Tauberian theorems for positive harmonic functions

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I. INTRODUCTION

1.1. Notation and terminology. Throughout this paper, \( n \) is a fixed positive integer, \( \mathbb{R}^n \) is \( n \)-dimensional euclidean space, and \( |x| \) denotes the euclidean norm of \( x \in \mathbb{R}^n \).

\[ B(r) = \{ x \in \mathbb{R}^n : |x| < r \} \], the open ball in \( \mathbb{R}^n \) with radius \( r \) and center 0.

\( m \) denotes Lebesgue measure on \( \mathbb{R}^n \), so normalized that \( m(B(r)) = r^n \) for \( 0 < r < \infty \).

\[ \mathbb{R}^{n+1}_+ = \{ (x, t) : x \in \mathbb{R}^n, t > 0 \} \], a half-space.

The Poisson kernel \( K(x, t) \) is defined on \( \mathbb{R}^{n+1} \) by

\[ K(x, t) = \frac{ct}{(|x|^2 + t^2)^{(n+1)/2}} \]

where \( c = c(n) \) is a constant, chosen so that

\[ \int_{\mathbb{R}^n} K(x, t) dm(x) = 1 \]

when \( t = 1 \). Since

\[ K(x, t) = t^{-n}K(t^{-1}x, 1) \]

it follows then that (2) actually holds for every \( t > 0 \).

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Let $\mu$ be a positive Borel measure on $\mathbb{R}^n$ whose Poisson integral

$$u(x, t) = \int_{\mathbb{R}^n} K(x - \xi, t) d\mu(\xi)$$

is finite for one (hence for all) $(x, t) \in (\mathbb{R}^{n+1})_+$. The symmetric derivative of $\mu$, at the origin of $\mathbb{R}^n$, is defined to be

$$D_{\text{sym}} \mu(0) = \lim_{r \to 0} \frac{\mu(B(r))}{m}$$

provided, of course, that this limit exists.

1.2. A well-known and easily proved analogue of a classical theorem of Fatou states that the existence of the limit (5) implies that $u$ has a "radial" limit at $x = 0$; more precisely,

$$\lim_{t \to 0} u(0, t) = (D_{\text{sym}} \mu)(0).$$

For this assertion, the positivity of $\mu$ is not needed. However, the positivity of $\mu$ is the tauberian condition which makes the following converse true:

**Theorem A.** If $u$ is the Poisson integral of a positive measure $\mu$, as in (4), and if there exists

$$\lim_{t \to 0} u(0, t) = L,$$

with $L < \infty$, then $(D_{\text{sym}} \mu)(0) = L$.

For $n = 1$, this was proved by Loomis [3]. Actually, both he and Gehring [1], [2] also considered non-radial limits. H. S. Shapiro [6] proved related tauberian theorems for holomorphic functions in products of half-planes. In the present paper, Theorem A is established, for arbitrary $n$, as a consequence of Wiener's tauberian theorem. As seems to happen very often in proofs in which Wiener's theorem is applied, most of the work consists in setting the stage for this application.

1.3. The hypothesis (7) can actually be weakened. This is done in Theorem B (stated below) which is the main result of the present paper.

The statement of Theorem B, and even more so its proof, relies on some concepts concerning harmonic analysis on the multiplicative group $G$ of the positive real numbers. A Haar measure on $G$ is $dt = r^{-1}dr$, where $dr$ denotes ordinary one-dimensional Lebesgue measure.

Convolution on $G$ is defined by

$$(f \ast g)(r) = \int_0^\infty f(r/s)g(s) \frac{ds}{s}$$
and the Fourier transform \( \hat{f} \) of a function \( f \in L^1(\tau) \) is

\[
(9) \quad \hat{f}(y) = \int_0^\infty f(r)r^{-\nu} \frac{dr}{r} \quad (-\infty < y < \infty).
\]

For \( 0 < \alpha < \infty \), define auxiliary functions \( H_\alpha \) by

\[
(10) \quad H_\alpha(r) = \begin{cases} \alpha r^{-\alpha} & \text{if } 1 < r < \infty, \\ 0 & \text{if } 0 < r < 1. \end{cases}
\]

Since \( \int_0^\infty H_\alpha dr = 1 \), it is clear that

\[
(11) \quad \lim_{t \to 0} (H_\alpha \ast v)(t) = \lim_{t \to 0} v(t)
\]

for every bounded measurable \( v \) for which \( \lim_{t \to 0} v(t) \) exists.

