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Categorical aspects are useful for topology—after 30 years

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Abstract

This is a survey paper presenting selected results on Embeddings of categories, Homeomorphisms of products and coproducts of spaces and Clones.

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Introduction

In 1976, at the IVth Prague Topological Symposium, M. Hušek and I delivered a joint talk "Categorical aspects are useful for topology". We were speaking about several themes where categorical aspects gave or at least inspired some topological results. In 1986, at the VIth Prague Topological Symposium, there was a continuation: "Categorical aspects are useful for topology—after 10 years". In my talk presented at Xth Prague Topological Symposium, "Categorical aspects are useful for topology—after 30 years", I tried to continue this tradition. I prepared to speak about three themes, namely

1. Embeddings of categories,
2. Homeomorphisms of products and coproducts of spaces,
3. Clones,

which were, more or less, influenced by categorical aspects. Because of time limitations, I was speaking only about themes 2 and 3.

The present paper is a survey of selected results from all three themes. Methods of the proofs are outlined only very seldom, but the references are always given so that the paper brings complete information about the results it describes. The concluding remarks outline a field of problems, possibly rich and promising, while single open problems appear within the text.

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Finally, let me mention that very little knowledge of category theory is expected from the reader: only the definitions of several basic notions.

1. Embeddings of categories

1.1. Older results on this topic are summarized in the monograph [16] by A. Pultr and myself. For the sake of completeness, I recall some facts from this book.

Definition. *A full embedding* F of a category $\mathcal{K}$ into a category $\mathcal{H}$ is a one-to-one functor of $\mathcal{K}$ onto a full subcategory of $\mathcal{H}$, i.e. for every pair a, b of objects of $\mathcal{K}$, F sends the set $\mathcal{K}(a, b)$ of all $\mathcal{K}$-morphisms with the domain a and the codomain b bijectively onto $\mathcal{H}(F(a), F(b))$.

1.2. Let me recall that a *concrete category* is a pair $(\mathcal{K}, U)$, where $\mathcal{K}$ is a category and $U : \mathcal{K} \to \mathrm{Set}$ is a faithful [i.e. one-to-one on every set $\mathcal{K}(a, b)$] functor of $\mathcal{K}$ into the category Set of all sets and all maps (sometimes called "forgetful functor"). For every object a of $\mathcal{K}$, $U(a)$ is called the *underlying set* of a.

A category $\mathcal{K}$ is called *concretizable* if there exists a functor $U : \mathcal{K} \to \mathrm{Set}$ such that $(\mathcal{K}, U)$ is a concrete category. Not all categories are concretizable—an example of non-concretizable category was given by J.R. Isbell in [6]. We say that a category $\mathcal{K}$ is *universal* iff every concretizable category can be fully embedded into $\mathcal{K}$.

1.3. Three basic topological results about full embeddings. The following three categories are universal:

Top_O, the category of all topological spaces and all their open continuous maps (see [16], p. 245),
$\mathrm{M\,Unif}_O$, the category of all metric spaces and all their open uniformly continuous maps (see [16], p. 232),
Top Smg, the category of all topological semigroups and all their continuous homomorphisms (see [40]).

1.4. Applications of the above results. For every universal category $\mathcal{K}$, there exists a congruence on it [i.e. a system of equivalences $\overset{a,b}{\sim}$ on every set $\mathcal{K}(a, b)$ such that they are preserved by the composition: $(c \xrightarrow{\gamma} a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b \xrightarrow{\delta} d)$ and $(\alpha \overset{a,b}{\sim} \beta)$ implies $(\delta\alpha\gamma \overset{c,d}{\sim} \delta\beta\gamma)$] such that the factor category $\mathcal{K}/{\sim}$ is *hyperuniversal* in the sense that it contains an isomorphic copy of *every* category (not only of a concretizable one) as a full subcategory. This follows immediately from the existence of a hyper-universal category $\mathcal{H}$ (in the Bernays–Gödel set-theory with the axiom of choice, its existence is proved in [26] where it is called universal) and the Kučera theorem (see [11]) stating that for every category $\mathcal{L}$ there exists a concretizable category $\mathcal{C}$ and a congruence $\sim$ on $\mathcal{C}$ such that $\mathcal{C}/{\sim}$ is isomorphic to $\mathcal{L}$. In fact, start from the hyperuniversal category $\mathcal{H}$, find a concretizable $\mathcal{C}$ and a congruence $\sim$ on $\mathcal{C}$ such that $\mathcal{C}/{\sim}$ is isomorphic to $\mathcal{H}$, by the Kučera theorem; then fully embed the concretizable $\mathcal{C}$ into a given universal category $\mathcal{K}$ and extend the congruence $\sim$ from the copy of $\mathcal{C}$ within $\mathcal{K}$ onto the whole $\mathcal{K}$ (we denote the extended congruence by $\sim$ again). Then $\mathcal{K}/\sim$ contains a copy of $\mathcal{H}$ as a full subcategory and hence it is hyperuniversal [41].

Thus each of the three categories Top_O, $\mathrm{M\,Unif}_O$ and Top Smg has a congruence on it such that the corresponding factor category is hyperuniversal.

Problem. Could such a congruence on Top_O or on $\mathrm{M\,Unif}_O$ or on Top Smg be described by means of the internal structure of objects of the category in question (such as, e.g., the homotopy classes of continuous maps or the conjugacy classes of homomorphisms)?

1.5. In topology, the basic choice of morphisms are all continuous maps. But the category Top of all topological spaces and all their continuous maps is *not* universal because of the constant maps, carefully avoided in Top_O and in $\mathrm{M\,Unif}_O$ by the restriction to open maps only, and in Top Smg by the restriction to homomorphisms. If one simply removes the constant maps, the resulting torso is generally not a category because the composite of two non-constant maps may well be constant. Hence the notion of full embedding has to be adapted to this situation.

