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Classification of Irreducible Modules for the Vertex Operator Algebra $M(1)^+$

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We classify the irreducible modules for the fixed point vertex operator subalgebra of the vertex operator algebra associated to the Heisenberg algebra with central charge 1 under the -1 automorphism.

1. INTRODUCTION

Let $\mathfrak{h}$ be a finite dimensional complex vector space of dimension d with a nondegenerate symmetric bilinear form and let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ be the corresponding affine algebra. Then the free bosonic Fock space $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ is a vertex operator algebra of central charge d (cf. [FLM]). If $d = 1$ the automorphism group of $M(1)$ is $\mathbb{Z}_2$ generated by θ (see Section 2.3). Then $M(1)$ has only two proper subalgebras, namely, $M(1)^+$ and the vertex operator subalgebra generated by the Virasoro algebra [DG]. In this paper we determine Zhu's algebra $A(M(1)^+)$ and classify the irreducible modules for $M(1)^+$. The classification result says that any irreducible $M(1)^+$ module is isomorphic to either a submodule of a $M(1)$ module or a submodule of a θ-twisted $M(1)$ module.

The vertex operator algebra $M(1)^+$ is closely related to W algebra. It was shown in [DG] that $M(1)^+$ is generated by the Virasoro element ω

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and a highest weight vector J of weight 4 (see Section 2). Thus $M(1)^+$ can be regarded as the vertex operator algebra associated to the W algebra $W(2,4)$ with central charge 1 (cf. [BFKNRV]). So this paper also gives a classification of irreducible modules for the W algebra $W(2,4)$ which can be lifted to modules for $M(1)^+$.

The result in this paper is fundamental in the classification of irreducible modules for the vertex operator algebras V_L^+ [DN]. Let L be a positive definite even lattice of rank 1. The corresponding vertex operator algebra V_L is a tensor product of $M(1)$ with the group algebra $\mathbb{C}[L]$. The structure and representation theory of V_L including the fusion rules are well understood (see [B, FLM, D, DL, DLM1]). Then θ can be extended to an automorphism of V_L of order 2. Moreover, the fixed point vertex operator subalgebra V_L^+ contains $M(1)^+$ as a subalgebra and V_L^+ is a completely reducible $M(1)^+$ module. The result of the present paper has been used in [DN] to determine Zhu's algebra $A(V_L^+)$ and to classify the irreducible modules for V_L^+.

It should be pointed out that conformal field theory associated to $M(1)^+$ is an orbifold theory (cf. [DVVV]) for the nonrational vertex operator algebra $M(1)$. Let V be a rational vertex operator algebra and let G be a finite group of automorphisms of V. The orbifold theory conjectures that any irreducible module of the fixed point vertex operator subalgebra V^G is isomorphic to a submodule of a g-twisted module for some $g \in G$. Our result in this paper suggests that it may be true even when V is not rational.

One important tool in the representation theory of vertex operator algebra is Zhu's algebra [Z]. In [Z] it was shown that for any vertex operator algebra V, there is an associative algebra $A(V)$ associated to V such that there is a one to one correspondence between the irreducible admissible V modules and irreducible $A(V)$ modules (see Section 2.2 for more detail). The main idea in the present paper is to determine Zhu's algebra $A(M(1)^+)$, which turns out to be a commutative algebra over $\mathbb{C}$ with two variables.

We should mention an important role played by a generalized PBW-type theorem in this paper. The classical PBW theorem gives a basis for the universal enveloping algebra of a Lie algebra and a nice spanning set for modules. For an arbitrary vertex operator algebra V, the component operators of the fixed generators of V in general do not form a Lie algebra, so one cannot use the classical PBW theorem to get a good spanning set in terms of the component operators of the generators. As mentioned before, $M(1)^+$ is generated by ω and J. Although the component operators of ω and J do not form a Lie algebra because as the commutators involve quadratic or higher products, we manage to obtain a kind of PBW-type result which is good enough to give nice spanning sets for $M(1)^+$ and $A(M(1)^+)$. The same idea and technique have been developed further in [DN] to yield nice spanning sets for V_L^+ and $A(V_L^+)$. A PBW-type generating property for general vertex operator algebras was given in [KL] recently.

The structure of this paper is as follows. In Section 2 we recall the definition of admissible twisted modules for a vertex operator algebra, the notion of Zhu's algebra and related results, and the construction of vertex operator algebra $M(1)^+$. In Section 3 we give the commutator relations for the component operators of ω and J, and we produce a kind of generalized PBW theorem. This enables us to get spanning sets for $M(1)^+$ and $A(M(1)^+)$. Section 4 shows how to evaluate the generators of $A(M(1)^+)$ on the top levels of the known irreducible modules for $M(1)^+$ to yield the relations which are good enough to determine the algebra structure of $A(M(1)^+)$. We then use $A(M(1)^+)$ to classify the irreducible modules for $M(1)^+$.

2. PRELIMINARIES

This section is divided into three parts. In the first part we recall various notions of (twisted) modules for a vertex operator algebra V (cf. [DLM2]). Zhu's algebra [18] and related results are explained in the second part. In the last part we review the vertex operator algebra $M(1)$ and its (twisted) modules (cf. [FLM]).

2.1. *Modules*

Let V be a vertex operator algebra (cf. [B, FLM]) and let g be an automorphism of V of finite order T. Denote the decomposition of V into eigenspaces with respect to the action of g as $V = \oplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r$ where $V^r = \{v \in V | gv = e^{-2\pi i r/T} v\}$.

An admissible g-twisted V module (cf. [DLM2, Z])

$$M = \sum_{n=0}^{\infty} M\left(\frac{n}{T}\right)$$

is an $(1/T)\mathbb{Z}$-graded vector space with the top level $M(0) \neq 0$ equipped with a linear map

$$\begin{aligned} &V \to (\mathrm{End}\, M)\{z\}, \\ &v \mapsto Y_M(v,z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \qquad (v_n \in \mathrm{End}\, M), \end{aligned}$$

which satisfies the following conditions: for all $0 \leq r \leq T-1, u \in V^r, v \in V, w \in M$,

$$Y_M(u,z) = \sum_{n \in r/T + \mathbb{Z}} u_n z^{-n-1},$$

$$u_n w = 0 \quad \text{for } n \gg 0,$$

$$Y_M(\mathbf{1}, z) = 1,$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1)$$

$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2), \tag{2.1}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ (elementary properties of the δ function can be found in [FLM]) and all binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable;

$$u_m M(n) \subset M(\mathrm{wt}(u) - m - 1 + n)$$

if u is homogeneous. If $g = 1$, this reduces to the definition of an admissible V module.