Hence Theorem A is a consequence of the following tauberian theorem:

**THEOREM B.** Suppose \( u \) is the Poisson integral of a positive measure \( \mu \), as in (4). Put \( v(t) = u(0, t) \), \( 0 < t < \infty \). If there is one \( \alpha \in (0, \infty) \) for which there exists

\[
(12) \quad \lim_{t \to 0} (H_\alpha \ast v)(t) = L,
\]

with \( L < \infty \), then \( (D_{\text{sym}}\mu)(0) = L \).

This will be proved in Part II.

1.4. To every positive harmonic function \( u \) in \((\mathbb{R}^{n+1})_+\) corresponds a positive measure \( \mu \) on \( \mathbb{R}^n \) and a constant \( A > 0 \) such that

\[
(13) \quad u(x, t) = At + \int_{\mathbb{R}^n} K(x - \xi, t)d\mu(\xi).
\]

This is certainly a known fact, but since I can find no proof of it in the literature (when \( n > 1 \)), I include one in Part IV. If (13) is combined with Theorem B and the Fatou theorem mentioned in § 1.2, one obtains the following tauberian corollary:

**COROLLARY.** If \( u \) is a positive harmonic function in \((\mathbb{R}^{n+1})_+\) and if

\[
(14) \quad \lim_{t \to 0} \int_0^t u(0, s)s^{\alpha-1}ds = L
\]

exists for one \( \alpha \in (0, \infty) \), with \( L < \infty \), then

\[
(15) \quad \lim_{t \to 0} u(0, t) = L.
\]

II. PROOF OF THEOREM B

2.1. A simple computation, using only the definitions (8) and (10), shows that

\[
(16) \quad (\beta - \alpha)H_\beta \ast H_\alpha = \beta H_\alpha - \alpha H_\beta
\]
for all \( x, \beta \in (0, \infty) \). If we convolve both sides of (16) with \( \nu \) and use (12) [which is written more explicitly in (14)] we see that the left side tends to \( (\beta - \alpha)L \) as \( t \to 0 \). Hence (12) implies that \( (H_\beta \ast \nu)(t) \to L \) as \( t \to 0 \), for every \( \beta \in (0, \infty) \). It involves therefore no loss of generality to assume (as we shall do) that (12) holds for the special value \( \alpha = \beta \).

Furthermore, \( \mu \) can be replaced by its restriction to any neighborhood of the origin, with no effect on either the hypothesis or the conclusion of Theorem B. It involves therefore no loss of generality to assume (as we shall do) that the total variation \( \|\mu\| = \mu(R^n) \) of \( \mu \) is finite.

2.2. The definitions of \( H_n \) and \( \nu \) give

\[
(H_n \ast \nu)(r) = \int_0^\infty H_n(r/s) \nu(s) \frac{ds}{s} = n \int_0^r (s/r)^n \frac{ds}{s} \int_{R^n} K(\xi, s) d\mu(\xi)
\]

or

\[
(H_n \ast \nu)(r) = nr^{-n} \int_{R^n} d\mu(\xi) \int_0^r K(\xi, s)s^{n-1} ds.
\]

The conclusion of Theorem B is that the averages

\[
M(r) = \frac{\mu}{m}(B(r)) = r^{-n} \mu(B(r))
\]

tend to \( L \) as \( r \to 0 \). As a first step, we shall now use (17) to prove that \( M(r) \) is bounded.

If \( |\xi| < r \) then \( K(\xi, s) > cs(r^2 + s^2)^{-(n+1)/2} \). The inner integral in (17) is therefore larger than

\[
\int_0^r \frac{cs^n ds}{(r^2 + s^2)^{(n+1)/2}} = \int_0^r \frac{ct^ndt}{(1 + t^2)^{(n+1)/2}} = c_1(n),
\]

say. Hence (17) implies that

\[
nc_1(n) M(r) < (H_n \ast \nu)(r) \quad (0 < r < \infty),
\]

since \( \int_{R^n} > \int_{B(r)} \), by positivity of \( \mu \).

By (12) and (19), \( M(r) \) is bounded as \( r \to 0 \). For large \( r \), the estimate

\[
M(r) = r^{-n} \mu(B(r)) < r^{-n} \|\mu\|
\]

applies.