Definition. A one-to-one functor $F : \mathcal{K} \to \mathrm{Top}$ of a category $\mathcal{K}$ into the category Top is called *almost full* if, for any $\mathcal{K}$-objects a, b, the functor F sends the set $\mathcal{K}(a, b)$ precisely onto the set of all non-constant continuous maps $F(a) \to F(b)$.

1.6. Three basic topological results about almost full embeddings. Every small category (i.e., such that all its morphisms form a set) can be almost fully embedded into

MTop, the category of all metric spaces and all their continuous maps ([27], see also [16], p. 218), and into
CompH, the category of all compact Hausdorff spaces and all their continuous maps ([29], see also [16], p. 257)

and every concretizable category can be almost fully embedded into

Par, the category of all paracompact spaces and all their continuous maps ([9], see also [16], p. 251).

1.7. An application of the above results. *Let the embedded category $\mathcal{K}$ be a discrete category*, i.e. all its morphisms are the unit morphisms only. Then its almost full image is a *rigid class of spaces*, i.e. it is a class $\mathcal{X}$ of spaces such that

(i) $\operatorname{card} X > 1$ for every $X \in \mathcal{X}$ [because the image of every unit morphism of $\mathcal{K}$ must be non-constant],
(ii) for every $X, Y \in \mathcal{X}$, $X \neq Y$, every continuous map $X \to Y$ is constant, and
(iii) for every $X \in \mathcal{X}$, every continuous map $X \to X$ is either the identity or constant.

By the basic results of 1.6, MTop and CompH contain arbitrarily large rigid sets of spaces and Par contains even a rigid proper class of spaces [because the category O of all one-element sets without non-identical morphisms is discrete and it is concretizable (because the inclusion functor $O \to \text{Set}$ is faithful) though O is not small].

Do MTop and CompH also contain a rigid proper class of spaces? The positive answer to this question is proved in [17] to be equivalent to the negation of the set-theoretical *Vopěnka Principle* VP. This principle is stated in the monograph [7] as follows:

for every finitary first order theory, every class of its models such that none of these models has an elementary embedding into another one, is a set.

Hence the categorical reasoning gives rather simple topological formulation of the negation of VP, namely the existence of a rigid proper class in CompH (or in MTop). In Par, the analogous statement is absolute.

1.8. Another application of the results in 1.6. Let the embedded category $\mathcal{K}$ have precisely one object, say a. Then the whole category $\mathcal{K}$ is in fact the monoid $\mathcal{K}(a, a)$. An almost full embedding of $\mathcal{K}$ into MTop and into CompH gives an essential strengthening of the classical result of de Groot [4] that

every group can be represented as the group of all homeomorphisms of a space onto itself.

This strengthening states that

every monoid can be represented as the monoid of all non-constant continuous selfmaps of a metric space and of a compact Hausdorff space.

Let me mention explicitly that all the non-constant continuous selfmaps of a space do not form a monoid in general because the composition of two such maps can be constant. But the above result states that for every monoid there exists a metric (and a compact Hausdorff) space such that its non-constant continuous selfmaps *are* closed with respect to the composition and they form a monoid isomorphic to the given one.

The reasoning gives a stronger result for countable monoids: then the space can be a compact metric space (see [29]).

1.9. Simultaneous representations are investigated in [37,38,48]. The constructions there allow representation of more monoids by a single space; for instance, for every monoid M_1 and its submonoid M_2,

- there is a Tychonoff space X such that M_2 is isomorphic to the monoid of all non-constant continuous selfmaps of X and M_1 of its β-compactification βX (see [38]) or, more generally,
- there exists a space X such that M_2 is represented by the non-constant continuous selfmaps of X and M_1 of its given reflection or its given coreflections whenever the reflection or the coreflection satisfy rather mild conditions (see [48]),

or every triple of monoids $M_1 \subseteq M_2 \subseteq M_3$ can be represented by a single metric space X such that its non-constant selfmaps which are non-expanding (i.e., distances-non-increasing) form a monoid isomorphic to M_1, uniformly continuous a monoid isomorphic to M_2 and continuous a monoid isomorphic to M_3 (see [37]).

1.10. As mentioned in 1.3, the category Top Smg of all topological semigroups is universal. Recently, V. Koubek started to investigate which varieties of topological algebras are universal. In [10], he characterized all the regular varieties V of unary algebras such that the corresponding category Top V is universal. For non-unary varieties of algebras, the analogous question remains open.

2. Homeomorphisms of products and coproducts of spaces

2.1. Let me start with the basic definition.

Definition. Let $\mathcal{K}$ be a category with finite products. A representation of a commutative semigroup $(S, \cdot)$ by products in $\mathcal{K}$ is any map

$$X : S \to \operatorname{obj} \mathcal{K}$$

of the underlying set S of the semigroup into the class obj $\mathcal{K}$ of all objects of $\mathcal{K}$ such that

(i) $X(s_1 \cdot s_2)$ is always isomorphic to $X(s_1) \times X(s_2)$ and
(ii) if $s \neq s'$, then $X(s)$ is not isomorphic to $X(s')$.

Hence, in fact, we create "a large semigroup" $\prod \mathcal{K}$: elements of $\prod \mathcal{K}$ are isomorphism types of objects of $\mathcal{K}$, the semigroup operation of $\prod \mathcal{K}$ is given by forming products in $\mathcal{K}$; the above representation of a semigroup $(S, \cdot)$ by products in $\mathcal{K}$ is then simply a semigroup embedding of $(S, \cdot)$ into $\prod \mathcal{K}$.