A *g-twisted V module* is an admissible g-twisted V module M which carries a $\mathbb{C}$ grading induced by the spectrum of $L(0)$. That is, we have

$$M = \coprod_{\lambda \in \mathbb{C}} M_\lambda,$$

where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$. Moreover, we require that $\dim M_\lambda$ is finite and for fixed λ, $M_{n/T+\lambda} = 0$ for all small enough integers n. Again, if $g = 1$, we get an ordinary V module.

2.2. *Zhu's Algebra*

Let us recall that a vertex operator algebra V is $\mathbb{Z}$-graded:

$$V = \coprod_{n \in \mathbb{Z}} V_n, \qquad v \in V_n, \qquad n = \mathrm{wt}(v).$$

Each $v \in V_n$ is called a homogeneous vector of weight n. To define Zhu's algebra $A(V)$ we need two products $*$ and $\circ$ on V. For $u \in V$ homogeneous and $v \in V$,

$$u * v = \mathrm{Res}_z\left(\frac{(1+z)^{\mathrm{wt}(u)}}{z} Y(u,z)v\right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)}{i} u_{i-1}v, \tag{2.2}$$

$$u \circ v = \mathrm{Res}_z\left(\frac{(1+z)^{\mathrm{wt}(u)}}{z^2} Y(u,z)v\right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)}{i} u_{i-2}v \tag{2.3}$$

and we extend both (2.2) and (2.3) to linear products on V. Define $O(V)$ to be the linear span of all $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. For $u \in V$ we denote by $o(u)$ the weight zero component operator of u on any admissible module. Then $o(u) = u_{\mathrm{wt}(u)-1}$ if u is homogeneous. The following theorem is essentially due to Zhu [Z].

THEOREM 2.1. (i) *The product* $*$ *induces an associative algebra structure on* $A(V)$ *with the identity* $\mathbf{1} + O(V)$. *Moreover,* $\omega + O(V)$ *is a central element of* $A(V)$.

(ii) *The map* $u \mapsto o(u)$ *gives a representation of* $A(V)$ *on* $M(0)$ *for any admissible* V *module* M. *Moreover, if any admissible* V *module is completely reducible, then* $A(V)$ *is a finite dimensional semisimple algebra.*

(iii) *The map* $M \mapsto M(0)$ *gives a bijection between the set of equivalence classes of simple admissible* V *modules and the set of equivalence classes of simple* $A(V)$ *modules.*

For convenience we write $[u] = u + O(V) \in A(V)$. We define $u \sim v$ for $u, v \in V$ if $[u] = [v]$. This induces a relation on $\mathrm{End}\, V$ such that $f, g \in \mathrm{End}\, V$, $f \sim g$ if and only if $fu \sim gu$ for all $u \in V$.

The following proposition is useful later (cf. [W, Z]).

PROPOSITION 2.2. (i) *Assume that* $u \in V$ *is homogenous,* $v \in V$, *and* $n \geq 0$. *Then*

$$\mathrm{Res}_z\left(\frac{(1+z)^{\mathrm{wt}(u)}}{z^{2+n}} Y(u,z)v\right) = \sum_{i=1}^{\infty} \binom{\mathrm{wt}(u)}{i} u_{i-n-2}v \in O(V).$$

(ii) *If* u *and* v *are homogeneous elements of* V, *then*

$$u * v \sim \mathrm{Res}_z\left(\frac{(1+z)^{\mathrm{wt}(v)-1}}{z} Y(v,z)u\right).$$

(iii) *For any* $n \geq 1$,

$$L(-n) \sim (-1)^n\{(n-1)(L(-2)+L(-1)) + L(0)\}, \quad (2.4)$$

where $L(n)$ *are the Virasoro operators given by* $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

2.3. *Vertex Operator Algebras* $M(1)$ *and* $M(1)^+$

Finally we discuss the construction of vertex operator algebra $M(1)$ and its (twisted) modules (cf. [FLM]). We also define the vertex operator subalgebra $M(1)^+$.

Let $\mathfrak{h}$ be a finite-dimensional vector space with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the corresponding affine Lie algebra. Let $\lambda \in \mathfrak{h}$ and consider the induced $\hat{\mathfrak{h}}$ module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C} \simeq S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \quad \text{(linearly)},$$

where $\mathfrak{h} \otimes t\mathbb{C}[t]$ acts trivially on $\mathbb{C}$, $\mathfrak{h}$ acts as $\langle \alpha, \lambda \rangle$ for $\alpha \in \mathfrak{h}$, and c acts as 1. For $\alpha \in \mathfrak{h}$ and $n \in \mathbb{Z}$, we write $\alpha(n)$ for the operator $\alpha \otimes t^n$ and put

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}.$$

Among $M(1, \lambda)$, $\lambda \in \mathfrak{h}$, $M(1) = M(1, 0)$ is of special interest because it has a natural vertex operator algebra structure as explained below. For $\alpha_1, \dots, \alpha_k \in \mathfrak{h}$, $n_1, \dots, n_k \in \mathbb{Z}$ $(n_i > 0)$, and $v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \in M(1)$, we define a vertex operator corresponding to v by

$$Y(v, z) = {\circ \atop \circ} \partial_z^{(n_1-1)}\alpha_1(z) \partial_z^{(n_2-1)}\alpha_2(z) \cdots \partial_z^{(n_k-1)}\alpha_k(z) {\circ \atop \circ},$$

where

$$\partial_z^{(n)} = \frac{1}{n!}\left(\frac{d}{dz}\right)^n$$

and a normal ordering procedure indicated by open colons signifies that all the factors in the expression above are to be reordered if necessary so that all the operators $\alpha(n)$ $(\alpha \in \mathfrak{h}, n < 0)$ are placed to the left of all the operators $\alpha(n)$ $(n \geq 0)$ before the expression is evaluated. We extend Y to all $v \in V$ by linearity. Let $\{\beta_1, \dots, \beta_d\}$ be an orthonormal basis of $\mathfrak{h}$. Set $\mathbf{1} = 1$ and $\omega = \frac{1}{2}\sum_{i=1}^{d} \beta_i(-1)^2$. The following theorem is well known (cf. [FLM).

THEOREM 2.3. *The space $M(1) = (M(1), Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra and $M(1, \lambda)$ for $\lambda \in \mathfrak{h}$ gives a complete list of inequivalent irreducible modules for $M(1)$.*

We define an automorphism θ of $M(1)$ by

$$\theta(\alpha_1(n_1) \cdots \alpha_k(n_k)) = (-1)^k \alpha_1(n_1) \cdots \alpha_k(n_k).$$

Then θ invariants $M(1)^+$ of $M(1)$ form a simple vertex operator subalgebra and the -1 eigenspace $M(1)^-$ is an irreducible $M(1)^+$ module (see [DM2, Theorem 2]). Clearly $M(1) = M(1)^+ \oplus M(1)^-$.

