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A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations

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Abstract

A Dirichlet boundary value problem for a delay parabolic differential equation is studied on a rectangular domain in the $x-t$ plane. The second-order space derivative is multiplied by a small singular perturbation parameter, which gives rise to parabolic boundary layers on the two lateral sides of the rectangle. A numerical method comprising a standard finite difference operator (centred in space, implicit in time) on a rectangular piecewise uniform fitted mesh of $N_x \times N_t$ elements condensing in the boundary layers is proved to be robust with respect to the small parameter, or parameter-uniform, in the sense that its numerical solutions converge in the maximum norm to the exact solution uniformly well for all values of the parameter in the half-open interval (0, 1]. More specifically, it is shown that the errors are bounded in the maximum norm by $C(N_x^{-2}\ln^2 N_x + N_t^{-1})$, where C is a constant independent not only of N_x and N_t but also of the small parameter. Numerical results are presented, which validate numerically this theoretical result and show that a numerical method consisting of the standard finite difference operator on a *uniform* mesh of $N_x \times N_t$ elements is not parameter-robust.

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1. Introduction

Singularly perturbed delay partial differential equations (DPDEs) provide more realistic models for phenomena in many areas of science (such as population dynamics) that display time-lag or after-effect than do conventional instantaneous singularly perturbed partial differential equations (PDEs).

Singularly perturbed PDEs relate an unknown function to its derivatives evaluated at the same instance. In contrast, singularly perturbed DPDEs model physical problems for which the evolution does not only depend on the present state of the system but also on the past history. Singularly perturbed PDEs have been studied extensively by many authors (see [1–8,10,14,15,17] and the references therein) and developed thoroughly over the last two decades. However, the theory and numerical solution of singularly perturbed DPDEs are still at the initial stage.

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The solutions of these singularly perturbed DPDEs and their dynamics are fundamentally different from those of singularly perturbed PDEs without time delay:

- The solution of singularly perturbed DPDEs is determined by an initial function $\phi_b(x,t)$ if $t-\tau<0$ rather than by a simple initial value $\phi_b(x,0)$ as happens in the case of singularly perturbed PDEs.
- The solutions of the equation may behave quite differently from the solution of the "approximating" singularly perturbed PDE obtained by replacing $u_\varepsilon(x,t-\tau)$ with the first terms of the Taylor series. Small lags can have large effects.
- Even if the function $\phi_b(x,t)$ in (1; see below) is sufficiently differentiable, in general the solution $u_\varepsilon(x,t)$ may have various derivative discontinuities along the integration interval in the time direction. This results from the fact that the initial function does not satisfy the DPDE. With every time step τ, however these discontinuities are smoothed out more and more.

DPDEs arise from many biological, chemical, and physical systems which are characterised by both spatial and temporal variables and exhibit various spatio-temporal patterns. Examples occur in population ecology (to describe the interaction of spatial diffusion and time delays), generic repression (taking into account time delays from processes of transcription and translation as well as spatial diffusion of reactants in the models), and modelling size-dependent cell growth and division. A characteristic example from numerical control is the equation

$$\frac{\partial u_\varepsilon}{\partial t}=\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}+\upsilon(g(u_\varepsilon(x,t-\tau)))\frac{\partial u_\varepsilon}{\partial x}+c[f(u_\varepsilon(x,t-\tau))-u_\varepsilon(x,t)],$$

which models a furnace used to process metal sheets. Here, u_ε is the temperature distribution in a metal sheet, moving at a velocity υ and heated by a source specified by the function f; both υ and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. An example from population dynamics is the so-called Britton-model,

$$\frac{\partial u_\varepsilon}{\partial t}=\varepsilon\Delta u_\varepsilon+u_\varepsilon(1-g*u_\varepsilon)$$

with

$$g*u_\varepsilon=\int_{t-\tau}^{t}\int_{\Omega}g(x-y,t-s)u_\varepsilon(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Here, $u_\varepsilon(x,t)$ denotes a population density, which evolves through random migration (modelled by the diffusion term) and reproduction (modelled by the nonlinear reaction term). The latter involves a convolution operator with a kernel $g(x,t)$, which models the distributed age-structure dependence of the evolution and its dependence on the population levels in the neighbourhood.

Boundary layers occur in the solution of singularly perturbed problems when the singular perturbation parameter, which multiplies terms involving the highest derivatives in the differential equation, tend to zero. These boundary layers are located in neighbourhoods of the boundary of the domain, where the solution has a very steep gradient. Away from any corner of the domain a boundary layer of either regular or parabolic type may occur. A boundary layer is said to be parabolic if the characteristics of the reduced equation (for $\varepsilon=0$) are parallel to the boundary, and regular if these characteristics are not parallel to the boundary. In the vicinity of a corner, a boundary layer is said to be of corner type.

A description of the contents of the paper follows. The problem is formulated in Section 2. The corresponding reduced problem is defined and the parabolic boundary layers are described. The maximum principle for the differential operator is stated and it is shown that this leads immediately to its ε-uniform stability. Sufficient compatibility conditions on the initial and boundary data to guarantee the existence, uniqueness and appropriate regularity of the solutions to the problem are then presented. In Section 3, both classical and new sharper ε-uniform bounds in the maximum norm for the derivatives of the solution are discussed. The latter are obtained by means of a new decomposition of the solution, which leads to a deceptively simple proof of the required results. The fitted mesh finite difference method is constructed in Section 4 and is proved to be an ε-uniform method in Section 5. In Section 6 numerical results are presented, which validate the theoretical results. It is also shown that a classical numerical method on a uniform mesh is not ε-uniform for the problem under consideration. The paper ends with Section 7 that summarise the main conclusions.

2. The one-dimensional time dependent problem with a delay term

Let $\Omega = (0, 1)$, $D = (0, 1) \times (0, T]$, and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and right sides of the rectangular D corresponding to $x = 0$ and 1, respectively, and $\Gamma_b = [0, 1] \times [-\tau, 0]$. The problem considered is the following singularly perturbed delay parabolic equation with Dirichlet boundary conditions on Γ:

$$\begin{aligned}&L_\varepsilon u_\varepsilon(x,t)m \equiv \left(\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} + au_\varepsilon\right)(x,t) = -b(x,t)u_\varepsilon(x,t-\tau) + f(x,t), \quad (x,t) \in D,\\ &u_\varepsilon(x,t) = \phi_l(t), \quad (x,t) \in \Gamma_l, \quad u_\varepsilon(x,t) = \phi_r(t), \quad (x,t) \in \Gamma_r,\\ &u_\varepsilon(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b,\end{aligned} \tag{1}$$

where $0 < \varepsilon \leqslant 1$ and $\tau > 0$ (note that $T = k\tau$ for some integer $k > 1$) are given constants, $a(x,t)$, $b(x,t)$, $f(x,t)$, $(x,t) \in \bar{D}$, and $\phi_l(t)$, $\phi_b(x,t)$ and $\phi_r(t)$, $(x,t) \in \Gamma$, are sufficiently smooth and bounded functions that satisfy

$$a(x,t) \geqslant 0, \quad b(x,t) \geqslant \beta \geqslant 0, \quad (x,t) \in \bar{D}.$$

The reduced problem corresponding to (1) is

$$\begin{aligned}&\frac{\partial u_0(x,t)}{\partial t} + a(x,t)u_0(x,t) = -b(x,t)u_0(x,t-\tau) + f(x,t), \quad (x,t) \in D,\\ &u_0(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b.\end{aligned} \tag{2}$$

Then it is clear that the solution of (1) has boundary layers on Γ_l and Γ_r. The characteristics of (2) are the vertical lines $x =$ constant, which implies that any boundary layers arising in the solution are of parabolic type.