2.2. Isomorphism types of structures and operations on them were investigated by A. Tarski and B. Jónsson, see [22], but they were oriented to algebra and they investigated mainly the algebraic structures. The first topological result on this topic was given by W. Hanf, who constructed a Boolean algebra B isomorphic to $B^3 = B \times B \times B$ but not isomorphic to $B^2 = B \times B$ (see [5]). Hence, he constructed a representation of the cyclic group $c_2 = \{0, 1\}$ by products in the category Bool of all Boolean algebras. In fact, $1 + 1 + 1 = 1$ in c_2 precisely corresponds to $B \times B \times B \simeq B$ in Bool; and, since B is not isomorphic to B^2, putting

$$X(1) = B \quad \text{and} \quad X(0) = B^2$$

we get precisely the representation of c_2 in Bool in the sense of the above definition.

2.3. Generalizing and handling the Hanf's method in more involved way, J. Adámek, V. Koubek and myself obtained a representation of every Abelian group in Bool (see [1]). The main result about this topic was proved by J. Ketonen, who constructed a representation of every countable commutative semigroup by products of countable Boolean algebras (see [8]). He does not use the Hanf's method but his proof is built on a deep analysis of the internal structure of countable Boolean algebras. As a byproduct, he solved the long-open "Tarski cube problem" of whether there exists a *countable* Boolean algebra B isomorphic to B^3 but not to B^2.

Problem. Which commutative semigroups have representation by products in Bool?

2.4. By means of the Stone Duality, we see that all the above results about representations by products of Boolean algebras are results about representations by coproducts of topological spaces.

Let us investigate the dual problems, i.e. *representations by products of topological spaces*. The first space X homeomorphic to X^3 but not to X^2 was constructed in 1973, see [28], and this space was already rather nice: it was metric, separable and locally compact. However, the dual to the above Ketonen's result is no more true.

2.5. YES and NO results. The Stone spaces of countable Boolean algebras are precisely compact metric 0-dimensional spaces. And every compact metric 0-dimensional space X homeomorphic to X^3 is already homeomorphic to X^2 (see [32]) so that the cyclic group c_2 does not have a representation by products within this class of spaces. But this NO result is rather tightly surrounded by YES results. Any pair of the three properties

(comp. + metr. + 0-dim.)

suffices.

(i) *Omitting the zero-dimensionality.* A compact metric space X homeomorphic to X^3 but not to X^2 was constructed in [30]. Later on, A. Orsatti and N. Rodino improved the result (see [15]): they constructed a compact metric connected space X admitting the structure of a compact Abelian group, X homeomorphic to X^3 but not to X^2.

(ii) *Omitting the metrizability.* The first compact Hausdorff 0-dimensional space X homeomorphic to X^3 but not to X^2 was constructed by V. Koubek and myself in [49]; later on, I added the separability (see [35]) (quite distinct construction was created there; in the construction, the spaces, named *Seq* now, were used).

(iii) *Omitting the compactness.* The metric 0-dimensional space X homeomorphic to X^3 but not to X^2 was constructed by J. Vinárek in [50]; later on, I added the separability, see [36].

2.6. More general results. In all the above cases, more general results were obtained. In the compact cases (i) and (ii), a representation of any finite Abelian group by products of the spaces in question was constructed. And in (iii), J. Vinárek even constructed a representation of *every* commutative semigroup by products of 0-dimensional metric spaces; and every countable commutative semigroup was proved to have a representation by products of separable such spaces. Also the NO result is more general: no commutative semigroup having a non-trivial group part has a representation by compact metric 0-dimensional spaces.

2.7. Sets of the Cantor discontinuum. Compact metric 0-dimensional spaces are, up to homeomorphism, just closed subsets of the Cantor discontinuum $\mathbb{C}$ and we have the NO result for them. But the construction of representations of all countable commutative semigroups in [36] proceeds in $\mathbb{C}$ as well and all the representing spaces are always $F_{\sigma\delta}$-subsets of $\mathbb{C}$. Thus, there is still a gap between these NO and YES results.

Problem. Does there exist an F_σ-subset X of $\mathbb{C}$ homeomorphic to X^3 but not to X^2?

2.8. The countable spaces. The situation concerning the representability by products of countable spaces is in essence clarified: every countable commutative semigroup has a representation

- by products of countable paracompact spaces, see [33] (in the construction, the spaces, called *Seq* now, are used);
- by products of countable Hausdorff spaces with countable weight, see [34].

In the last result, Hausdorff spaces cannot be replaced by T_3-spaces, otherwise we get NO result: such spaces are already metrizable and a countable metric space X homeomorphic to X^3 is already homeomorphic to X^2, see [31].

2.9. Simultaneous representations by products. Let $\mathcal{K}$ and $\mathcal{H}$ be categories with finite products and let $\prod\mathcal{K}$ and $\prod\mathcal{H}$ be their "large semigroups" as in 2.1. If a functor $F:\mathcal{K}\to\mathcal{H}$ preserves finite products, it determines a semigroup homomorphism $\prod\mathcal{K}\to\prod\mathcal{H}$ in the evident way; let us denote it by $\prod F$.

Given a semigroup homomorphism $h:S\to S'$, its *simultaneous representation by products in* $F:\mathcal{K}\to\mathcal{H}$ consists of semigroup embeddings $g:S\to\prod\mathcal{K}$ and $g':S'\to\prod\mathcal{H}$ such that the square

$$\begin{array}{ccc} S & \xrightarrow{\;h\;} & S' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ \prod\mathcal{K} & \xrightarrow{\;\prod F\;} & \prod\mathcal{H} \end{array}$$

commutes, i.e. $g'\circ h=\prod F\circ g$.