Following [DM1], we define $\theta \circ M(1, \lambda) = (\theta \circ M(1, \lambda), Y_\theta)$, where $\theta \circ M(1, \lambda) = M(1, \lambda)$ as vector spaces and $Y_\theta(v, z) = Y(\theta v, z)$. Then $\theta \circ M(1, \lambda)$ is also an irreducible $M(1)$ module isomorphic to $M(1, -\lambda)$.

The following proposition is a direct consequence of Theorem 6.1 of [DM2].

PROPOSITION 2.4. *If* $\lambda \neq 0$, *then* $M(1, \lambda)$ *and* $M(1, -\lambda)$ *are isomorphic and irreducible* $M(1)^+$ *modules*.

Next we turn our attention to the θ-twisted $M(1)$ modules (cf. [FLM]). The twisted affine algebra is defined to be $\hat{\mathfrak{h}}[-1] = \sum_{n\in\mathbb{Z}} \mathfrak{h} \otimes t^{1/2+n} \oplus \mathbb{C}c$ and its canonical irreducible module is

$$M(1)(\theta) = U\big(\hat{\mathfrak{h}}[-1]\big) \otimes_{U(\mathfrak{h}\otimes t^{1/2}\mathbb{C}[t]\oplus \mathbb{C}c)} \mathbb{C} \simeq S\big(\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]\big),$$

where $\mathfrak{h} \otimes t^{1/2}\mathbb{C}[t]$ acts trivially on $\mathbb{C}$ and c acts like 1. As before, there is an action of θ on $M(1)(\theta)$ by $\theta(\alpha_1(n_1)\cdots \alpha_k(n_k)) = (-1)^k\alpha_1(n_1)\cdots \alpha_k(n_k)$, where $\alpha_i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}$, and $\alpha(n) = \alpha \otimes t^n$. We denote the ± 1 eigenspace of $M(1)(\theta)$ under θ by $M(1)(\theta)^{\pm}$.

Let $v = \alpha_1(-n_1)\cdots \alpha_k(-n_k) \in M(1)$. We define

$$W_\theta(v,z) = {}^{\circ}_{\circ}\, \partial_z^{(n_1-1)}\alpha_1(z)\, \partial_z^{(n_2-1)}\alpha_2(z)\cdots \partial_z^{(n_k-1)}\alpha_k(z)\, {}^{\circ}_{\circ},$$

where the right side is an operator on $M(1)(\theta)$, namely,

$$\alpha(z) = \sum_{n\in 1/2+\mathbb{Z}} \alpha(n) z^{-n-1},$$

and the normal ordering notation is as before. Furthermore, we extend this to all $v \in M(1)$ by linearity. Define constants $c_{mn} \in \mathbb{Q}$ for $m, n \geq 0$ by the formula

$$\sum_{m,n\geq 0} c_{mn} x^m y^n = -\log\left(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2}\right).$$

Set

$$\Delta_z = \sum_{m,n\geq 0} \sum_{i=1}^{d} c_{mn}\, \beta_i(m)\beta_i(n) z^{-m-n}.$$

Now we define *twisted vertex operators* $Y_\theta(v,z)$ for $v \in M(1)$ as

$$Y_\theta(v,z) = W_\theta(e^{\Delta_z} v, z).$$

Then we have

THEOREM 2.5. (i) $(M(1)(\theta), Y_\theta)$ *is the unique irreducible* θ*-twisted* $M(1)$ *module*.

(ii) $M(1)(\theta)^{\pm}$ *are irreducible* $M(1)^+$ *modules*.

Part (i) is a result of Chapter 9 of [15] and part (ii) follows Theorem 5.5 of [DLi].

In the paper, we mainly consider the case that $\mathfrak{h}$ is one dimensional. From now on we always assume that $\mathfrak{h} = \mathbb{C}h$ with the normalized inner product $\langle h, h \rangle = 1$.

Remark 2.6. It is easy to see in this case that the automorphism group of $M(1)$ is generated by θ. It was pointed out in [DG] that $M(1)^+$ is the only proper vertex operator subalgebra of $M(1)$ which differs from the vertex operator subalgebra generated by ω.

For later use we need to know the first few coefficients of z in Δ_z. Note that

$$
\begin{aligned}
-\log&\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right)\\
&= -\frac{1}{4}x-\frac{1}{4}y+\frac{3}{32}x^2+\frac{1}{16}xy+\frac{3}{32}y^2\\
&\quad -\frac{5}{96}x^3-\frac{1}{32}x^2y-\frac{1}{32}xy^2-\frac{5}{96}y^3\\
&\quad +\frac{35}{1024}x^4+\frac{5}{256}x^3y+\frac{9}{512}x^2y^2+\frac{5}{256}xy^3+\frac{35}{1024}y^4+\cdots.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta_z = {}& -\tfrac{1}{2}h(0)h(1)z^{-1}+\left(\tfrac{3}{16}h(0)h(2)+\tfrac{1}{16}h(1)^2\right)z^{-2}\\
&+\left(-\tfrac{5}{48}h(0)h(3)-\tfrac{1}{16}h(1)h(2)\right)z^{-3}\\
&+\left(\tfrac{35}{512}h(0)h(4)+\tfrac{5}{128}h(1)h(3)+\tfrac{9}{512}h(2)^2\right)z^{-4}+\cdots. \qquad (2.5)
\end{aligned}
$$

3. A SPANNING SET OF $A(M(1)^+)$

In this section we use a result in [DG] to yield a spanning set of $M(1)^+$ and then use it to produce a spanning set of $A(M(1)^+)$. We also list known irreducible modules for $M(1)^+$ and the actions of $L(0)$ and $o(J)$ on the top levels of these modules, where J is a singular vector of $M(1)^+$ of weight 4 defined in Section 3.1.

3.1. *Some Commutator Relations*

Recall that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, where the component operators $L(n)$ together with 1 spanned a Virasoro algebra of central charge 1 on $M(1)$. It is well known that $M(1)$ is a unitary representation for the Virasoro algebra and $M(1)^+$, as the submodule for the Virasoro algebra, is a direct sum of irreducible modules

$$M(1)^+ = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L(1, 4m^2) \tag{3.1}$$

where $L(1, 4m^2)$ is an irreducible highest weight Virasoro module with highest weight $4m^2$ and central charge 1 (see [DG, Theorem 2.7(1)].