For a nonlinear example, one may consider Hutchinson's equation

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} = \varepsilon\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} + \lambda u_\varepsilon(x,t)[1 - u_\varepsilon(x,t-\tau)], \tag{3}$$

which arises in population dynamics as a rough model for the evolution of a population in mathematical ecology with density $u_\varepsilon(x,t)$. In many practical applications, the parameter ε is usually small. Murray [13] describes the derivation of the diffusion equation for an assemblage of particles, for example, cells, bacteria, chemicals, animals and so on. The diffusion process is based on a density and random walk of the particles. The diffusion coefficient or diffusivity ε of the particles is a measure of how efficiently the particles disperse from a high to a low density. For example, in blood, hemoglobin molecules have a diffusion coefficient of the order of $10^{-7}\,\text{cm}^2/\text{sec}$, while that for oxygen in blood is of the order of $10^{-5}\,\text{cm}^2/\text{sec}$, see [13]. Other typical biological values for ε of the order of 10^{-9}–$10^{-11}\,\text{cm}^2/\text{sec}$ are studied in [12]. The parameter λ represents the growth rate of the population. A set of examples illustrating the wide range of existing DPDE models can be found in [18].

The differential operator L_ε in (1) satisfies the following minimum principle.

Minimum principle. *Assume that $a \in C^0(\bar{D})$ and let $\psi \in C^2(D) \cap C^0(\bar{D})$. Suppose that $\psi \geqslant 0$ on Γ. Then $L_\varepsilon\psi \geqslant 0$ in D implies that $\psi \geqslant 0$ in $\bar{D}$.*

The stability of L_ε and an ε-uniform bound for the solution of (1) in the maximum norm are established by the following theorem.

Theorem 1. *Let v be any function in the domain of definition of the differential operator L_ε in* (1). *Then*

$$\|v\| \leqslant (1 + \alpha T)\max\{\|L_\varepsilon v\|, \|v\|_\Gamma\},$$

and any solution u_ε of (1) *has the ε-uniform upper bound*

$$\|u_\varepsilon\| \leqslant (1 + \alpha T)\max\{\|f\|, \|\phi\|_\Gamma\},$$

where the constant $\alpha = \max_{\bar{D}}\{0, 1 - a\} \leqslant 1$.

Here ϕ on Γ means the notation

$$\phi = \begin{cases} \phi_l & \text{on } \Gamma_l, \\ \phi_r & \text{on } \Gamma_r, \\ \phi_b & \text{on } \Gamma_b. \end{cases}$$

For equations of the type (1), where the delay values $t-\tau$ are bounded away from t by a positive constant, we will verify the existence of the solution by methods of steps.

Suppose that the solution is known, say

$$u_\varepsilon(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b.$$

Then $u_\varepsilon(x,t-\tau)$ is a known function of $(x,t) \in [0,1] \times [0,\tau]$ and (1) becomes a classical PDE, which can be treated by known existence theories (see [9]). Then we know $u_\varepsilon(x,t)$, $(x,t) \in [0,1] \times [0,\tau]$ and can compute the solution for $(x,t) \in [0,1] \times [\tau, 2\tau]$ and so on. Then this method of steps yields the existence and uniqueness results for all $(x,t) \in \bar{D}$. The existence and uniqueness of a solution of (1) can be established under the assumption that the data are Hölder continuous and also satisfy appropriate compatibility conditions at the corner points $(0,0)$, $(1,0)$, $(0,-\tau)$ and $(1,-\tau)$. Then the required compatibility conditions are

$$\phi_b(0,0) = \phi_l(0), \quad \phi_b(1,0) = \phi_r(0), \tag{4}$$

and

$$\begin{aligned} &\frac{d\phi_l(0)}{dt} - \varepsilon \frac{\partial^2 \phi_b(0,0)}{\partial x^2} + a(0,0)\phi_b(0,0) = -b(0,0)\phi_b(0,-\tau) + f(0,0), \\ &\frac{d\phi_r(0)}{dt} - \varepsilon \frac{\partial^2 \phi_b(1,0)}{\partial x^2} + a(1,0)\phi_b(1,0) = -b(1,0)\phi_b(1,-\tau) + f(1,0). \end{aligned} \tag{5}$$

Note that $\phi_l(t)$, $\phi_b(x,t)$ and $\phi_r(t)$ are assumed to be smooth for (5) to make sense, namely, $\phi_l \in C^1([0,T])$, $\phi_b \in C^{(2,1)}(\Gamma_b)$, $\phi_r \in C^1([0,T])$.

The following classical theorem gives sufficient conditions for the existence of a unique solution (see [9]).

Theorem 2. *Let* $a, b, f \in C^{\alpha}(\bar{D})$, $\phi_l \in C^{1+\alpha/2}([0,T])$, $\phi_b \in C^{(2+\alpha, 1+\alpha/2)}(\Gamma_b)$, $\phi_r \in C^{1+\alpha/2}([0,T])$, *and assume that the compatibility conditions* (4) *and* (5) *are fulfilled. Then* (1) *has a unique solution* u_ε *and* $u_\varepsilon \in C^{(2+\alpha,1+\alpha/2)}(\bar{D})$.

3. Bounds on the solution and its derivatives

The error estimate for the fitted mesh finite difference method, which will be described below, is proved under the assumption that the solution of (1) is more regular than is guaranteed by the result in Theorem 2. To obtain this greater regularity, stronger compatibility conditions are imposed at the corners.

For sufficiently small t, $t \leqslant \tau$, the function $u_\varepsilon(x,t)$ is a solution of an initial-boundary value problem for the parabolic equation

$$\begin{aligned} &L_\varepsilon u_\varepsilon(x,t) = F(x,t), \quad (x,t) \in D, \quad t \leqslant \tau, \\ &u_\varepsilon(x,t) = \phi_l(t), \quad (x,t) \in \Gamma_l, \quad u_\varepsilon(x,t) = \phi_r(t), \quad (x,t) \in \Gamma_r, \\ &u_\varepsilon(x,t) = \Phi(x), \quad (x,t) \in \Gamma_0, \quad t \leqslant \tau. \end{aligned} \tag{6}$$

Here $\Gamma_0 = [0,1] \times \{t=0\}$, $F(x,t) = f(x,t) - b(x,t)\phi_b(x,t-\tau)$, $(x,t) \in \overline{D}$, $\Phi(x) = \phi_b(x,0)$, $x \in [0,1]$.

Assume that, for the data of problem (6), compatibility conditions are fulfilled [9], which ensure the required smoothness of u_ε in a neighbourhood of the set $S^* = \{(0,0) \cup (1,0)\}$

$$u_\varepsilon \in C^{\beta,\beta/2}(\overline{D}^{\delta}), \tag{7}$$

where $\overline{D}^{\delta}$ is an δ-neighbourhood of the set S^*, δ is a sufficiently small number, β is a parameter that ensures the required smoothness of the solution. The existence of a smooth solution for the problem is now established in the following theorem:

Theorem 3. *Let the data* $a, b, f \in C^{(2+\alpha,1+\alpha/2)}(\bar{D})$, $\phi_1 \in C^{2+\alpha/2}([0,T])$, $\phi_b \in C^{(4+\alpha,2+\alpha/2)}(\Gamma_b)$, $\phi_r \in C^{2+\alpha/2}([0,T])$, $\alpha \in (0,1)$, *and assume that the condition* (7), *where* $\beta = 4$, *is fulfilled. Then* (1) *has a unique solution* u_ε *and* $u_\varepsilon \in C^{(4+\alpha,2+\alpha/2)}(\bar{D})$. *Furthermore, the derivatives of the solution* u_ε *satisfy, for all non-negative integers* i, j *such that* $0 \leqslant i + 2j \leqslant 4$,

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leqslant C\varepsilon^{-i/2},$$

where the constant C *is independent of* ε.

