27\varsigma^6+447\omega^2\varsigma^4+519\omega^4\varsigma^2+99\omega^6)\sigma_g^6-192\omega^4\varsigma(\omega^2+\varsigma^2)\sigma_g^4+32\omega^6(7\omega^2+3\varsigma^2)\sigma_g^2-256\omega^8\varsigma,$$

and almost-sure stability boundary in the third-order perturbation.

$$A=-15,\quad B(\sigma_g)=9(\omega^2-5\varsigma^2)\sigma_g^2,\quad C(\sigma_g)=-(45\varsigma^4+18\varsigma^2+\omega^4)\sigma_g^4+64\omega^6,$$
$$D(\sigma_g)=-(15\varsigma^6+27\omega^2\varsigma^4-13\omega^4\varsigma^2-\omega^6)\sigma_g^6+64\omega^4(\omega^2+\varsigma^2)\sigma_g^2-512\omega^8\varsigma.$$

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