Let

$$J = h(-1)^4 \mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \tfrac{3}{2} h(-2)^2 \mathbf{1}, \tag{3.2}$$

which is a singular vector of weight 4 for the Virasoro algebra. Then the field

$$J(z) = {\circ \atop \circ} h(z)^4 {\circ \atop \circ} - {\circ \atop \circ} \partial_z^2 h(z) h(z) {\circ \atop \circ} + \tfrac{3}{2} {\circ \atop \circ} (\partial_z h(z))^2 {\circ \atop \circ}$$

is a primary field. We have commutation relations

$$[L(m), J(z)] = z^m (z\partial_z + 4(m+1)) J(z) \qquad (m \in \mathbb{Z}),$$

which follow from the Jacobi identity (2.1) and which are equivalent to

$$[L(m), J_n] = (3(m+1) - n) J_{n+m} \qquad (m, n \in \mathbb{Z}), \tag{3.3}$$

where $J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$.

Next we compute the commutator $[J_m, J_n]$ for $m, n \in \mathbb{Z}$. Again by the Jacobi identity (2.1) we know

$$[J_m, J_n] = \sum_{i=0}^{\infty} \binom{m}{i} (J_i J)_{m+n-i}.$$

Since the weight of J is 4, we see that $\mathrm{wt}(J_i J) = 7 - i \leq 7$. Then it follows from the decomposition (3.1) that for any $i \in \mathbb{Z}_{\geq 0}$, we have $J_i J \in L(1, 0) \oplus L(1, 4)$ and then all these are expressed as linear combinations of

$$L(-m_1) \cdots L(-m_s)\mathbf{1}, \qquad L(-n_1) \cdots L(-n_t) J,$$

where $m_1 \geq m_2 \geq \cdots \geq m_2 \geq 2$, $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$, and $s, t \leq 3$. Note that for any vertex operator algebra V, $u, v \in V$ and $m, n \in \mathbb{Z}$, $(u_m v)_n$ is a linear combination of operators $u_s v_t$ and $v_t u_s$ for $s, t \in \mathbb{Z}$. Using (3.3) we obtain the following lemma

LEMMA 3.1. *For any $m, n \in \mathbb{Z}$, commutators $[J_m, J_n]$ are expressed as linear combinations of*

$$L(p_1) \cdots L(p_s), \qquad L(q_1) \cdots L(q_t) J_r,$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t, r \in \mathbb{Z}$ and $s, t \leq 3$.

3.2. *A Spanning Set for $M(1)^+$*

We first note the following theorem.

THEOREM 3.2 [DG, Theorem 2.7(2)]. *As a vertex operator algebra, $M(1)^+$ is generated by the Virasoro element ω and any singular vector of weight greater than 0. In particular, $M(1)^+$ is generated by ω and J.*

From this theorem we see that $M(1)^+$ is spanned by

$$\left\{u^1_{m_1} \cdots u^k_{m_k}\mathbf{1} | u^i = \omega, J, m_i \in \mathbb{Z}\right\},$$

which are not necessarily linearly independent. We say that an expression $u^1_{m_1} \cdots u^k_{m_k}\mathbf{1}$ has length t with respect to J, which we write $\ell_J(u^1_{m_1} \cdots u^k_{m_k}\mathbf{1}) = t$, if $\{i | u^i = J\}$ has cardinality t. Note that $\omega_i = L(i-1)$. An induction on $\ell_J(u^1_{m_1} \cdots u^k_{m_k}\mathbf{1})$ using (3.3) and Lemma 3.1 shows that $u^1_{m_1} \cdots u^k_{m_k}\mathbf{1}$ is a linear combination of vectors of type

$$\left\{L(m_1)L(m_2) \cdots L(m_s)J_{n_1}J_{n_2} \cdots J_{n_t}\mathbf{1} | m_1, n_b \in \mathbb{Z}\right\}.$$

Thus $M(1)^+$ is spanned by those vectors.

Using the commutator relations (3.3) and the fact that $L(m)\mathbf{1} = 0$, $m \geq -1$, we get the following lemma.

LEMMA 3.3. *Let W be the subspace of $M(1)^+$ spanned by $J_{n_1} \ldots J_{n_t}\mathbf{1}$ with $n_i \in \mathbb{Z}$. Then W is invariant under the action of $L(m)$, $m \geq -1$.*

PROPOSITION 3.4. *The vertex operator algebra $M(1)^+$ is spanned by the vectors*

$$L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1},$$

where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$.

Proof. We have already shown that $M(1)^+$ is spanned by

$$L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1}$$

where $m_a, n_b \in \mathbb{Z}$. Using the PBW theorem for the Virasoro algebra, we can assume that $m_1 \geq \cdots \geq m_s$. By Lemma 3.3 we can further assume that $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$. We proceed by induction on the length with respect to J that $v = L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1}$ can be spanned by the indicated vectors in the proposition.

If the length is 0, it is clear. Suppose that it is true for all monomials v such that $\ell_J(v) < t$. Since $J_k\mathbf{1} = 0$ for $k \geq 0$, we can assume $n_t \geq 1$. If $n_1 \geq \cdots \geq n_t$, we are done. Otherwise there exists n_a such that $n_{a+1} \geq \cdots \geq n_t$, but $n_a < n_{a+1}$. There are two cases $n_a \leq 0$ and $n_a > 0$ which are dealt with separately. If $n_a \leq 0$, then $J_{-n_a}\mathbf{1} = 0$ and

$$L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1}$$
$$= \sum_{j=a+1}^{t} L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots \check{J}_{-n_a} \cdots \left[J_{-n_a}, J_{-n_j}\right] \cdots J_{-n_t}\mathbf{1},$$

where $\check{J}_{-n_a}$ means that we omit the term J_{-n_a}. However, by Lemma 3.1, $[J_{-n_a}, J_{-n_j}]$ are linear combinations of operators of type

$$L(p_1) \cdots L(p_{s'}), \qquad L(q_1) \cdots L(q_{t'}) J_r.$$

By substituting these into the above and using commutation relation (3.3) again, the right-hand side is a linear combination of monomials whose lengths with respect to J are less than or equal to $t - 1$. Thus by induction hypothesis, this is expressed as linear combinations of expected monomials.