Proof. The proof of the first part is given in Ladyzhenskaya et al. [9, Chapter IV, p. 320]. The bounds on the derivatives are obtained as follows. By transforming the variable x to the stretched variable $\tilde{x} = x/\sqrt{\varepsilon}$ the problem (1) is transformed to the problem

$$\left(\frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{a}\tilde{u}\right)(\tilde{x}, t) = -\tilde{b}(\tilde{x}, t)\tilde{u}(\tilde{x}, t-\tau) + \tilde{f}(\tilde{x}, t), \quad (\tilde{x}, t) \in \tilde{D}_\varepsilon,$$
$$\tilde{u}(\tilde{x}, t) = \phi_{1,\varepsilon}(t), \quad (\tilde{x}, t) \in \tilde{\Gamma}_{1,\varepsilon}, \quad \tilde{u}(\tilde{x}, t) = \phi_{r,\varepsilon}(t), \quad (\tilde{x}, t) \in \tilde{\Gamma}_{r,\varepsilon}, \tag{8}$$
$$\tilde{u}(\tilde{x}, t) = \phi_{b,\varepsilon}(\tilde{x}, t), \quad (\tilde{x}, t) \in \tilde{\Gamma}_{b,\varepsilon},$$

where $\tilde{D}_\varepsilon = (0, 1/\sqrt{\varepsilon}) \times (0, T]$ and $\tilde{\Gamma}_\varepsilon$ is its boundary analogous to Γ. The differential equation in (8) is independent of ε. Applying the estimate (10.5) from [9, p. 352] gives, for all non-negative integers i, j such that $0 \leqslant i + 2j \leqslant 4$, and all $\tilde{N}_\delta$ in $\tilde{D}_\varepsilon$,

$$\left\| \frac{\partial^{i+j} \tilde{u}}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leqslant C(1 + \|\tilde{u}\|_{\tilde{N}_{2\delta}}).$$

Here the constant C is independent of $\tilde{N}_\delta$ where, for any $\lambda > 0$, $\tilde{N}_\lambda$ is a neighbourhood of diameter λ in $\tilde{D}_\varepsilon$. Returning to the original variable x yields

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leqslant C\varepsilon^{-i/2}(1 + \|u_\varepsilon\|_{\bar{D}}).$$

The proof is completed by using the bound on u_ε in Theorem 1. □

The bounds on the derivatives of the solution given in Theorem 3 were derived from classical results. It turns out, however, that they are not adequate for the proof of the ε-uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in [16]. The key step is to decompose the solution u_ε into its smooth and singular components.

Let u_ε be the solution of (1) and write

$$u_\varepsilon = v_\varepsilon + w_\varepsilon, \tag{9}$$

where v_ε and w_ε are smooth and singular components of u_ε defined in the following way. The smooth component is further decomposed into the sum

$$v_\varepsilon = v_0 + \varepsilon v_1,$$

where v_0, v_1 are defined by

$$\frac{\partial v_0}{\partial t} + a v_0 = -b v_0(x, t-\tau) + f, \quad (x, t) \in D,$$

$$\upsilon_0(x,t)=\phi_b(x,t),\quad (x,t)\in\Gamma_b,$$

$$L_\varepsilon\upsilon_1=-b\upsilon_1(x,t-\tau)+\frac{\partial^2\upsilon_0}{\partial x^2},\quad (x,t)\in D,$$

$$\upsilon_1(x,t)=0,\quad (x,t)\in\Gamma.$$

The function υ_0 is the solution of the reduced problem. Furthermore υ_ε satisfies

$$L_\varepsilon\upsilon_\varepsilon=-b\upsilon_\varepsilon(x,t-\tau)+f,\quad (x,t)\in D,\ \ \upsilon_\varepsilon=\phi_b(x,t),\ \ (x,t)\in\Gamma_b,$$

$$\upsilon_\varepsilon(0,t)=\upsilon_0(0,t),\quad (x,t)\in\Gamma_\mathrm{l},\ \ \upsilon_\varepsilon(1,t)=\upsilon_0(1,t),\ \ (x,t)\in\Gamma_\mathrm{r}.$$

With υ_ε thus defined, it follows that w_ε is determined and that it satisfies

$$L_\varepsilon w_\varepsilon=-bw_\varepsilon(x,t-\tau),\quad (x,t)\in D,\ \ w_\varepsilon(x,t)=0,\ \ (x,t)\in\Gamma_b,$$

$$w_\varepsilon(0,t)=\phi_\mathrm{l}(t)-\upsilon_0(0,t),\quad (x,t)\in\Gamma_\mathrm{l},\ \ w_\varepsilon(1,t)=\phi_\mathrm{r}(t)-\upsilon_0(1,t),\ \ (x,t)\in\Gamma_\mathrm{r}.$$

It is also convenient to write

$$w_\varepsilon=w_\mathrm{l}+w_\mathrm{r},$$

where w_l and w_r are defined by

$$L_\varepsilon w_\mathrm{l}=-bw_\mathrm{l}(x,t-\tau),\quad (x,t)\in D,$$

$$w_\mathrm{l}(0,t)=\phi_\mathrm{l}(t)-\upsilon_0(0,t),\quad (x,t)\in\Gamma_\mathrm{l},\ \ w_\mathrm{l}=0,\ \ (x,t)\in\Gamma_b\cup\Gamma_\mathrm{r},$$

$$L_\varepsilon w_\mathrm{r}=-bw_\mathrm{r}(x,t-\tau),\quad (x,t)\in D,$$

$$w_\mathrm{r}(1,t)=\phi_\mathrm{r}(t)-\upsilon_0(1,t),\quad (x,t)\in\Gamma_\mathrm{r},\ \ w_\mathrm{r}=0,\ \ (x,t)\in\Gamma_\mathrm{l}\cup\Gamma_b.$$

It is clear that w_l and w_r correspond, respectively, to the boundary layers on Γ_l and Γ_r.

For simplicity, we assume that the following condition holds

$$\frac{\partial^i}{\partial x^i}\Phi(x)=0,\quad i\leqslant l,\ \ \frac{\partial^{i+j}}{\partial x^i\partial t^j}F(x,t)=0,\ \ i+2j\leqslant l-2,\ \ (x,t)\in S^*. \tag{10}$$

The required non-classical bounds on υ_ε and w_ε, and their derivatives, are established in the following theorem:

Theorem 4. *Assume that* $a,b,f\in C^{(4+\alpha,2+\alpha/2)}(\overline{D})$, $\phi_\mathrm{l}\in C^{3+\alpha/2}([0,T])$, $\phi_\mathrm{r}\in C^{3+\alpha/2}([0,T])$, $\phi_b\in C^{(6+\alpha,3+\alpha/2)}(\Gamma_b)$, $\alpha\in(0,1)$, *and let the conditions* (7) *and* (10), *where* $\beta=l=6$, *be satisfied. Then, for integers* i,j *such that* $0\leqslant i+2j\leqslant 4$, *we have the estimates*

$$\left\|\frac{\partial^{i+j}\upsilon_\varepsilon}{\partial x^i\partial t^j}\right\|_{\overline{D}}\leqslant C(1+\varepsilon^{1-i/2}), \tag{11}$$

$$\left|\frac{\partial^{i+j}w_\mathrm{l}(x,t)}{\partial x^i\partial t^j}\right|\leqslant C\varepsilon^{-i/2}\mathrm{e}^{-x/\sqrt{\varepsilon}}, \tag{12}$$

$$\left|\frac{\partial^{i+j}w_\mathrm{r}(x,t)}{\partial x^i\partial t^j}\right|\leqslant C\varepsilon^{-i/2}\mathrm{e}^{-(1-x)/\sqrt{\varepsilon}},\quad (x,t)\in\overline{D}, \tag{13}$$

where C is independent of ε.