In [39], every semigroup homomorphism $h:S\to S'$ is proved to have a simultaneous representation by products in the functor

$$\mathrm{Metr}\xrightarrow{\;c\;}\mathrm{Metr}^c,$$

where Metr denotes the category of all metric spaces and all their non-expanding maps, Metr^c is its subcategory of all complete spaces and c is the completion functor (see [39]).

More complex forms of simultaneous representations by products are introduced in [39] (the definition is evident) and every diagram

$$S_1 \xrightarrow{h_1} S_2 \xrightarrow{h_2} S_3$$

consisting of arbitrary commutative semigroups S_1, S_2, S_3 and their arbitrary homomorphisms h_1, h_2 is proved in [39] to have a simultaneous representation by products in the diagram

$$\mathrm{Metr} \xrightarrow{F_m} \mathrm{M\,Unif} \xrightarrow{F_u} \mathrm{M\,Top},$$

where Metr or M Unif or M Top are the categories of all metric spaces and their non-expanding or uniformly continuous or continuous maps and F_m, F_u are the corresponding forgetful functors.

A suitable choice of S_1, S_2, S_3 and h_1, h_2 gives, for example, the following statement: For every triple of natural numbers a, b, c there exists a metric space X such that

X^p is homeomorphic to X^q iff $p \equiv q \mod a$,
X^p is unif. homeomorphic to X^q iff $p \equiv q \mod ab$,
X^p is isometric to X^q iff $p \equiv q \mod abc$.

3. Clones

3.1. In 1991, I met John R. Isbell at the category theory conference in Montreal. He gave me the following problem: whether there exist two topological spaces with isomorphic monoids of all continuous selfmaps and non-isomorphic clones. And I have learned that *the clone* clo X *of a topological space* X is the full subcategory of Top generated by all its finite powers $X^0, X, X^2, \ldots, X^n, \ldots, n \in \omega$. Its *n-segment* $\mathrm{clo}_n\, X$ is its full subcategory generated by $X^0, \ldots, X^n$.

3.2. Let me recall that the monoid $M(X)$ of all continuous selfmaps of X carries considerable information about the space X itself. You can always recover the underlying set of X: points of X can be identified with constant selfmaps of X and these are precisely the left zeros of $M(X)$ [i.e., a map $c : X \to X$ is constant iff $c \circ f = c$ for every map $f : X \to X$]. The topology of X can be recovered from $M(X)$ whenever X contains a subspace inductively generating its topology, e.g., two-point space for 0-dimensional Hausdorff spaces or an arc for Tychonoff spaces (see [14]). Hence to find spaces X, Y with $M(X)$ isomorphic to $M(Y)$ but clo X non-isomorphic to clo Y, one has to look for them outside these classes of spaces. I succeeded in solving of the Isbell's problem in [42], using semirigid spaces.

3.3. Semirigid spaces

Definition. Let B be a closed subset of a space X. Then X is called *B-semirigid* if every continuous $f : X \to X$ is either the identity or constant or $f(X) \subseteq B$.

And a space $X = (P, t)$ is called *extremally B-semirigid* if B is its closed discrete subset and for *arbitrary* Hausdorff topology τ on B, the space $(P, \bar{\tau})$ is B-semirigid whenever it is a Hausdorff space and $\bar{\tau}$ is an extension of τ onto P coarser than t and coinciding with t on the subset $P \setminus B$.

The main result of [42] states that, whenever B is a subset of a set P such that $\mathrm{card}\, B = \mathrm{card}(P \setminus B) \geqslant 2^{\aleph_0}$, then a metric ρ on P exists such that $X = (P, \rho)$ is extremally B-semirigid. In the construction, Cook continuum [2] is used. But once having the extremally B-semirigid space X constructed, you can forget its internal structure and you can work with it as with a "black box".

3.4. The principal advantage of extremally B-semirigid spaces is that they allow one to construct B-semirigid spaces satisfying some *recursive conditions*. Let us describe it in its simplest case: given a one-to-one map h of P onto B with $\mathrm{card}\, B = \mathrm{card}(P \setminus B) \geqslant 2^{\aleph_0}$, you can construct a topology t on P such that h is a homeomorphism of (P, t) onto $(B, t/B)$ [this is *the* recursive condition because t appears both in the domain and in the codomain of h] and still the space (P, t) is B-semirigid. The topology t is constructed by iterating the following procedure: if, for an ordinal α, t_α is already constructed, you shift it by h onto B, and $t_{\alpha+1}$ is its "suitable" extension. If this procedure stops, you have a topology t such that h is a homeomorphism of (P, t) onto $(B, t/B)$. *Important:* if the starting topological space (P, t_0) is extremally B-semirigid and the "suitable" extensions in the steps from t_α to $t_{\alpha+1}$ are done so that the requirements in the definition of the extremal B-semirigidity are satisfied [particularly, $t_{\alpha+1}$ restricted to $P \setminus B$ coincides with the restriction of t_α (hence of t_0) on this set], then the resulting space (P, t) is B-semirigid.

Suitable (more complex and more sophisticated) recursive conditions together with the B-semirigidity of the resulting space allow one to obtain spaces whose clones have some required properties. This method is used in almost all subsequent constructions.

3.5. Problem 1 of the monograph [23] sounds similarly to the Isbell's problem. W. Taylor asks there whether the first order language of clone theory has, for topological spaces, a more expressive power than the first order language of monoid theory, i.e., whether there exist spaces X, Y with the monoids $M(X)$ and $M(Y)$ elementarily equivalent in the language of monoid theory such that their clones are not elementarily equivalent in the language of clone theory. Let me recall briefly some data about these languages (for details see [24]).