If $n_a > 0$, then either $n_a < n_t$ or there exists b with $t > b > a$ so that $n_b > n_a > n_{b+1}$. Then we have either

$$L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1}$$
$$= \sum_{j=a+1}^{t} L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots \check{J}_{-n_a} \cdots \left[J_{-n_a}, J_{-n_j}\right] \cdots J_{-n_t}\mathbf{1}$$
$$+ L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots \check{J}_{-n_a} \cdots J_{-n_t} J_{-n_a}\mathbf{1}$$

or

$$L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t}\mathbf{1}$$
$$= \sum_{j=a+1}^{b} L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots \check{J}_{-n_a} \cdots \left[J_{-n_a}, J_{-n_j}\right] \cdots J_{-n_t}\mathbf{1}$$
$$+ L(-m_1) \cdots L(-m_s) J_{-n_1} \cdots \check{J}_{-n_a} \cdots J_{-n_b} J_{-n_a} J_{-n_{b+1}} \cdots J_{-n_t}\mathbf{1}.$$

From the discussion of case $n_a \le 0$ it is enough to show that either

$$L(-m_1)\cdots L(-m_s)J_{-n_1}\cdots \check{J}_{-n_a}\cdots J_{-n_t}J_{-n_a}\mathbf{1}$$

or

$$L(-m_1)\cdots L(-m_s)J_{-n_1}\cdots \check{J}_{-n_a}\cdots J_{-n_b}J_{-n_a}J_{-n_{b+1}}\cdots J_{-n_t}\mathbf{1}$$

can be expressed as linear combinations of desired vectors. This follows from an induction on a. ∎

3.3. *A Spanning Set for $A(M(1)^+)$*

For short we set

$$v^{*s} = \overbrace{v * \cdots * v}^{s}$$

for $v \in M(1)^+$. Recalling $[v] = v + O(M(1)^+)$ for $v \in M(1)^+$, we will also use a similar notation $[v]^s$. Then it is easy to see that $[v^{*t}] = [v]^{*t}$.

THEOREM 3.5. *Zhu's algebra $A(M(1)^+)$ is spanned by $\mathscr{S} = \{[\omega]^{*s} * [J]^{*t} | s, t \ge 0\}$.*

Proof. By Proposition 3.4, it is enough to show that for any monomial

$$v = L(-m_1)\cdots L(-m_s)J_{-n_1}\cdots J_{-n_t}\mathbf{1},$$

where $m_1 \ge m_2 \ge \cdots \ge m_1 \ge 2$, $n_1 \ge n_2 \ge \cdots \ge n_t \ge 1$, and $[v]$ is a linear combination of $\mathscr{S}$. We prove by induction on $\ell_J(v)$ that $[v]$ is spanned by vectors $[\omega]^{*p} * [J]^{*q}$ in $\mathscr{S}$ such that $q \le t$ and weights of its homogeneous components are less than or equal to the weight of v.

In the case that $\ell_J(v) = 0$, then $v = L(-m_1)\cdots L(-m_s)\mathbf{1}$, which is spanned by $\{[\omega]^{*s} | s \ge 0\}$ (cf. [FZ]). Now let $t > 0$ and assume that the statement is true for all v with $\ell_J(v) < t$. We will prove by induction on the weight of v that $[v]$ is a linear combination of $\mathscr{S}$. Clearly, the smallest weight is $t\,\mathrm{wt}(J)$ and the corresponding v has the form

$$v = \overbrace{J_{-1}\cdots J_{-1}}^{s}\mathbf{1}.$$

Then by (2.2),

$$J^{*t} - v = \sum_{\substack{n_i \in \{-1,0,1,2,3\} \\ (n_i) \ne (-1,\dots,-1)}} a_{n_1 n_2 \cdots n_t} J_{n_1} J_{n_2} \cdots J_{n_t}\mathbf{1}.$$

Since each term appearing in the right-hand side involves J_{n_i} for some nonnegative n_i, we can write the right-hand side as a linear combination of spanning vectors in Proposition 3.4 whose lengths are strictly less than t. Thus by induction hypothesis, the image of the right-hand side in $A(M(1)^+)$ is spanned by $\mathscr{S}$ and so is $[v]$.

Now consider general $v = L(-m_1)\cdots L(-m_s)J_{-n_1}\cdots J_{-n_t}\mathbf{1}$. Without loss of generality, we can assume that $m_1 = m_2 = \cdots = m_s = 2$, namely,

$$v = \overbrace{L(-2)\cdots L(-2)}^{s} J_{-n_1}\cdots J_{-n_t}\mathbf{1}$$

since if there exists m_i such that $m_i \geq 3$, then $m_1 \geq 3$ and by (2.4),

$$v \sim (-1)^{m_1}\{(m_1-1)(L(-2)+L(-1))+L(0)\}L(-m_2)\cdots L(-m_s)J_{-n_1}\cdots J_{-n_t}\mathbf{1},$$

which is a sum of three homogeneous vectors of weight strictly less than $\mathrm{wt}(v)$. Then we see

$$v = \omega^{*s} * (J_{-n_1}\mathbf{1}) * (J_{-n_2}\cdots J_{-n_t}\mathbf{1}) + v',$$

where $\mathrm{wt}(v') < \mathrm{wt}(v)$. Then again by using induction hypothesis about weight, it is enough to show that the image of

$$v = \omega^{*s} * (J_{-n_1}\mathbf{1}) * (J_{-n_2}\cdots J_{-n_t}\mathbf{1})$$

in $A(M(1)^+)$ is spanned by $\mathscr{S}$. Since ω is a central element in $A(M(1)^+)$, we have

$$\begin{aligned} v &= (J_{-n_1}\mathbf{1}) * \omega^{*s} * (J_{-n_2}\cdots J_{-n_t}\mathbf{1}) \\ &= J_{-n_1}\big(\omega^{*s} * (J_{-n_2}\cdots J_{-n_t}\mathbf{1})\big) + v', \end{aligned}$$

where $\mathrm{wt}(v') < \mathrm{wt}(v)$. If $n_1 > 1$, we can use the fact that $J_{-n_1}u$ is congruent to a sum of vectors whose lengths are less than or equal to t and whose weights are smaller than $\mathrm{wt}(v)$ (cf. (2.3)) to show that $[v]$ is spanned by $\mathscr{S}$. If $n_1 = 1$, then $n_2 = \cdots = n_t = 1$ and

$$v = \omega^{*s} * J^{*t} + \text{lower weight terms}.$$

Again it is done by induction assumption. ∎

Remark 3.6. From the proof of Theorem 3.5, we see that v is spanned by $\omega^{*s} * J^{*t}$ with $2s + 4t \le \mathrm{wt}(v)$.

3.4. *List of Irreducible Modules*

As mentioned in Section 2.3, $M(1)^+$ has irreducible modules

$$M(1)^+, M(1)^-, M(1,\lambda)(0 \neq \lambda \in \mathbb{Z}), M(1)(\theta)^+, M(1)(\theta)^-.$$

Recall that $M(1,\lambda)$ and $M(1)(\theta)$ are symmetric algebras on $\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]$, respectively, as vectors spaces.

The following table gives the action of ω and J on the top levels of these modules.