Proof. See [9, Chapter 4] for the existence and regularity results. The bounds on the functions and their derivatives are proved as follows.

The reduced solution υ_0 is the solution of a first-order differential equation and a classical argument leads to the estimate

$$\left\| \frac{\partial^{i+j}\upsilon_0}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leqslant C. \tag{14}$$

Furthermore, the function υ_1 is the solution of a problem of the form to which Theorem 3 applies. It follows that

$$\left\| \frac{\partial^{i+j}\upsilon_1}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leqslant C\varepsilon^{-i/2}. \tag{15}$$

Since

$$\frac{\partial^{i+j}\upsilon_\varepsilon}{\partial x^i \partial t^j} = \frac{\partial^{i+j}\upsilon_0}{\partial x^i \partial t^j} + \varepsilon \frac{\partial^{i+j}\upsilon_1}{\partial x^i \partial t^j},$$

the required estimates of the smooth component υ_ε and its derivatives follow by using (14) and (15).

The required bounds on w_l and w_r and their derivatives can be obtained analogously. The proof is, therefore, only given for w_l and its derivatives. To bound w_l, define

$$\psi^{\pm}(x,t) = C\mathrm{e}^{-x/\sqrt{\varepsilon}}\mathrm{e}^{\alpha t} \pm w_l(x,t).$$

Then, if C is chosen sufficiently large and $\alpha \geqslant 0$,

$$\psi^{\pm}(x,0) = C\mathrm{e}^{-x/\sqrt{\varepsilon}} \geqslant 0,$$

$$\psi^{\pm}(0,t) = C\mathrm{e}^{\alpha t} \pm (\phi_l(t) - \upsilon_0(0,t)) \geqslant 0,$$

$$\psi^{\pm}(1,t) = C\mathrm{e}^{-1/\sqrt{\varepsilon}}\mathrm{e}^{\alpha t} \geqslant 0,$$

and

$$L_\varepsilon \psi^{\pm}(x,t) = C(a - 1 + \alpha)\mathrm{e}^{-x/\sqrt{\varepsilon}}\mathrm{e}^{\alpha t} \geqslant 0$$

if α is chosen as in Theorem 1 to be $\alpha = \max_{\bar{D}}\{0, 1-a\}$. It follows from the maximum principle that for all $(x,t) \in \bar{D}$

$$|w_l(x,t)| \leqslant C\mathrm{e}^{-x/\sqrt{\varepsilon}}\mathrm{e}^{\alpha t} \leqslant C\mathrm{e}^{-x/\sqrt{\varepsilon}}$$

as required.

The bounds on the derivatives of w_l are obtained as follows. First, a transformation is made from x to the stretched variable $\tilde{x} = x/\sqrt{\varepsilon}$. Using the variables $(\tilde{x}, t)$ the parameter ε does not enter into the differential equation and so the appropriate results in [9, Section 4.10] are applicable to its solution $\tilde{w}_l$. Note that the domain of the stretched variable $\tilde{x}$ is clearly $(0, 1/\sqrt{\varepsilon})$. The argument divides into two cases corresponding to the position of $\tilde{x}$. For each neighbourhood $\tilde{N}_\delta$ in $(2, 1/\sqrt{\varepsilon}) \times (0, T]$, from [9, Section 4.10] we have

$$\left\| \frac{\partial^{i+j}\tilde{w}_l}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leqslant C\|\tilde{w}_l\|_{\tilde{N}_{2\delta}},$$

and the required bound follows by transforming back to the variable x and using the bound just obtained on w_l.

Likewise, for each neighbourhood $\tilde{N}_\delta$ in $(0, 2] \times (0, T]$, from [9, Section 4.10] we have

$$\left\| \frac{\partial^{i+j}\tilde{w}_l}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leqslant C(1 + \|\tilde{w}_l\|_{\tilde{N}_{2\delta}}),$$

and the required bound follows by again transforming back to the variable x, using the bound on w_l and noting that $\mathrm{e}^{-x/\sqrt{\varepsilon}} \geqslant \mathrm{e}^{-2} = C$ for $\tilde{x} \leqslant 2$. This completes the proof. □

4. Numerical method

In this section, problem (1) is discretised using a fitted numerical method composed of a standard finite difference operator on a fitted piecewise uniform mesh. The finite difference operator has a centred difference quotient in space and a backward difference quotient in time. The fitted piecewise uniform mesh is constructed by dividing $\bar{\Omega}$ into three subintervals

$$\bar{\Omega} = \bar{\Omega}_l \cup \bar{\Omega}_c \cup \bar{\Omega}_r,$$

where $\Omega_l = (0, \sigma)$, $\Omega_c = (\sigma, 1-\sigma)$, $\Omega_r = (1-\sigma, 1)$, and the fitting factor σ is chosen to satisfy

$$\sigma = \min\{\tfrac{1}{4}, 2\sqrt{\varepsilon}\ln N_x\}, \tag{16}$$

where N_x denotes the number of mesh elements used in the x-direction. The multi-index notation $N = (N_x, N_t)$ is also used, where N_t is the number of mesh elements in the t-direction.

A piecewise uniform mesh $\Omega_\sigma^{N_x}$ on Ω with N_x mesh elements ($N_x \geqslant 4$) is obtained by placing a uniform mesh with $N_x/4$ mesh elements on both Ω_l and Ω_r and a uniform mesh with $N_x/2$ mesh elements on Ω_c. Uniform meshes with step-size Δt, Ω^{N_t} and Ω^{m_τ} with N_t and m_τ mesh elements are used on $[0, T]$ and $[-\tau, 0)$, respectively. The fitted piecewise uniform meshes D_σ^N on D and $\Gamma_{b,\sigma}^N$ on Γ_b are then defined as the tensor products

$$D_\sigma^N = \Omega_\sigma^{N_x} \times \Omega^{N_t}, \quad \Gamma_{b,\sigma}^N = \Omega_\sigma^{N_x} \times \Omega^{m_\tau}$$

and the boundary points Γ_σ^N of D_σ^N are $\Gamma_\sigma^N = \bar{D}_\sigma^N \cap \Gamma$. We put $\Gamma_{l,\sigma}^N = \Gamma_\sigma^N \cap \Gamma_l$ and $\Gamma_{r,\sigma}^N = \Gamma_\sigma^N \cap \Gamma_r$. Note that, whenever $\sigma = \frac{1}{4}$ the mesh is uniform and on the other hand when $\sigma = 2\sqrt{\varepsilon}\ln N_x$ the mesh is condensing near the edges Γ_l and Γ_r.