3.6. The first order language $\mathcal{L}$ of clone theory

(a) *has ω sorts of variables*; for clo X, the variables of the nth sort range over the set of all continuous maps $X^n \to X$;
(b) *it has n constants of the sort n*, for every $n \in \omega$; for clo X, these are precisely the product projections $\pi_i^n : X^n \to X$, $i \in n$;
(c) *it has countably many operational symbols S_m^n*, acting on a variable of the mth sort and an m-tuple of variables of the nth sort resulting in a variable of the nth sort; for clo X, S_m^n just substitutes, in continuous map $g : X^m \to X$, an m-tuple of continuous maps $f_0, \dots, f_{m-1} : X^n \to X$, i.e. in clo X, $S_m^n(g; f_0, \dots, f_{m-1}) = h$ with

$$h(x_0, \dots, x_{n-1}) = g\big(f_0(x_0, \dots, x_{n-1}), \dots, f_{m-1}(x_0, \dots, x_{n-1})\big).$$

The basic formulas of $\mathcal{L}$ are the equalities of correctly formed expressions in the above data and, as usual, more complex formulas are created from the basic formulas by means of logical connectives and quantifiers; sentences are closed formulas and elementary equivalence means that precisely the same sentences are satisfied.

The first order language $\mathcal{M}$ of monoid theory is a fragment of $\mathcal{L}$: it uses only variables of the 1st sort (for clo X, ranging over continuous maps $X \to X$), the operation S_1^1 only (which is just the composition) and the unique constant $\pi_0^{(1)}$ (which is just the identity). The first order languages $\mathcal{L}_n$ of n-segments of clones are "in between" variants of $\mathcal{L}$ and $\mathcal{M}$.

3.7. In [42], for every $n \in \omega$, I constructed metric spaces $X = (P, \rho)$ and $Y = (P, \sigma)$ such that

the n-segments $\text{clo}_n X$ and $\text{clo}_n Y$ of their clones are equal

(*equal* in the sense that they are formed by the same maps among the powers $P^0, \dots, P^n$ of their common underlying set P) but

the $(n+1)$-segments $\text{clo}_{n+1} X$ and $\text{clo}_{n+1} Y$ are not elementarily equivalent in the language $\mathcal{L}_{n+1}$.

This solves both the Isbell's problem and the Taylor's problem because the equality implies the isomorphism and the isomorphism implies the elementary equivalence.

However, the Isbell's problem and the Taylor's problem are distinct, as shown by J. Sichler and myself, see below.

Theorem. *(See [18].) For every triple of positive natural numbers $n_1 \leqslant n_2 \leqslant n_3$, there exist metric spaces X and Y on a joint underlying set such that the n-segments of their clones are*

equal	*iff*	$n \leqslant n_1$,
isomorphic	*iff*	$n \leqslant n_2$,
elementarily equivalent	*iff*	$n \leqslant n_3$.

3.8. Suitable modifications of the method described in 3.4 led to the following theorems:

Theorem. *(See [25].) For every $n \in \omega$ there exists a space X such that $\operatorname{clo}_k X$ is isomorphic to $\operatorname{clo}_k Y$ precisely when $k \leqslant n$ where Y is a Tychonoff modification (or compactly generated modification or sequential modification or countable modification) of X.*

Theorem. *(See [43].) For every $n \in \omega$ there exists a metric space X such that every continuous map $X^k \to X$ is uniformly continuous precisely when $k \leqslant n$.*

In [44], also non-expanding maps $X^k \to X$ are investigated.

3.9. The theorem of 3.7 and both theorems in 3.8 are special cases of the following *general situation*:

Let $(\mathcal{K}, U)$ and $(\mathcal{H}, V)$ be concrete categories with concrete finite products, let $F : \mathcal{K} \to \mathcal{H}$ be a functor with $V \circ F = U$, i.e. preserving the underlying sets and maps. For any pair X_1, X_2 of objects of $\mathcal{K}$ with $U(X_1) = U(X_2)$ we can compare clones of the following four pairs of objects:

$$X_1 \text{ and } X_2; \quad X_1 \text{ and } FX_1; \quad X_2 \text{ and } FX_2; \quad FX_1 \text{ and } FX_2.$$

We compare the n-segments of the clones with respect to the elementary equivalence, the isomorphism and, since all the objects X_1, X_2, FX_1, FX_2 have the same underlying set, also to the equality [equality in the sense of 3.7]. For brevity, denote

$\overset{1}{\sim}$ the equality,

$\overset{2}{\sim}$ the isomorphism, and

$\overset{3}{\sim}$ the elementary equivalence,

and put

$$r^{(i)} = \sup\{n; \operatorname{clo}_n X_1 \overset{i}{\sim} \operatorname{clo}_n X_2\},$$
$$s^{(i)} = \sup\{n; n \operatorname{clo}_n FX_1 \overset{i}{\sim} \operatorname{clo}_n FX_2\},$$
$$t_j^{(i)} = \sup\{n; \operatorname{clo}_n X_j \overset{i}{\sim} \operatorname{clo}_n FX_j\}, \quad j = 1, 2.$$

Hence we have a 3×4 matrix

$$M = \begin{pmatrix} r^{(1)} & s^{(1)} & t_1^{(1)} & t_2^{(1)} \\ r^{(2)} & s^{(2)} & t_1^{(2)} & t_2^{(2)} \\ r^{(3)} & s^{(3)} & t_1^{(3)} & t_2^{(3)} \end{pmatrix}$$

of elements of $\{0, 1, \ldots, \infty\}$. The matrix M necessarily has the following properties:

(α) *all columns are non-decreasing*; this is a trivial consequence of the fact that the equality implies the isomorphism and the isomorphism implies the elementary equivalence;

(β) *all rows are grounded* in the sense that no entry of the quadruple is strictly smaller than the remaining three; this is a trivial consequence of transitivity: for instance, if $n \leqslant t_j^{(i)}$, $j = 1, 2$, and $n \leqslant s^{(i)}$, i.e. $\operatorname{clo}_n X_j \overset{i}{\sim} \operatorname{clo}_n FX_j$ and $\operatorname{clo}_n FX_1 \overset{i}{\sim} \operatorname{clo}_n FX_2$, the mere transitivity of $\overset{i}{\sim}$ gives $\operatorname{clo}_n X_1 \overset{i}{\sim} \operatorname{clo}_n X_2$, and hence $n \leqslant r^{(i)}$.