	$M(1)^+$	$M(1)^-$	$M(1,\lambda)$, $\lambda \in \mathbb{C}^\times$	$M(1)(\theta)^+$	$M(1)(\theta)^-$
$M(0)$	$\mathbb{C}\mathbf{1}$	$\mathbb{C}h(-1)\mathbf{1}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}h(-1/2)$
ω	0	1	$\lambda^2/2$	$1/16$	$9/16$
J	0	-6	$\lambda^4 - \lambda^2/2$	$3/128$	$-45/128$

Here we give some explanations on how to get the table. The actions of ω and J on these spaces except $M(1)(\theta)^\pm$ are easily verified. From the definition of $Y_\theta(u,z)$, we see

$$Y_\theta(\omega, z) = \tfrac{1}{2} {}^\circ_\circ h(z)^2 {}^\circ_\circ + \tfrac{1}{16} z^{-2}.$$

Recall the expression of J from (3.2). Then by using (2.5), we get

$$e^{\Delta_z} J = J + \tfrac{3}{4} h(-1)^2 \mathbf{1} z^{-2} + \tfrac{3}{128} z^{-4}$$

and thus

$$Y_\theta(J, z) = J(z) + \tfrac{3}{4} {}^\circ_\circ h(z)^2 {}^\circ_\circ z^{-2} + \tfrac{3}{128} z^{-4},$$

where $h(z) = \sum_{n \in 1/2 + \mathbb{Z}} h(n) z^{-n-1}$. The actions of ω and J on the top levels of $M(1)(\theta)^+$ and $M(1)(\theta)^-$ are immediately derived.

4. CLASSIFICATION OF IRREDUCIBLE MODULES FOR $M(1)^+$

In this section we explicitly determine the algebra structure of $A(M(1)^+)$ and use this result to prove that the list of the irreducible modules in Section 3.4 is complete.

4.1. *The Structure of $A(M(1)^+)$*

It was proved in Section 3.3 that Zhu's algebra $A(M(1)^+)$ as an associative algebra is generated by $[\omega]$ and $[J]$. Since $[\omega]$ is a central element, $A(M(1)^+)$ is a commutative associative algebra and must be isomorphic to a quotient of the polynomial algebra $\mathbb{C}[x, y]$ with variables x and y modulo an ideal I. We still need to determine the ideal explicitly. For this purpose we will find relations between $[\omega]$ and $[J]$ in $A(M(1)^+)$.

For convenience we simply write u instead of $[u]$ for $u \in M(1)^+$ and uv instead of $u * v$.

PROPOSITION 4.1. *In $A(M(1)^+)$,*

$$J^2 = p(\omega) + q(\omega)J,$$

where

$$p(x) = \tfrac{1816}{35}x^4 - \tfrac{212}{5}x^3 + \tfrac{89}{10}x^2 - \tfrac{27}{70}x, \qquad q(x) = -\tfrac{314}{35}x^2 + \tfrac{89}{14}x - \tfrac{27}{70}.$$

Equivalently,

$$(J + \omega - 4\omega^2)(70J + 908\omega^2 - 515\omega + 27) = 0.$$

Proof. Recall that as a module for the Virasoro algebra, $M(1)^+$ has the decomposition $M(1)^+ = \bigoplus_{m \geq 0} L(1, 4m^2)$. Since J is the singular vector with weight 4, we see

$$J^2 = \sum_{i \geq 0} \binom{4}{i} J_{i-1}J \in L(1,0) \oplus L(1,4).$$

Therefore, from Remark 3.6, we get

$$J^2 = p(\omega) + q(\omega)J, \tag{4.1}$$

where p and q are polynomials of degrees less than or equal to 4 and 2, respectively. Let

$$p(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon \quad \text{and} \quad q(x) = ax^2 + bx + c.$$

To determine the coefficients of $p(x)$ and $q(x)$, we evaluate both sides of (4.1) on modules listed in Section 3.4.

Since $\omega = J = 0$ on the top level of $M(1)^+$, we have $\epsilon = 0$. On the top level of $M(1)^+$, $\omega = 1$ and $J = -6$ give $\alpha + \beta + \gamma + \delta - 6(a + b + c) = 36$. Furthermore, on the top levels of $M(1, \lambda)$ for $\lambda \in \mathbb{C}^\times$ we know $\omega = \lambda^2/2$ and $J = \lambda^4 - \lambda^2/2$. Comparing the coefficients of λ^is tells us

$$\alpha + 4a = 16, \qquad \beta + 4b - a = -8, \qquad \gamma + 4c - b = 1, \qquad \delta - c = 0.$$

Finally, we get two more equations by substituting $\omega = 1/16$, $J = 3/128$ on $M(1)(\theta)^+$ and $\omega = 9/16$, $J = -45/128$ on $M(1)(\theta)^-$. Solving this linear system gives the desired result. ∎

PROPOSITION 4.2. *In* $A(M(1)^+)$,

$$(\omega - 1)\left(\omega - \tfrac{1}{16}\right)\left(\omega - \tfrac{9}{16}\right)\left(J + \omega - 4\omega^2\right) = 0.$$

As a vertex operator algebra, $M(1)^+$ has the weight space decomposition $M(1) + = \oplus_{m \geq 0} M(1)_m^+$. The list of dim $M(1)_m^+$ for m up to 10 is

m	0	1	2	3	4	5	6	7	8	9	10
dim $M(1)_m^+$	1	0	1	1	3	3	6	7	12	14	22

To produce the second relation, we need the following lemma, whose proof is given in the Appendix.

LEMMA 4.3. *The vectors*

$$L(-1)M(1)_9^+, \qquad L(-3)M(1)_7^+, \qquad h(-1)_{-3}^4 h(-1)^4\mathbf{1}, \qquad L(-2)^5\mathbf{1} \tag{4.2}$$

span $M(1)_{10}^+$.

Now we can prove Proposition 4.2. First note that any weight 10 vector is contained in $L(1,0) \oplus L(1,4)$ and is a linear combination of vectors of type $L(-n_1)\cdots L(-n_s), L(-m_1)\cdots L(-m_t)J$, where $n_1 \geq \cdots \geq n_s \geq 2, m_1 \geq \cdots \geq m_t \geq 1, \sum n_a = 10$, and $\sum m_b + 4 = 10$. From the proof of Theorem 3.5, images of this kind of vectors in $A(M(1)^+)$ can be expressed as linear combinations of ω^i $(i = 0, 1, \dots, 5)$ and $\omega^i J$ $(i = 0, 1, 2, 3)$.

By Proposition 2.2(i) and (iii), $L(-1)M(1)_9^+, L(-3)M(1)_7^-$, and $h(-1)_{-3}^4 h(-1)^4\mathbf{1}$ are congruent to vectors whose homogeneous components have weights less than 10. Note that $\omega^i = L(-2)^i\mathbf{1} +$ lower weight terms. Then it follows from Remark 3.6 and Proposition 4.1 that $L(-1)M(1)_9^+, L(-3)M(1)_7^+$, and $h(-1)_{-3}^4 h(-1)^4\mathbf{1}$ are congruent to vectors spanned by ω_i $(i = 0, 1, \dots, 4)$ and $\omega^i J$ $(i = 0, 1, 2)$. Thus by Lemma 4.3 we see that $M(1)_{10}^+ + O(M(1)^+)$ is spanned by ω^i $(i = 0, 1, \dots, 5)$ and $\omega^i J$ $(i = 0, 1, 2)$. As a result we have

$$\omega^3 J = P(\omega) + Q(\omega)J,$$

where det $P \leq 5$ and deg $Q \leq 2$. Evaluating this equation on the top levels of the modules listed in Section 3.4 gives the desired result.