The resulting fitted mesh finite difference method for (1) is then

$$\begin{aligned}
&L_\varepsilon^N U_\varepsilon \equiv D_t^- U_\varepsilon - \varepsilon\delta_x^2 U_\varepsilon + aU_\varepsilon = -bU_\varepsilon(x_i, t_{j-m_\tau}) + f, \quad (x_i, t_j) \in D_\sigma^N,\\
&U_\varepsilon(x_0, t_j) = \phi_l(t_j), \quad (x_0, t_j) \in \Gamma_{l,\sigma}^N, \quad U_\varepsilon(x_{N_x}, t_j) = \phi_r(t_j), \quad (x_{N_x}, t_j) \in \Gamma_{r,\sigma}^N,\\
&U_\varepsilon(x_i, t_j) = \phi_b(x_i, t_j), \quad (x_i, t_j) \in \Gamma_{b,\sigma}^N,
\end{aligned} \tag{17}$$

where the step length Δt satisfies the constraint $\tau = m_\tau \Delta t$, where m_τ is a positive integer, $t_j = j\Delta t$, $j \geqslant -m_\tau$, and for any mesh function $V_{i,j} = V(x_i, t_j)$

$$\delta_x^2 V_{i,j} = \frac{(D_x^+ - D_x^-)V_{i,j}}{(x_{i+1} - x_{i-1})/2}$$

with

$$D_x^+ V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{x_{i+1} - x_i}, \quad D_x^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{x_i - x_{i-1}}$$

and an analogous definition of D_t^-.

The finite difference operator L_ε^N in (17)

$$L_\varepsilon^N = D_t^- - \varepsilon\delta_x^2 + aI,$$

satisfies the following well-known discrete minimum principle on $\bar{D}_\sigma^N$.

Discrete minimum principle. *Assume that Ψ satisfies $\Psi \geqslant 0$ on Γ_σ^N. Then $L_\varepsilon^N \Psi \geqslant 0$ on D_ε^N implies that $\Psi \geqslant 0$ at each point of $\bar{D}_\sigma^N$.*

An immediate consequence of the discrete minimum principle is the following ε-uniform stability property of the operator L_ε^N.

Lemma 1. *If Z is any mesh function in the domain of definition of the finite difference operator L_ε^N in* (17). *Then*

$$\|Z\| \leqslant (1 + \alpha T)\max\{\|L_\varepsilon^N Z\|, \|Z\|_{\Gamma_\sigma^N}\}.$$

5. Convergence of the numerical method

The ε-uniform error estimate for the numerical solution is contained in the following theorem.

Theorem 5. *Let the estimates* (11)–(13) *be fulfilled for the components defined in* (9). *Then the fitted mesh finite difference method* (17) *consisting of the standard finite difference operator* L_ε^N *and the fitted piecewise uniform mesh* $D_\sigma^N \cup \Gamma_{b,\sigma}^N$, *condensing in neighbourhoods of the edges* Γ_l *and* Γ_r, *is ε-uniform for problem* (1) *provided that the fitting factor σ is chosen according to the formula* (16) *above. Moreover, the solution* u_ε *of* (1) *and the numerical solution* U_ε *of* (17) *satisfy the following ε-uniform error estimate for all* $N_x \geqslant 4$:

$$\sup_{0<\varepsilon\leqslant 1} \|U_\varepsilon - u_\varepsilon\|_{\bar{D}_\sigma^N} \leqslant C((N_x^{-1}\ln N_x)^2 + N_t^{-1}),$$

where C is a constant independent of N_x, N_t *and* ε.

Proof. We start by noting that on the first interval $[0,\tau]$, the right-hand side of (1) becomes $f(x,t)-b(x,t)\phi_b(x,t-\tau)$, being independent of ε, and hence the result in [11] is applicable to the difference scheme and we thus obtain

$$\sup_{0<\varepsilon\leqslant 1} \|U_\varepsilon - u_\varepsilon\|_{\bar{D}_{\sigma,\tau}^N} \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}), \tag{18}$$

where $\bar{D}_{\sigma,\tau}^N = \Omega_\sigma^{N_x} \times \Omega_1^{N_\tau}$ ($\Omega_1^{N_\tau}$ is the uniform mesh with $N_\tau = m_\tau$ mesh elements used on $[0,\tau)$) and C is independent of N_x, N_τ and ε.

The solution $U_\varepsilon(x_i,t_j)$ found on $\bar{D}_{\sigma,\tau}^N$ is denoted by $U_{\varepsilon,\tau}(x_i,t_j)$, $(x_i,t_j)\in \bar{D}_{\sigma,\tau}^N$.

On the second interval $[\tau,2\tau]$, and also for $t\geqslant 2\tau$, it is not possible to apply immediately the result of [11] due to the fact that the delay term $u_\varepsilon(x,t-\tau)$ depends on ε. For this reason, we examine the detailed proof of the estimate for the difference between the numerical solution U_ε and the solution u_ε itself over the interval $[\tau,2\tau]$.

Consider the following singularly perturbed delay parabolic equation:

$$\begin{aligned} &L_\varepsilon u_\varepsilon(x,t) \equiv \left(\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} + a u_\varepsilon\right)(x,t) = -b(x,t)u_\varepsilon(x,t-\tau) + f(x,t), \\ &(x,t)\in D_2 = (0,1)\times(\tau,2\tau], \quad u_\varepsilon(x,\tau) = u_\varepsilon(x,t_{m_\tau}), \quad x\in\Omega, \\ &u_\varepsilon(0,t) = \phi_\mathrm{l}(t), \quad t\in[\tau,2\tau], \quad u_\varepsilon(1,t)=\phi_\mathrm{r}(t), \quad t\in[\tau,2\tau]. \end{aligned} \tag{19}$$

We determine the numerical solution U_ε of (19) at $(x_i,t_j)\in \bar{D}_{\sigma,2\tau}^N = \Omega_\sigma^{N_x}\times\Omega_2^{N_\tau}$ ($\Omega_2^{N_\tau}$ is the uniform mesh with $N_\tau = m_\tau$ mesh elements used on $[\tau,2\tau)$) by

$$\begin{aligned} &L_\varepsilon^N U_\varepsilon \equiv D_t^- U_\varepsilon - \varepsilon\delta_x^2 U_\varepsilon + aU_\varepsilon = -bU_\varepsilon(x_i,t_{j-m_\tau}) + f, \quad (x_i,t_j)\in D_{\sigma,2\tau}^N, \\ &U_\varepsilon(0,t_j) = \phi_\mathrm{l}(t_j), \quad t_j\in\Omega_2^{N_\tau}, \quad U_\varepsilon(1,t_j) = \phi_\mathrm{r}(t_j), \quad t_j\in\Omega_2^{N_\tau}, \\ &U_\varepsilon(x_i,t_j) = U_{\varepsilon,\tau}(x_i,t_j), \quad (x_i,t_j)\in D_{\sigma,\tau}^N. \end{aligned} \tag{20}$$

The solution u_ε of (19) is decomposed into smooth and singular components $u_\varepsilon = y_\varepsilon + z_\varepsilon$. The smooth component is further decomposed into the sum $y_\varepsilon = y_0 + \varepsilon y_1$, where y_0 and y_1 are defined by

$$\begin{aligned} &\frac{\partial y_0}{\partial t} + a y_0 = -b y_0(x,t-\tau) + f, \quad (x,t)\in D_2, \\ &y_0(x,t) = u_\varepsilon(x,t), \quad (x,t)\in\Omega\times[0,\tau], \\ &L_\varepsilon y_1 = -b y_1(x,t-\tau) + \frac{\partial^2 y_0}{\partial x^2}, \quad (x,t)\in D_2, \\ &y_1(x,t) = 0, \quad (x,t)\in\Omega\times[0,\tau], \quad y_1(0,t) = y_1(1,t) = 0, \quad t\in[\tau,2\tau]. \end{aligned}$$

The function y_0 is the solution of the reduced problem. Furthermore, y_ε satisfies

$$L_\varepsilon y_\varepsilon = -by_\varepsilon(x, t-\tau) + f, \quad (x,t) \in D_2, \ \ y_\varepsilon = u_\varepsilon(x,t), \ \ (x,t) \in \Omega \times [0,\tau],$$

$$y_\varepsilon(0,t) = y_0(0,t), \quad y_\varepsilon(1,t) = y_0(1,t), \ \ t \in [\tau, 2\tau].$$