Definition. A 3×4 matrix A of elements of $\{0, 1, \ldots, \infty\}$ is called *admissible* if it satisfies (α) and (β).

The main question. For which functors $F : \mathcal{K} \to \mathcal{H}$ every admissible matrix A can be *realized by* F (in the evident sense that there exist objects X_1, X_2 in $\mathcal{K}$ such that the matrix M obtained from them by the above described procedure is equal to the given matrix A)? In [19], J. Sichler and I gave some partial answers in the theorems quoted below.

Theorem. *(See [19].) Every admissible matrix can be realized by the forgetful functor* M Unif → M Top.

3.10. Topological modifications. Let $\mathcal{K}$ be a full subcategory of Top closed with respect to finite products. A functor

$$m : \mathcal{K} \to \mathcal{K}$$

is called a lower (or an upper) topological modification if $m \circ m = m$, m preserves the underlying sets and maps and, for every space X in $\mathcal{K}$, the identity map of its underlying set is continuous as a map $mX \to X$ (or as a map $X \to mX$) (see [3]).

Lower Modification Theorem. *(See [19].) Let $m : \mathcal{K} \to \mathcal{K}$ be a lower topological modification such that*

(a) *$\mathcal{K}$ contains all metrizable spaces and $mX = X$ for them*;
(b) *$\mathcal{K}$ is closed with respect to closed subspaces and m preserves them*;
(c) *there exists a Hausdorff totally disconnected space X_0 in $\mathcal{K}$ such that $mX_0 \neq X_0$ and mX_0 is metrizable.*

Then m realizes all admissible matrices.

The reader can verify herself (himself) that many lower modifications satisfy the mild conditions in the theorem.

There is also an Upper Modification Theorem in [19], but since its formulation is rather complicated I omit it here. Suffice it to say that the Tychonoff modification satisfies its hypothesis, so that it also realizes all admissible matrices.

Problem. Which uniform modifications (i.e. modification on subcategories of the category Unif of all uniform spaces and all uniformly continuous maps) realize all the admissible matrices? This problem has not yet been attacked.

3.11. Clone properties and monoid properties. Every sentence s of the first order language of clone theory (or of monoid theory) determines a topological property because the class of all spaces X for which clo X satisfies s is closed with respect to homeomorphic images. Let us call a topological property a *clone property* (or a *monoid property*) iff it can be expressed by a sentence s of the language of clone theory (or monoid theory).

In the previous proofs, clone properties were heavily used, but sentences determining them were never written out in whole because they would be too long. There is a natural question whether there is a short and simple sentence s of the language of clone theory such that the corresponding clone property is distinct from all the monoid properties. Here is such a sentence (where $f^{(2)}$ denotes a variable of the sort 2 and $g^{(1)}$ a variable of the sort 1):

$$(\forall f^{(2)})(\exists g^{(1)})\big(\big(f^{(2)} = S_1^2(g^{(1)}; \pi_0^{(2)})\big) \vee \big(f^{(2)} = S_1^2(g^{(1)}; \pi_1^{(2)})\big)\big).$$

Informally. Every continuous map $f : X^2 \to X$ factorizes through a product projection.

Spaces with this property were called in [45,46]

coconnected spaces.

Why use this funny name?

3.12. The duality in category theory consists of reversing arrows and adding the syllable CO to the corresponding notions: products × coproducts.

Hence, entering the “dual world” we have coclones instead of clones, the first order language of coclone theory and coclone properties (but monoid properties remain monoid properties). Connectedness is a coclone property determined by the following sentence of coclone theory (written informally): a space X is connected iff

every continuous $f : X \to X + X$ factorizes through a coproduct injection.

This is precisely the sentence dual to the above sentence of the clone theory defining the coconnected spaces. Thus

connectedness is a coclone property and
coconnectedness is a clone property dual to it.

3.13. Let us stay in the "dual world" for the following question: is the connectedness distinct from all the monoid properties? The answer is trivial

YES within the class of all spaces: the discrete and the indiscrete spaces on a set P have the same monoid of all continuous selfmaps and if card $P > 1$, the indiscrete space is connected and the discrete space is not.

NO within the class of all T_1-spaces. It is expressed, e.g., by the following monoid sentence (since all the variables are of the sort 1, we omit the superscript (1)):

$$\begin{aligned}&\neg((\exists g)(\exists a)(\exists b)((\neg(a=b)) \wedge c(a) \wedge c(b)\\ &\quad \wedge ((\forall x)(c(x) \Longrightarrow (S_1^1(g,x)=a) \vee S_1^1(g,x)=b))))\\ &\quad \wedge ((\exists y)(S_1^1(g,y)=a)) \wedge (\exists z)(S_1^1(g,z)=b)))),\end{aligned}$$

where $c(u) \equiv (\forall v)(S_1^1(u,v)=u)$.

Informally. There exists no continuous $g : X \to X$ such that the image Im g consists precisely of two distinct points. The answer is NO already in the class of all T_0-spaces, but there the monoid sentence expressing connectedness is longer.