Now we can state our first main theorem.

THEOREM 4.4. *We have the algebra isomorphism*

$$\mathbb{C}[x, y]/\langle P, Q\rangle \cong A\left(M(1)^+\right),$$

where

$$P = (y + x - 4x^2)(70y + 908x^2 - 515x + 27),$$
$$Q = (x - 1)(x - \tfrac{1}{16})(x - \tfrac{9}{16})(y + x - 4x^2).$$

Proof. By Theorem 3.5, we have a surjective algebra homomorphism

$$\varphi: \mathbb{C}[x, y] \to A(M(1)^+), \quad x \mapsto \omega, \quad y \mapsto J.$$

Let $K(x, y) \in \operatorname{Ker} \varphi$ and regard $K(x, y)$ as a polynomial in variable y. Note that $P(x, y)$ has degree 2 in y. Using the division algorithm we can write $K(x, y) = A(x, y)P(x, y) + R(x, y)$, where $A(x, y), R(x, y) \in \mathbb{C}[x, y]$ so that $R(x, y)$ has degree 1 in y. We can express $R(x, y)$ as $R(x, y) = B(x)(y + x - 4x^2) + C(x)$. By Proposition 4.1, $P(x, y) \in \operatorname{Ker} \varphi$. So we have

$$B(\omega)(J + \omega - 4\omega^2) + C(\omega) = 0. \tag{4.3}$$

Evaluating (4.3) on the top levels of modules $M(1, \lambda)$ yields $C(\lambda^2/2) = 0$, since $J + \omega - 4\omega^2 = 0$ on the top level of $M(1, \lambda)$ for all $\lambda \in \mathbb{C}^\times$. Thus $C(x) = 0$ as a polynomial. Further evaluating (4.3) on the top levels of $M(1)^-, M(1)(\theta)^\pm$ and noting that $J + \omega - 4\omega^2 \neq 0$, we get $B(1) = B(1/16) = B(9/16) = 0$. This implies $(x - 1)(x - 1/16)(x - 9/16) | B(x)$. Thus we reach

$$K(x, y) = A(x, y)P(x, y) + D(x)Q(x, y)$$

for some polynomial $D(x)$. Since $Q(x, y)$ lies in $\operatorname{Ker} \varphi$ already by Proposition 4.2, we conclude that $\operatorname{Ker} \varphi = \langle P(x, y), Q(x, y) \rangle$. ∎

4.2. *Classification of Irreducible Modules for $M(1)^+$*

Finally we can use $A(M(1)^+)$, whose algebra structure was determined in the previous section, to classify the irreducible modules for $M(1)^+$.

THEOREM 4.5. *The set*

$$\{M(1)^\pm, M(1)(\theta)^\pm, M(1, \lambda) \cong M(1, -\lambda), \lambda \in \mathbb{C}^\times\}$$

gives a complete list of inequivalent irreducible $M(1)^+$ modules. Moreover, any irreducible admissible $M(1)^+$ module is an ordinary module.

Proof. Let $M = \bigoplus_{n \geq 0} M(n)$ be an irreducible admissible $M(1)^+$ module with $M(0) \neq 0$. Then $M(0)$ is an irreducible $A(M(1)^+)$ module. Since $A(M(1)^+)$ is commutative, $M(0)$ is one dimensional. So both ω and J act as scalars α and β on $M(0)$. From Theorem 4.4 we have

$$(\beta + \alpha - 4\alpha^2)(70\beta + 908\alpha^2 - 515\alpha + 27) = 0$$

and

$$(\alpha - 1)\left(\alpha - \tfrac{1}{16}\right)\left(\alpha - \tfrac{9}{16}\right)(\beta + \alpha - 4\alpha^2) = 0.$$

If $\beta + \alpha - 4\alpha^2 = 0$ and $\alpha \neq 0$, then $M(0)$ is isomorphic to the top level of $M(1, \sqrt{2\alpha})$ and M is isomorphic to $M(1, \sqrt{2\alpha})$. If $\beta + \alpha - 4\alpha^2 = 0$ and $\alpha = 0$, then M is isomorphic to $M(1)^+$. Otherwise we have $(\alpha - 1)(\alpha - 1/16)(\alpha - 9/16) = 0$ and $70\beta + 908\alpha^2 - 515\alpha + 27 = 0$. One can easily verify that M is isomorphic to $M(1)^-$, $M(1)(\theta)^+$, and $M(1)(\theta)^-$ when $\alpha = 1, 1/16$, and $9/16$. ▮

APPENDIX

Here we give the details of a proof of Lemma 4.3. First, we list bases of $M(1)_7^+$, $M(1)_9^+$, and $M(1)_{10}^+$ which have dimensions 7, 14, and 22, respectively.

A basis of $M(1)_7^+$:

$$e_1 = h(-6)h(-1)\mathbf{1}, \qquad e_2 = h(-5)h(-2)\mathbf{1},$$

$$e_3 = h(-4)h(-3)\mathbf{1}, \qquad e_4 = h(-4)h(-1)^3\mathbf{1},$$

$$e_5 = h(-3)h(-2)h(-1)^2\mathbf{1}, \qquad e_6 = h(-2)^3h(-1)\mathbf{1},$$

$$e_7 = h(-2)h(-1)^5\mathbf{1}.$$

A basis of $M(1)_9^+$:

$$f_1 = h(-8)h(-1)\mathbf{1}, \qquad f_2 = h(-7)h(-2)\mathbf{1},$$

$$f_3 = h(-6)h(-3)\mathbf{1}, \qquad f_4 = h(-6)h(-1)^3\mathbf{1},$$

$$f_5 = h(-5)h(-4)\mathbf{1}, \qquad f_6 = h(-5)h(-2)h(-1)^2\mathbf{1},$$

$$f_7 = h(-4)h(-3)h(-1)^2\mathbf{1}, \qquad f_8 = h(-4)h(-2)^2h(-1)\mathbf{1},$$

$$f_9 = h(-4)h(-1)^5\mathbf{1}, \qquad f_{10} = h(-3)^2h(-2)h(-1)\mathbf{1},$$

$$f_{11} = h(-3)h(-2)^3\mathbf{1}, \qquad f_{12} = h(-3)h(-2)h(-1)^4\mathbf{1},$$