We conclude: \( M(r) \) is bounded on \((0, \infty)\).

2.3. The function

\[
k(t) = nct(1 + t^2)^{-(n+1)/2} \quad (0 < t < \infty)
\]

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satisfies

(21) \[ nK(\xi, |\xi| t) = |\xi|^{-n} k(t) \]

when \( \xi \neq 0 \). Since \( M(r) \) is bounded and \( \mu(B(r)) = r^n M(r) \), \( \mu \) has no mass at the origin. If we replace \( s \) by \( |\xi| t \) in (17), and use (21), we therefore obtain

\[
(H_n \star v)(r) = \int_{R^n} d\mu(\xi) \frac{r|\xi|}{t} (t/r)^n k(t) \frac{dt}{t} = \int_0^\infty (t/r)^n \mu(B(r/t)) k(t) \frac{dt}{t}
\]
or

(22) \[ H_n \star v = M \star k. \]

Formula (22) is the principal step in the proof of Theorem B. In conjunction with (12), it tells us that the bounded function \( M \) satisfies

(23) \[ \lim_{r \to 0} (M \star k)(r) = L. \]

2.4. In order to apply Wiener’s theorem, we now have to compute the Fourier transform of \( k \). By (20) and (9),

(24) \[
\hat{k}(y) = nc \int_0^\infty (1 + t^2)^{-(n+1)/2} t^{-y} dt.
\]

The change of variables \( 1 + t^2 = 1/s \) transforms this integral into a beta-function; explicitly,

(25) \[
\hat{k}(y) = \frac{nc}{2} \frac{\Gamma\left(\frac{n+i y}{2}\right) \Gamma\left(\frac{1-i y}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.
\]

This has no zeros.

Moreover, our choice of \( c \) shows that

\[
\frac{1}{c} = \int_{R^n} (1 + |\xi|^2)^{-(n+1)/2} dm(\xi) = n \int_0^\infty (1 + r^2)^{-(n+1)/2} r^{n-1} dr,
\]

by switching to polar coordinates. If \( r \) is replaced by \( 1/t \) in the last integral, comparison with (24) shows that

(26) \[ \int_0^\infty k(t) \frac{dt}{t} = \hat{k}(0) = 1. \]

2.5. In view of (23), (25), and (26), Wiener’s tauberian theorem tells us now that

(27) \[ \lim_{r \to 0} (M \star f)(r) = L \]

for every \( f \in L^1(\tau) \) with \( \int_0^\infty f(r)r^{-1} dr = 1. \)
The desired conclusion of Theorem B, namely

\[ (28) \quad \lim_{r \to 0} M(r) = L \]

can now be reached by a familiar end game.

Pick \( \gamma > 1 \), close to 1. Choose positive functions \( f_i \in L^1(\tau) \), \( i = 1, 2 \), with integral 1, such that \( f_1 \) vanishes outside the interval \([1, \gamma]\) and \( f_2 \) vanishes outside \([\gamma^{-1}, 1]\). By (18), \( r^n M(r) = \mu(B(r)) \), a nondecreasing function of \( r \). Hence

\[
M\left(\frac{r}{t}\right) \leq t^n M(r) < \gamma^n M(r) \quad (1 < t < \gamma)
\]

and

\[
M\left(\frac{r}{t}\right) > t^n M(r) > \gamma^{-n} M(r) \quad (\gamma^{-1} < t < 1).
\]

It follows that

\[ (29) \quad \gamma^{-n}(M \ast f_1)(r) < M(r) < \gamma^n(M \ast f_2)(r). \]

Letting \( r \to 0 \) in (29), we see by (27) that the upper and lower limits of \( M(r) \), as \( r \to 0 \), lie in \([\gamma^{-n}L, \gamma^nL]\), for every \( \gamma > 1 \). This gives (28), and completes the proof of Theorem B.