With y_ε thus defined, it follows that z_ε is determined and that it satisfies

$$L_\varepsilon z_\varepsilon = -bz_\varepsilon(x, t-\tau), \quad (x,t) \in D_2, \ \ z_\varepsilon(x,t) = 0, \ \ (x,t) \in \Omega \times [0,\tau],$$

$$z_\varepsilon(0,t) = \phi_\mathrm{l}(t) - y_0(0,t), \quad z_\varepsilon(1,t) = \phi_\mathrm{r}(t) - y_0(1,t), \ \ t \in [\tau, 2\tau].$$

It is also convenient to write

$$z_\varepsilon = z_\mathrm{l} + z_\mathrm{r},$$

where z_l and z_r are defined by

$$L_\varepsilon z_\mathrm{l} = -bz_\mathrm{l}(x, t-\tau), \quad (x,t) \in D_2, \ \ z_\mathrm{l}(x,t) = 0, \ \ (x,t) \in \Omega \times [0,\tau],$$

$$z_\mathrm{l}(0,t) = \phi_\mathrm{l}(t) - y_0(0,t), \quad z_\mathrm{l}(1,t) = 0, \ \ t \in [\tau, 2\tau],$$

$$L_\varepsilon z_\mathrm{r} = -bz_\mathrm{r}(x, t-\tau), \quad (x,t) \in D_2, \ \ z_\mathrm{r}(x,t) = 0, \ \ (x,t) \in \Omega \times [0,\tau],$$

$$z_\mathrm{r}(1,t) = \phi_\mathrm{r}(t) - y_0(1,t), \quad z_\mathrm{r}(0,t) = 0, \ \ t \in [\tau, 2\tau].$$

The required non-classical bounds on y_ε and z_ε, and their derivatives, are contained in Theorem 4.

The solution U_ε of (20) is decomposed into smooth and singular components in an analogous manner to the decomposition of the solution u_ε of (19). Thus

$$U_\varepsilon = Y_\varepsilon + Z_\varepsilon,$$

where Y_ε is the solution of the inhomogeneous problem

$$L_\varepsilon^N Y_\varepsilon = -bY_\varepsilon(x_i, t_{j-m_\tau}) + f, \quad (x_i, t_j) \in D^N_{\sigma,2\tau}, \ \ Y_\varepsilon = U_{\varepsilon,\tau}, \ \ (x_i,t_j) \in D^N_{\sigma,\tau},$$

$$Y_\varepsilon(0,t_j) = y_\varepsilon(0,t_j), \quad Y_\varepsilon(1,t_j) = y_\varepsilon(1,t_j), \ \ t_j \in \Omega_2^{N_\tau}$$

and, therefore, Z_ε must satisfy

$$L_\varepsilon^N Z_\varepsilon = -bZ_\varepsilon(x_i, t_{j-m_\tau}), \quad (x_i, t_j) \in D^N_{\sigma,2\tau}, \ \ Z_\varepsilon(x_i,t_j) = 0, \ \ (x_i,t_j) \in D^N_{\sigma,\tau},$$

$$Z_\varepsilon(0,t_j) = \phi_\mathrm{l}(t_j) - y_\varepsilon(0,t_j), \quad Z_\varepsilon(1,t_j) = \phi_\mathrm{r}(t_j) - y_\varepsilon(1,t_j), \ \ t_j \in \Omega_2^{N_\tau}.$$

The error can then be written in the form

$$U_\varepsilon - u_\varepsilon = (Y_\varepsilon - y_\varepsilon) + (Z_\varepsilon - z_\varepsilon),$$

and so the smooth and singular components of the error can be estimated separately.

The smooth component of the error is estimated as follows by a classical argument. From the differential and difference equations it is easy to see that

$$\begin{aligned} L_\varepsilon^N(Y_\varepsilon - y_\varepsilon) &= -bY_\varepsilon(x_i, t_{j-m_\tau}) + f - L_\varepsilon^N y_\varepsilon \\ &= b(y_\varepsilon(x_i, t_{j-m_\tau}) - Y_\varepsilon(x_i, t_{j-m_\tau})) + (L_\varepsilon - L_\varepsilon^N)y_\varepsilon \\ &= b(u_\varepsilon(x_i, t_{j-m_\tau}) - U_{\varepsilon,\tau}(x_i, t_{j-m_\tau})) + (L_\varepsilon - L_\varepsilon^N)y_\varepsilon \end{aligned}$$

and so

$$L_\varepsilon^N(Y_\varepsilon - y_\varepsilon) = b(u_\varepsilon(x_i, t_{j-m_\tau}) - U_{\varepsilon,\tau}(x_i, t_{j-m_\tau})) - \varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)y_\varepsilon + \left(\frac{\partial}{\partial t} - D_t^-\right)y_\varepsilon.$$

It follows from (18) and classical estimates (see, for example [10, p. 21]) that, at each point (x_i, t_j) in $D^N_{\sigma,2\tau}$,

$$|L^N_\varepsilon(Y_\varepsilon - y_\varepsilon)(x_i,t_j)| \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}) + \begin{cases} \dfrac{\varepsilon}{3}(x_{i+1}-x_{i-1})\left\|\dfrac{\partial^3 y_\varepsilon}{\partial x^3}\right\| + \dfrac{1}{2}(t_j - t_{j-1})\left\|\dfrac{\partial^2 y_\varepsilon}{\partial t^2}\right\| & \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma, \\ \dfrac{\varepsilon}{12}(x_i - x_{i-1})^2\left\|\dfrac{\partial^4 y_\varepsilon}{\partial x^4}\right\| + \dfrac{1}{2}(t_j - t_{j-1})\left\|\dfrac{\partial^2 y_\varepsilon}{\partial t^2}\right\| & \text{otherwise.}\end{cases}$$

Using the estimates of the derivatives of y_ε in Theorem 4, and since $x_i - x_{i-1} \leqslant 2N_x^{-1}$, $x_{i+1} - x_{i-1} \leqslant 4N_x^{-1}$ and $t_j - t_{j-1} \leqslant N_\tau^{-1}$, we obtain

$$|L^N_\varepsilon(Y_\varepsilon - y_\varepsilon)(x_i,t_j)| \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}) + \begin{cases} C[\sqrt{\varepsilon}N_x^{-1} + N_\tau^{-1}] & \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma, \\ C(N_x^{-2} + N_\tau^{-1}) & \text{otherwise.}\end{cases}$$

The proof of the estimate for the smooth component of the error is completed as in [11] by introducing barrier functions and applying the discrete minimum principle, and it follows that

$$|Y_\varepsilon - y_\varepsilon| \leqslant C(N_x^{-2}\ln N_x + N_\tau^{-1}). \tag{21}$$

To estimate the singular component of the error, in an analogous way to that for z_ε, the singular component Z_ε is written in the form

$$Z_\varepsilon = Z_\mathrm{l} + Z_\mathrm{r},$$

where Z_l and Z_r are defined by

$$L^N_\varepsilon Z_\mathrm{l} = -bZ_\mathrm{l}(x_i, t_{j-m_\tau}), \quad (x_i,t_j) \in D^N_{\sigma,2\tau}, \quad Z_\mathrm{l}(x_i,t_j) = 0, \quad (x_i,t_j) \in D^N_{\sigma,\tau},$$

$$Z_\mathrm{l}(0,t_j) = \phi_\mathrm{l}(t_j) - y_0(0,t_j), \quad Z_\mathrm{l}(1,t_j) = 0, \quad t_j \in \Omega_2^{N_\tau},$$

$$L^N_\varepsilon Z_\mathrm{r} = -bZ_\mathrm{r}(x_i, t_{j-m_\tau}), \quad (x_i,t_j) \in D^N_{\sigma,2\tau}, \quad Z_\mathrm{r}(x_i,t_j) = 0, \quad (x_i,t_j) \in D^N_{\sigma,\tau},$$

$$Z_\mathrm{r}(0,t_j) = 0, \quad Z_\mathrm{r}(1,t_j) = \phi_\mathrm{r}(t_j) - y_0(1,t_j), \quad t_j \in \Omega_2^{N_\tau}.$$

The error can then be written in the form

$$Z_\varepsilon - z_\varepsilon = (Z_\mathrm{l} - z_\mathrm{l}) + (Z_\mathrm{r} - z_\mathrm{r}),$$

and the errors $Z_\mathrm{l} - z_\mathrm{l}$ and $Z_\mathrm{r} - z_\mathrm{r}$, associated, respectively, with the boundary layers of Γ_l and Γ_r, can be estimated separately.