$$f_{13} = h(-2)^3h(-1)^3\mathbf{1}, \qquad f_{14} = h(-2)h(-1)^7\mathbf{1}.$$

A basis of $M(1)^+_{10}$:

$$g_1 = h(-9)h(-1)\mathbf{1}, \qquad g_2 = h(-8)h(-2)\mathbf{1},$$
$$g_3 = h(-7)h(-3)\mathbf{1}, \qquad g_4 = h(-7)h(-1)^3\mathbf{1},$$
$$g_5 = h(-6)h(-4)\mathbf{1}, \qquad g_6 = h(-6)h(-2)h(-1)^2\mathbf{1},$$
$$g_7 = h(-5)^2\mathbf{1}, \qquad g_8 = h(-5)h(-3)h(-1)^2\mathbf{1},$$
$$g_9 = h(-5)h(-2)^2h(-1)\mathbf{1}, \qquad g_{10} = h(-5)h(-1)^5\mathbf{1},$$
$$g_{11} = h(-4)^2h(-1)^{\mathbf{2}}\mathbf{1}, \qquad g_{12} = h(-4)h(-3)h(-2)h(-1)\mathbf{1},$$
$$g_{13} = h(-4)h(-2)^{\mathbf{3}}\mathbf{1}, \qquad g_{14} = h(-4)h(-2)h(-1)^4\mathbf{1},$$
$$g_{15} = h(-3)^3h(-1)\mathbf{1}, \qquad g_{16} = h(-3)^2h(-2)^2\mathbf{1},$$
$$g_{17} = h(-3)^2h(-1)^4\mathbf{1}, \qquad g_{18} = h(-3)h(-2)^2h(-1)^3\mathbf{1},$$
$$g_{19} = h(-3)h(-1)^7\mathbf{1}, \qquad g_{20} = h(-2)^4h(-1)^2\mathbf{1},$$
$$g_{21} = h(-2)^2h(-1)^6\mathbf{1}, \qquad g_{22} = h(-1)^{10}\mathbf{1}.$$

TABLE A1

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
$L(-1)f_1$	8	1	0	0	0	0	0	0	0	0	0
$L(-1)f_2$	0	7	2	0	0	0	0	0	0	0	0
$L(-1)f_3$	0	0	6	0	3	0	0	0	0	0	0
$L(-1)f_4$	0	0	0	6	0	3	0	0	0	0	0
$L(-1)f_5$	0	0	0	0	5	0	4	0	0	0	0
$L(-1)f_6$	0	0	0	0	0	5	0	2	2	0	0
$L(-1)f_7$	0	0	0	0	0	0	0	4	0	0	3
$L(-1)f_8$	0	0	0	0	0	0	0	0	4	0	0
$L(-1)f_9$	0	0	0	0	0	0	0	0	0	4	0
$L(-1)f_{10}$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_{11}$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_{12}$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_{13}$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_{14}$	0	0	0	0	0	0	0	0	0	0	0
$L(-3)e_1$	6	0	0	0	1	1	0	0	0	0	0
$L(-3)e_2$	0	5	0	0	0	0	2	0	1	0	0
$L(-3)e_3$	0	0	4	0	3	0	0	0	0	0	0
$L(-3)e_4$	0	0	0	4	0	0	0	0	0	0	3
$L(-3)e_5$	0	0	0	0	0	3	0	2	0	0	0
$L(-3)e_6$	0	0	0	0	0	0	0	0	6	0	0
$L(-3)e_7$	0	0	0	0	0	0	0	0	0	2	0
$h(-1)^4_{-3}h(-1)^4\mathbf{1}$	96	0	0	144	0	144	0	144	0	48	72

It is easy to see that

$$
\begin{aligned}
&h(-1)^4_{-3}h(-1)^4\mathbf{1}\\
&\quad = 96h(-9)h(-1)\mathbf{1} + 144h(-7)h(-1)^3\mathbf{1}\\
&\qquad + 144h(-6)h(-2)h(-1)^2\mathbf{1} + 144h(-5)h(-3)h(-1)^2\mathbf{1}Q\\
&\qquad + 72h(-4)^2h(-1)^2 + 48h(-5)h(-1)^5\mathbf{1}\\
&\qquad + 96h(-4)h(-2)h(-1)^4\mathbf{1} + 48h(-3)^2h(-1)^4\mathbf{1}\\
&\qquad + 48h(-3)h(-2)^2h(-1)^3\mathbf{1} + 4h(-3)h(-1)^7\mathbf{1}\\
&\qquad + 6h(-2)^2h(-1)^6\mathbf{1}.
\end{aligned}
$$

Tables A1 and A2 give the precise linear combinations of certain vectors in terms of g_i for $i = 1, \dots, 22$. For example, $L(-1)f_1 = 8g_1 + g_2$. We know from the tables that the vectors in (4.2) without $L(-2)^5\mathbf{1}$ span a 21 dimensional subspace of $M(1)^+_{10}$ and none of these vectors involves the term $h(-1)^{10}\mathbf{1}$. On the other hand, $L(-2)^5\mathbf{1}$ involves the term $h(-1)^{10}\mathbf{1}$. Thus the vectors in (4.2) span $M(1)^+_{10}$, as expected.

TABLE A2

	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}
$L(-1)f_1$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_2$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_3$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_4$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_5$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_6$	0	0	0	0	0	0	0	0	0	0	0
$L(-1)f_7$	2	0	0	0	0	0	0	0	0	0	0
$L(-1)f_8$	4	1	0	0	0	0	0	0	0	0	0
$L(-1)f_9$	0	0	5	0	0	0	0	0	0	0	0
$L(-1)f_{10}$	6	0	0	2	1	0	0	0	0	0	0
$L(-1)f_{11}$	0	3	0	0	6	0	0	0	0	0	0
$L(-1)f_{12}$	0	0	3	0	0	2	4	0	0	0	0
$L(-1)f_{13}$	0	0	0	0	0	0	6	0	3	0	0
$L(-1)f_{14}$	0	0	0	0	0	0	0	2	0	7	0
$L(-3)e_1$	0	0	0	0	0	0	0	0	0	0	0
$L(-3)e_2$	0	0	0	0	0	0	0	0	0	0	0
$L(-3)e_3$	1	0	0	0	0	0	0	0	0	0	0
$L(-3)e_4$	0	0	1	0	0	0	0	0	0	0	0
$L(-3)e_5$	2	0	0	0	0	0	1	0	0	0	0
$L(-3)e_6$	0	1	0	0	0	0	0	0	1	0	0
$L(-3)e_7$	0	0	5	0	0	0	0	0	0	1	0
$h(-1)^4_{-3}h(-1)^4\mathbf{1}$	0	0	96	0	0	48	48	4	0	6	0

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