3.14. Let us return to our "non-trivial" world now. In which classes of spaces is coconnectedness distinct from all monoid properties? Spaces X and Y with the same monoids $M(X)$ and $M(Y)$ such that Y is coconnected but X is not were constructed

within the class of Hausdorff spaces, in 2000 (see [45]),
within the class of Tychonoff spaces, in 2004 (see [20]),
within the class of metrizable spaces, recently (see [46]).

3.15. Coconnectedness makes sense in any category with finite products.

Definition. Let $(\mathcal{K}, U)$ be a concrete category with finite products. It is called *c-comprehensive* (= comprehensive with respect to coconnectedness) if it has objects X and Y with the same underlying set and the same endomorphism monoids such that precisely one of them is coconnected.

Theorem. *(See [20].) The categories* Tych*,* Unif *and* Metr *are c-comprehensive.*

3.16. A kind of "simultaneous c-comprehensivity" is given by the following definition:

Let $(\mathcal{K}, U)$ and $(\mathcal{H}, V)$ be concrete categories with finite products, $F : \mathcal{K} \to \mathcal{H}$ a finite-products-preserving functor such that $V \circ F = U$. The functor F is called *c-comprehensive* iff there are objects X, Y in $\mathcal{K}$ such that

$$U(X) = U(Y) = V(FX) = V(FY),$$

all the endomorphism monoids $M(X)$, $M(Y)$, $M(FX)$, $M(FY)$ are formed by the same selfmaps of the set $U(X)$ and

X *is not* coconnected in $\mathcal{K}$ (hence FX in $\mathcal{H}$) and
FY *is* coconnected in $\mathcal{H}$ (hence Y in $\mathcal{K}$).

Clearly, if F is c-comprehensive then both the categories $\mathcal{K}$ and $\mathcal{H}$ are c-comprehensive. The converse is not true in general, see the theorem below.

Theorem. *(See [46].) Though all the categories*

Metr, M Unif, M Top

are c-comprehensive, the forgetful functors Metr $\to$ M Unif *and* Metr $\to$ M Top *are not c-comprehensive, while the forgetful functor* M Unif $\to$ M Top *is c-comprehensive.*

3.17. Concluding remarks. There are many interesting clone and monoid properties, e.g., the *image determining property* ID (see [47]):

if continuous maps $f, g : X \to X$ satisfy $\operatorname{Im} f = \operatorname{Im} g$ then $f = g$,
and its n-variant n-ID,
if continuous maps $f, g : X^n \to X$ satisfy $\operatorname{Im} f = \operatorname{Im} g$, then
$f = g \circ p$, where p only permutes coordinates.

It is not difficult to prove that

$$m\text{-ID} \Longrightarrow n\text{-ID} \quad \text{whenever } n \leqslant m.$$

Is the converse implication also true? Or does there exist a space X which is ID but not 2-ID? In which classes of spaces such a space exists? Within which classes of spaces is 2-ID a monoid property? There are many such questions. Although A. Barkhudaryan observed that none of the "usual" topological properties is a clone property [because there are rigid spaces X and Y on a set P with $\operatorname{card} P = 2^{\aleph_0}$ (hence they have the same clone, see [23]) such that X is compact metric while Y is not a Hausdorff space; thus no class of spaces contained in the class of Hausdorff spaces and containing the class of all compact metric spaces is determined by a sentence of the language $\mathcal{L}$], above results on clone properties give a deeper insight into them. E.g., given a sentence s of the language $\mathcal{L}$, in which classes of spaces it is equivalent to a monoid sentence, or s is equivalent to another given sentence s'; or do there exist sentences s and s' equivalent within a class of spaces, but not equivalent within a larger class. Thus, the investigation of clone properties offers a rich and possibly interesting field of problems.

Finally, last but not least, this field of problems is greatly relevant to other parts of mathematics: clones on a set and abstract clones are heavily used in universal algebra (see, e.g., [21]), and under the name "algebraic theory" in the categorical approach to universal algebra (see [12,13]). This motivated W. Taylor in [23] to initiate his investigation of clones in topology.