Consider the error $Z_\mathrm{l} - z_\mathrm{l}$. From the differential and difference equations it is easy to see that

$$L^N_\varepsilon(Z_\mathrm{l} - z_\mathrm{l}) = (L_\varepsilon - L^N_\varepsilon)z_\mathrm{l} = -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)z_\mathrm{l} + \left(\frac{\partial}{\partial t} - D_t^-\right)z_\mathrm{l}. \tag{22}$$

A classical estimate gives the estimate for all $(x_i, t_j) \in D^N_{\sigma,2\tau}$

$$|L^N_\varepsilon(Z_\mathrm{l} - z_\mathrm{l})(x_i,t_j)| \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}).$$

Using Lemma 1 then gives for all $(x_i, t_j) \in \bar{D}^N_{\sigma,2\tau}$

$$|(Z_\mathrm{l} - z_\mathrm{l})(x_i,t_j)| \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}). \tag{23}$$

For more details we refer to [11]. Completely analogous arguments lead to the estimate for the error corresponding to the boundary layer for all $(x_i, t_j) \in \Gamma_\mathrm{r}$

$$|(Z_\mathrm{r} - z_\mathrm{r})(x_i,t_j)| \leqslant C((N_x^{-1}\ln N_x)^2 + N_\tau^{-1}). \tag{24}$$

Combining (21), (23) and (24) completes the proof on the second interval $[\tau, 2\tau]$ and we can prove the theorem by induction. □

6. Numerical experiments

In this section, we show the results obtained for two examples.

Example 1.

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t}-\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}=-2\mathrm{e}^{-1}u_\varepsilon(x,t-1),\quad (x,t)\in(0,1)\times(0,2],\\ &u_\varepsilon(x,t)=\mathrm{e}^{-(t+x/\sqrt{\varepsilon})},\quad (x,t)\in[0,1]\times[-1,0],\\ &u_\varepsilon(0,t)=\mathrm{e}^{-t},\quad u_\varepsilon(1,t)=\mathrm{e}^{-(t+1/\sqrt{\varepsilon})},\quad t\in(0,2].\end{aligned}\tag{25}$$

The exact solution of this problem is $u_\varepsilon(x,t)=\mathrm{e}-(t+x/\sqrt{\varepsilon})$. It is clear that there is a parabolic boundary layer in a neighbourhood of Γ_1, but because of the boundary values there is no boundary layer on Γ_r.

In what follows, the problem is solved using numerical method (17) comprising standard finite difference operators (centred in space, implicit in time) on either uniform meshes with $N_x\times N_t$ elements or fitted meshes with $N_x\times N_t$ elements. The fitted meshes used in these computations are of the form described in Section 4, and so they condense on both Γ_1 and Γ_r. But because there is no boundary layer on Γ_r, there is no need for the mesh to condense on Γ_r. This means that equally good numerical results could have been obtained for this problem using a mesh condensing on Γ_1 alone and, therefore, requiring fewer mesh points. The reasons for not removing the mesh condensation on Γ_r was because the available code was written for the more general case and the optimal mesh was not investigated. In the remainder of this section, it is assumed that $N_x=N_t=N$ (note that $m_\tau=N_t/2$ and $\Delta t=1/m_\tau$).

The maximum errors $E(\varepsilon,N)=\max_{t_j}(\max_{x_i}|u_\varepsilon(x_i,t_j)-U_\varepsilon(x_i,t_j)|)$ and the convergence rates $R(\varepsilon,N)=\log_2(E(\varepsilon,N)/E(\varepsilon,2N))$ in the numerical solutions using meshes with $N=64$, 128, 256, 512, and 1024 and values of ε from 2^{-4} to 2^{-30} are presented in Table 1.

The last row of the table contains $E(N)=\max_\varepsilon E(\varepsilon,N)$ occurring in the rows above it. Since these maxima occur along a diagonal of the table, and do not decrease significantly as N increases, it is clear that there is a persistent maximum error of about 2.7% no matter how large N is. This shows numerically that this numerical method is not ε-uniform. Another feature of this behaviour is that when a value of ε is chosen that is below the diagonal, then the error grows with increasing N until the diagonal is reached. This behaviour is not in accord with the properties expected of a satisfactory numerical method.

On the other hand, the analogous results on the appropriate fitted meshes are presented in Table 2. In this table, the maxima of the columns occur in the row corresponding to $\varepsilon=2^{-12}$ and these maxima decrease rapidly as N increases. This behaviour is in complete agreement with the theoretical result in Theorem 5. Note that with this ε-uniform method, when $N=64$, the maximum error in that column is less than 0.5%, which cannot be achieved for any value of N using a uniform mesh.

Example 2.

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t}-\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}+\frac{1+x^2}{2}u_\varepsilon=t^3-u_\varepsilon(x,t-1),\quad (x,t)\in(0,1)\times(0,2],\\ &u_\varepsilon(x,t)=0,\quad (x,t)\in[0,1]\times[-1,0],\quad u_\varepsilon(0,t)=u_\varepsilon(1,t)=0,\quad t\in(0,2],\end{aligned}\tag{26}$$

for which the exact solution is unknown.

Fig. 1 shows an approximation of the solution for $\varepsilon=2^{-20}$. Now, for each value of ε, we estimate the maximum errors by

$$e_\varepsilon^{N,\Delta t}=\max_{t_j}\left(\max_{x_i}|\tilde{U}^{2N,\Delta t/2}(x_{2i},t_j)-U^{N,\Delta t}(x_i,t_j)|\right),$$