III. AN EXAMPLE

The following very natural question was asked by Dan Shea: Is the condition \( L < \infty \) really needed in Theorems A and B? The following example shows that the answer is affirmative:

*For every \( n > 0 \), there is a finite positive Borel measure \( \mu \) on \( \mathbb{R}^n \) whose Poisson integral \( u \) satisfies

\[ (30) \quad u(0,t) \to +\infty \text{ as } t \to 0 \]

although

\[ (31) \quad \liminf_{r \to 0} \frac{\mu}{m}(B(r)) = 0. \]

To construct \( \mu \), let \( \delta_k \) and \( w_k \) \( (k = 1, 2, 3, \ldots) \) be positive numbers such that

\[ (32) \quad \frac{1}{\delta_k} \sum_{i=k+1}^{\infty} w_i \to 0 \text{ as } k \to \infty \]

but

\[ (33) \quad (\delta_k)^{-2} w_k w_{k+1} \to \infty \text{ as } k \to \infty. \]
For example, \( w_k = 1/k! \) and \( \delta_k = k^{1/2} w_{k+1} \) satisfy these requirements. Let \( \mu_k \) be a positive measure, of total mass \( w_k \), concentrated on the boundary of \( B(r_k) \).

Define \( \mu = \sum_{k=1}^{\infty} \mu_k \).

Since \( (\mu/m)(B(r_k)) = (r_k)^{-\alpha} \mu(B(r_k)) \) is the left side of (32), \( \mu \) satisfies (31).

To prove (30), fix \( t \in (0, r_1) \), then choose \( k \) so that \( r_{k+1} < t < r_k \). Then

\[
 c^{-1} u(0, t) > \frac{t w_k}{r_k^{n+1}} + w_{k+1} \frac{r_k^{n+1}}{t^n}.
\]

Since \( r_k^2 + t^2 < 2r_k^2 \) and \( r_{k+1}^2 + t^2 < 2t^2 \), it follows that

\[
 2^{(n-1)/2} c^{-1} u(0, t) > \frac{t w_k}{r_k^{n+1}} + \frac{w_{k+1}}{t^n} \geq \frac{(w_k w_{k+1})^{1/2}}{r_k^2} \frac{1}{\delta_k}
\]

which tends to \( \infty \) as \( k \to \infty \), by (33). Thus \( \mu \) satisfies (30).

IV. THE POISSON INTEGRAL REPRESENTATION

We conclude with a proof of the theorem that was stated in § 1.4. The proof is similar to the one given by Loomis and Widder [4] for positive harmonic functions in a half plane; the point of their paper was the avoidance of the usual conformal mapping of a half plane onto a disc.

**Lemma 1.** Suppose \( f \) is harmonic in \( \mathbb{R}^{n+1} \), \( f(x, 0) = 0 \) for all \( x \in \mathbb{R}^n \), and \( f(x, t) > 0 \) when \( t > 0 \). Then there is a constant \( A \) such that \( f(x, t) = A t \).

**Proof.** Expand \( f \) in a series

\[
 f(x, t) = \sum_{p=1}^{\infty} H_p(x, t)
\]

in which each \( H_p \) is a (real) harmonic polynomial, homogeneous of degree \( p \). The reflection principle implies that \( f(x, -t) = -f(x, t) \). Combined with the uniqueness of the expansion (34), this implies that

\[
 H_p(x, -t) = -H_p(x, t)
\]

for all \( (x, t) \in \mathbb{R}^{n+1} \) and for all \( p \). We note some consequences of (35):

(i) \( H_1(x, t) = A t \); also, \( A = \frac{\partial f}{\partial t}(0, 0) > 0 \).

(ii) By (i), \( fH_1 > 0 \) on \( \mathbb{R}^{n+1} \).

(iii) Since \( H_p(x, 0) = 0 \), (i) shows that \( H_1 \) divides \( H_p \) in the ring of polynomials on \( \mathbb{R}^{n+1} \). Hence there are constants \( \gamma(p) < \infty \) (which, a priori, may depend on \( f \)) such that

\[
 |H_p(x, t)| < \gamma(p)|H_1(x, t)| \quad ((x, t) \in S)
\]

where \( S \) is the unit sphere in \( \mathbb{R}^{n+1} \).
Being harmonic homogeneous polynomials, the $H_p$'s are orthogonal to each other in $L^2(\sigma)$, where $\sigma$ is the rotation-invariant probability measure on $S$. (For a very simple proof of this, see Theorem 1 in [5].) For $0 < r < \infty$, we have

\[(37) \quad f_r(x, t) = f(rx, rt) = \sum_{p=1}^{\infty} r^p H_p(x, t) \]

(this defines $f_r$), and the above-mentioned orthogonality, combined with (ii) and (36), gives

\[r^p \int_S H_p^2 d\sigma = \int_S f_r H_p d\sigma < \gamma(p) \int_S f_r H_1 d\sigma = \gamma(p) r \int_S H_1 d\sigma.\]

Letting $r \to \infty$, it follows that $H_p = 0$ for all $p > 1$. This proves the lemma.