References

[1] J. Adámek, V. Koubek, V. Trnková, Sums of Boolean spaces represent any group, Pacific J. Math. 61 (1975) 1–7.
[2] H. Cook, Continua which admit only the identity mapping onto non-degenerate sub-continua, Fund. Math. 60 (1967) 241–249.
[3] E. Čech, Topological Spaces, Revised by Z. Frolík and M. Katětov, Academia, Prague, 1966.
[4] J. de Groot, Groups represented by homeomorphism groups, Math. Ann. 138 (1959) 80–102.
[5] W. Hanf, On some fundamental problems concerning isomorphisms of Boolean algebras, Math. Scand. 5 (1957) 205–217.
[6] J.R. Isbell, Two set-theoretical theorems in categories, Fund. Math. 53 (1963) 43–49.
[7] T. Jech, Set Theory, Academic Press, New York, 1978.
[8] J. Ketonen, The structure of countable Boolean algebras, Ann. of Math. 108 (1978) 41–89.
[9] V. Koubek, Each concrete category has a representation by T_2 paracompact topological spaces, Comment. Math. Univ. Carolinae 15 (1974) 655–664.
[10] V. Koubek, Universal topological unary varieties, Topology Appl. 127 (2003) 425–446.
[11] L. Kučera, Every category is a factorization of a concrete one, J. Pure Appl. Algebra 1 (1971) 373–376.
[12] F.W. Lawvere, Functorial semantics of algebraic theories, Proc. Natl. Acad. Sci. USA 50 (1963) 869–872.
[13] F.W. Lawvere, Some algebraic problems in context of functorial semantics of algebraic theories, in: Lecture Notes in Math., vol. 61, Springer-Verlag, 1968, pp. 41–46.
[14] K.D. Magill Jr., A survey of semigroups of continuous selfmaps, Semigroup Forum 11 (1975/76) 189–282.
[15] A. Orsatti, N. Rodino, Homeomorphisms between finite powers of topological spaces, Topology Appl. 23 (1986) 271–277.
[16] A. Pultr, V. Trnková, Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories, North-Holland/Academia, Amsterdam/Prague, 1980.
[17] J. Rosický, V. Trnková, Representability of concrete categories by non-constant morphisms, Arch. Math. (Brno) 25 (1989) 115–118.
[18] J. Sichler, V. Trnková, On elementary equivalence and isomorphism of clone segments, Period. Math. Hungar. 32 (1996) 113–128.
[19] J. Sichler, V. Trnková, Clone segment independence in topology and algebra, Acta Sci. Math. (Szeged) 68 (2002) 611–672.
[20] J. Sichler, V. Trnková, Maps between a space and its square, Topology Appl. 142 (2004) 159–179.
[21] Á Szendrei, Clones in Universal Algebra, Les Presses de L'Université de Montréal, 1986.
[22] A. Tarski, Cardinal algebras; with an appendix by B. Jónsson and A. Tarski, Cardinal products of isomorphism types, New York, 1949.
[23] W. Taylor, The Clone of a Topological Space, Research and Exposition in Math., vol. 13, Helderman Verlag, Berlin, 1986.
[24] W. Taylor, Abstract clone theory, in: I.G. Rosenberg, G. Sabidussi (Eds.), Algebras and Order, Proceedings of the NATO Advanced Study Institute, Montréal 1991, Kluwer Academic Publishers, 1993, pp. 507–530.
[25] A. Tozzi, V. Trnková, Clone segments of the Tychonoff modification of a space, Appl. Cat. Structures 8 (2000) 407–424.
[26] V. Trnková, Universal categories, Comment. Math. Univ. Carolinae 7 (1966) 143–206.

[27] V. Trnková, Non-constant continuous mappings of metric or compact Hausdorff spaces, Comment. Math. Univ. Carolinae 13 (1972) 283–295.
[28] V. Trnková, X^m is homeomorphic to X^n iff $m \sim n$ where $\sim$ is a congruence on natural numbers, Fund. Math. 80 (1973) 51–56.
[29] V. Trnková, All small categories are representable by continuous non-constant maps of compact spaces, DAN SSSR 230 (1976) 789–791 (in Russian); Transl. AMS: Soviet. Math. Doklady 17 (1976) 1403–1406.
[30] V. Trnková, Categorial aspects are useful for topology, in: General Topology and its Relation to Modern Analysis and Algebra IV, in: Lecture Notes in Math., vol. 609, Springer-Verlag, 1977, pp. 211–225.
[31] V. Trnková, Homeomorphisms of powers of metric spaces, Comment. Math. Univ. Carolinae 21 (1980) 41–53.
[32] V. Trnková, Isomorphisms of sums of countable Boolean algebras, Proc. Amer. Math. Soc. 261 (1980) 463–482.
[33] V. Trnková, Homeomorphisms of products of countable spaces, DAN SSSR 263 (1982) 47–51 (in Russian); Transl. AMS: Soviet. Math. Dokl. 25 (1982) 304–307.
[34] V. Trnková, Countable Hausdorff spaces with countable weight, Comment. Math. Univ. Carolinae 26 (1985) 749–770.
[35] V. Trnková, Homeomorphisms of products of Boolean separable spaces, Fund. Math. 126 (1985) 45–61.
[36] V. Trnková, Homeomorphisms of products of subsets of the Cantor discontinuum, in: Dissertationes Mathematicae (Rozprawy matematyczne), vol. 268, Polish Academy of Sciences, Warszawa, 1988, pp. 1–40.
[37] V. Trnková, Simultaneous representations by metric spaces, Cahiers Topologie Géom. Différentielle 29 (3) (1988) 217–239.
[38] V. Trnková, Nonconstant continuous maps of spaces and of their β-compactifications, Topology Appl. 33 (1989) 410–431.
[39] V. Trnková, Simultaneous representations by products in categories, in: J. Adámek, S. MacLane (Eds.), Categorical Topology and its Relations to Analysis, Algebra and Combinatorics, World Scientific, 1989, pp. 410–431.
[40] V. Trnková, Universal concrete categories and functors, Cahiers Topologie Géom. Différentielle Catégoriques 34 (1993) 239–256.
[41] V. Trnková, Universalities, Appl. Cat. Structures 2 (1994) 173–185.
[42] V. Trnková, Semirigid spaces, Trans. Amer. Math. Soc. 343 (1994) 305–329.
[43] V. Trnková, Continuous and uniformly continuous maps of powers of metric spaces, Topology Appl. 63 (1995) 189–200.
[44] V. Trnková, Maps of finite powers of metric spaces (recursive conditions for spaces at work), Topology Appl. 108 (2000) 277–301.
[45] V. Trnková, Clones, coclones and coconnected spaces, Acta Math. Univ. Commenianae 69 (2000) 241–259.
[46] V.Trnková, Metrizability and coconnectedness, Appl. Cat. Structures, in press.
[47] V. Trnková, Clone properties of topological spaces, Arch. Mat. (Brno) 42 (2006) 427–440.
[48] V. Trnková, M. Hušek, Non-constant continuous maps of modifications of topological spaces, Comment. Math. Univ. Carolinae 29 (1988) 747–765.
[49] V. Trnková, V. Koubek, Isomorphisms of sums of Boolean algebras, Proc. Amer. Math. Soc. 66 (1977) 231–236.
[50] J. Vinárek, Representations of commutative semigroups by products of metric 0-dimensional spaces, Comment. Math. Univ. Carolinae 23 (1982) 715–726.