Table 1
Maximum errors and the convergence rates using classical uniform meshes

$\varepsilon\backslash N$	64	128	256	512	1024
2^{-4}	2.035E −03	1.013E −03	5.054E −04	2.524E −04	1.261E−04
	1.006	1.003	1.002	1.001	
2^{-6}	2.158E−03	1.045E−03	5.138E−04	2.547E−04	1.268E−04
	1.046	1.025	1.013	1.006	
2^{-8}	2.628E−03	1.169E−03	5.449E−04	2.625E−04	1.287E−04
	1.169	1.101	1.054	1.028	
2^{-10}	4.505E−03	1.652E−03	6.696E−04	2.938E−04	1.366E−04
	1.447	1.303	1.188	1.105	
2^{-12}	1.144E−02	3.571E−03	1.161E−03	4.191E−04	1.680E−04
	1.680	1.621	1.470	1.319	
2^{-14}	2.642E−02	1.067E−02	3.100E−03	9.141E−04	2.937E−04
	1.309	1.783	1.762	1.638	
2^{-16}	2.611E−02	2.619E−02	1.027E−02	2.864E−03	7.905E−04
	−.004	1.350	1.843	1.857	
2^{-18}	1.021E−02	2.630E−02	2.607E−02	1.008E−02	2.746E−03
	−1.365	0.013	1.372	1.876	
2^{-20}	2.664E−03	1.032E−02	2.640E−02	2.601E−02	9.977E−03
	−1.953	−1.355	0.021	1.383	
2^{-22}	6.697E−04	2.693E−03	1.037E−02	2.645E−02	2.598E−02
	−2.008	−1.946	−1.351	0.026	
2^{-24}	1.677E−04	6.768E−04	2.707E−03	1.040E−02	2.648E−02
	−2.013	−2.000	−1.942	−1.348	
2^{-26}	4.193E−05	1.694E−04	6.804E−04	2.714E−03	1.041E−02
	−2.015	−2.006	−1.996	−1.940	
2^{-28}	1.048E−05	4.237E−05	1.703E−04	6.822E−04	2.718E−03
	−2.015	−2.007	−2.002	−1.994	
2^{-30}	2.621E−06	1.059E−05	4.260E−05	1.708E−04	6.831E−04
	−2.015	−2.007	−2.003	−2.000	
$E(N)$	2.642E−02	2.630E−02	2.640E−02	2.645E−02	2.648E−02

Table 2
Maximum errors and the convergence rates using fitted piecewise uniform meshes

$\varepsilon\backslash N$	64	128	256	512	1024
2^{-4}	2.035E − 03	1.013E − 03	5.054E − 04	2.524E − 04	1.261E − 04
	1.006	1.003	1.002	1.001	
2^{-6}	2.158E − 03	1.045E − 03	5.138E − 04	2.547E − 04	1.268E − 04
	1.046	1.025	1.013	1.006	
2^{-8}	2.628E − 03	1.169E − 03	5.449E − 04	2.625E − 04	1.287E − 04
	1.169	1.101	1.054	1.028	
2^{-10}	4.505E − 03	1.652E − 03	6.696E − 04	2.938E − 04	1.366E − 04
	1.447	1.303	1.188	1.105	
2^{-12}	4.718E − 03	1.959E − 03	8.212E − 04	3.536E − 04	1.576E − 04
	1.268	1.254	1.216	1.166	
2^{-14}	4.718E − 03	1.959E − 03	8.212E − 04	3.536E − 04	1.576E − 04
	1.268	1.254	1.216	1.166	
...	...	...	...	...	...
...	...	...	...	...	...
2^{-30}	4.718E − 03	1.959E − 03	8.212E − 04	3.536E − 04	1.576E − 04
	1.268	1.254	1.216	1.166	
$E(N)$	4.718E − 03	1.959E − 03	8.212E − 04	3.536E − 04	1.576E − 04

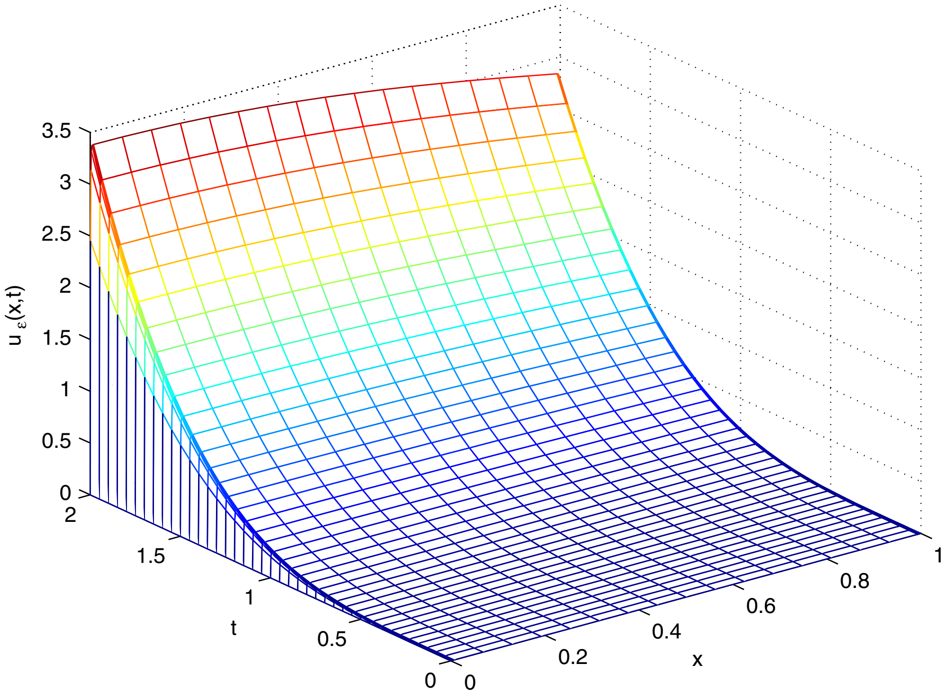

Fig. 1. Numerical solution of (26) for $\varepsilon = 2^{-20}$ taking $N = 32$ and $\Delta t = 0.05$.

Table 3
Maximum errors and the convergence rates for problem (26)

$\varepsilon \backslash N\Delta t$	$N = 64$, $\Delta t = 0.1$	$N = 128$, $\Delta t = 0.1/2$	$N = 256$, $\Delta t = 0.1/2^2$	$N = 512$, $\Delta t = 0.1/2^3$	$N = 1024$, $\Delta t = 0.1/2^4$
2^{-4}	7.205E − 02	3.675E − 02	1.856E − 02	9.325E − 03	4.674E − 03
	0.971	0.986	0.993	0.996	
2^{-6}	1.007E − 01	5.094E − 02	2.562E − 02	1.285E − 02	6.434E − 03
	0.982	0.991	0.996	0.998	
2^{-8}	1.087E − 01	5.487E − 02	2.756E − 02	1.381E − 02	6.912E − 03
	0.986	0.994	0.997	0.998	
2^{-10}	1.118E − 01	5.637E − 02	2.829E − 02	1.417E − 02	7.093E − 03
	0.988	0.994	0.997	0.999	
2^{-12}	1.130E − 01	5.690E − 02	2.855E − 02	1.430E − 02	7.156E − 03
	0.990	0.995	0.997	0.999	
2^{-14}	1.134E − 01	5.710E − 02	2.864E − 02	1.434E − 02	7.177E − 03
	0.990	0.995	0.998	0.999	
2^{-16}	1.136E − 01	5.716E − 02	2.867E − 02	1.436E − 02	7.183E − 03
	0.991	0.996	0.998	0.999	
...	...	...	...	...	...
...	...	...	...	...	...
2^{-30}	1.136E − 01	5.718E − 02	2.868E − 02	1.436E − 02	7.186E − 03
	0.991	0.996	0.998	0.999	
$E(N)$	1.136E − 01	5.718E − 02	2.868E − 02	1.436E − 02	7.186E − 03

where $\tilde{U}^{2N,\Delta t/2}(x_i, t_j)$ is the numerical solution on a mesh containing the mesh points (x_i, t_j) of D_σ^N and also the midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$, $t_{j+1/2} = (t_j + t_{j+1})/2$, $i, j = 0, 1, \ldots, N-1$. Table 3 displays the results obtained using (17). From this table, we deduce the first order of ε-uniform convergence.

7. Conclusion

A singularly perturbed Dirichlet boundary value problem for a linear delay parabolic differential equation having parabolic boundary layers was formulated. A fitted mesh finite difference method was constructed and was proved to be an ε-uniform method for this problem. Numerical results were presented, which numerically validate this theoretical result and show that a method using the standard finite difference operator on a uniform mesh is not an ε-uniform method.

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