**Lemma 2.** Suppose $f$ is continuous and $> 0$ on the closure of $(R^{n+1})_+$ and $f$ is harmonic in $(R^{n+1})_+$. Then there is a constant $A$ such that

\[(38) \quad f(x, t) = At + \int_{R^n} K(x-\xi, t)f(\xi, 0)dm(\xi).\]

Moreover,

\[(39) \quad A = \lim_{t \to \infty} t^{-1}f(0, t).\]

(The term "closure" refers of course to the space $R^{n+1}$.)

**Proof.** Let $P[f](x, t)$ denote the Poisson integral in (38).

Let $\varphi$ be any continuous function on $R^n$, with compact support, such that $0 < \varphi(\xi) < f(\xi, 0)$. Fix $\epsilon > 0$. Then there exists $r$, so large that $P[\varphi](x, t) < \epsilon$ for all $(x, t)$ with $|x|^2 + \rho^2 = r^2, t > 0$. Thus $f - P[\varphi] > - \epsilon$ for these $(x, t)$. Since $f - P[\varphi]$ is harmonic and has continuous nonnegative boundary values, we have $f - P[\varphi] > - \epsilon$ whenever $|x|^2 + \rho^2 < r^2, t > 0$. It follows that $f > P[\varphi]$. Apply this to a sequence $\{\varphi_i\}$ such that $\varphi_i(\xi)$ converges monotonically to $f(\xi, 0)$. This shows that $f > P[f]$ on $(R^{n+1})_+$. By the reflection principle, $f - P[f]$ extends to a harmonic function on $R^{n+1}$ to which Lemma 1 can be applied. This gives (38). Finally, (39) is an immediate consequence of (38) and (1).

**Proof of Theorem.** We are given a positive harmonic function $f$ in $(R^{n+1})_+$. For each $\delta > 0$, Lemma 2 applies to the function $f_\delta$ defined by $f_\delta(x, t) = f(x, t + \delta)$, and shows that

\[(40) \quad f(x, t + \delta) = A_\delta t + \int_{R^n} K(x - \xi, t)f(\xi, \delta)dm(\xi)\]

for $(x, t) \in (R^{n+1})_+$. Since $t/(t + \delta) \to 1$ as $t \to \infty$,

\[A_\delta = \lim_{t \to \infty} t^{-1}f_\delta(0, t) = \lim_{t \to \infty} (t + \delta)^{-1}f(0, t + \delta).\]
Thus $A_\delta$ is independent of $\delta$, and (40) can be rewritten in the form

\[(41) \quad f(x, t+\delta) = \bar{A} t + \int_{\mathbb{R}^n} \frac{K(x-\xi, t)}{K(\xi, 1)} d\lambda_\delta(\xi) \]

where $d\lambda_\delta(\xi) = K(\xi, 1)f(\xi, \delta)dm(\xi)$. Since

\[(42) \quad f(0, 1+\delta) = \bar{A} + ||\lambda_\delta||,\]

the measures $\lambda_\delta$ are bounded in the total variation norm, for $0<\delta<1$, say. By (41) and (42),

\[(43) \quad f(x, t+\delta) - tf(0, 1+\delta) = \int_{\mathbb{R}^n} \left\{ \frac{K(x-\xi, t)}{K(\xi, 1)} - t \right\} d\lambda_\delta(\xi).\]

The integrand in (43) belongs to $C_0(\mathbb{R}^n)$, for every $(x, t) \in (\mathbb{R}^{n+1})_+$. The boundedness of $\{||\lambda_\delta||\}$ shows that there is a sequence $\delta_t \to 0$ such that $\lambda_{\delta_t}$ converges to some measure $\lambda_0$ in the weak $\star$-topology of the dual of $C_0(\mathbb{R}^n)$. Thus $\delta$ can be replaced by $0$ in (43), and we obtain the desired representation

\[(44) \quad f(x, t) = \bar{A} t + \int_{\mathbb{R}^n} K(x-\xi, t)d\mu(\xi)\]

in which $d\mu(\xi) = K(\xi, 1)^{-1}d\lambda_0(\xi)$, $\bar{A} = f(0, 1) - ||\lambda_0||$.

REFERENCES