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The dominance hierarchy in root systems of Coxeter groups

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ABSTRACT

If x and y are roots in the root system with respect to the standard (Tits) geometric realization of a Coxeter group W, we say that x *dominates* y if for all $w \in W$, wy is a negative root whenever wx is a negative root. We call a positive root *elementary* if it does not dominate any positive root other than itself. The set of all elementary roots is denoted by $\mathscr{E}$. It has been proved by B. Brink and R.B. Howlett [B. Brink, R.B. Howlett, A finiteness property and an automatic structure of Coxeter groups, Math. Ann. 296 (1993) 179–190] that $\mathscr{E}$ is finite if (and only if) W is a finite-rank Coxeter group. Amongst other things, this finiteness property enabled Brink and Howlett to establish the automaticity of all finite-rank Coxeter groups. Later Brink has also given a complete description of the set $\mathscr{E}$ for arbitrary finite-rank Coxeter groups [B. Brink, The set of dominance-minimal roots, J. Algebra 206 (1998) 371–412]. However the set of non-elementary positive roots has received little attention in the literature. In this paper we answer a collection of questions concerning the dominance behavior between such non-elementary positive roots. In particular, we show that for any finite-rank Coxeter group and for any non-negative integer n, the set of roots each dominating precisely n other positive roots is finite. We give upper and lower bounds for the sizes of all such sets as well as an inductive algorithm for their computation.

1. Summary of background material

Definition 1.1. (See Krammer [12].) Suppose that V is a vector space over $\mathbb{R}$ and let $(\,,\,)$ be a bilinear form on V, and let Π be a subset of V. Then Π is called a *root basis* if the following conditions are satisfied:

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(C1) $(a,a)=1$ for all $a\in\Pi$, and if a, b are distinct elements of Π then either $(a,b)=-\cos(\pi/m_{ab})$ for some integer $m_{ab}=m_{ba}\geqslant 2$, or else $(a,b)\leqslant -1$ (in which case we define $m_{ab}=m_{ba}=\infty$);
(C2) $0\notin \mathrm{PLC}(\Pi)$, where for any set A, $\mathrm{PLC}(A)$ denotes the set

$$\left\{\sum_{a\in A}\lambda_a a \;\middle|\; \lambda_a\geqslant 0 \text{ for all } a\in A \text{ and } \lambda_{a'}>0 \text{ for some } a'\in A\right\}.$$

If Π is a root basis, then we call the triple $\mathscr{C}=(V,\Pi,(\,,\,))$ a *Coxeter datum*. Throughout this paper we fix a particular Coxeter datum $\mathscr{C}$. Observe that (C1) implies that for each $a\in\Pi$, $a\notin \mathrm{PLC}(\Pi\setminus\{a\})$. Furthermore, (C1) together with (C2) yield that whenever $a,b\in\Pi$ are distinct then $\{a,b\}$ is linearly independent. For each $a\in\Pi$ define $\rho_a\in \mathrm{GL}(V)$ by the rule: $\rho_a x=x-2(x,a)a$, for all $x\in V$. Note that ρ_a is an involution, and $\rho_a a=-a$. The following proposition summarizes a few useful results:

Proposition 1.2. *(See [9, Lecture 1].)*

(i) *Suppose that $a,b\in\Pi$ are distinct such that $m_{ab}\neq\infty$. Set $\theta=\pi/m_{ab}$. Then for each integer i,*

$$(\rho_a\rho_b)^i a=\frac{\sin(2i+1)\theta}{\sin\theta}a+\frac{\sin 2i\theta}{\sin\theta}b,$$

and in particular, $\rho_a\rho_b$ has order m_{ab}.

(ii) *Suppose that $a,b\in\Pi$ are distinct such that $m_{ab}=\infty$. Set $\theta=\cosh^{-1}(-(a,b))$. Then for each integer i,*

$$(\rho_a\rho_b)^i a=\begin{cases}\frac{\sinh(2i+1)\theta}{\sinh\theta}a+\frac{\sinh 2i\theta}{\sinh\theta}b, & \text{if } (a,b)\neq -1,\\ (2i+1)a+2ib, & \text{if } (a,b)=-1,\end{cases}$$

and in particular, $\rho_a\rho_b$ has infinite order.

Let $G_{\mathscr{C}}$ be the subgroup of $\mathrm{GL}(V)$ generated by the involutions in the set $\{\rho_a\mid a\in\Pi\}$. Let (W,R) be a Coxeter system in the sense of [2], [8] or [11] with $R=\{r_a\mid a\in\Pi\}$ being a set of involutions generating W subject to the condition that $(r_ar_b)^{m_{ab}}=1$ for all distinct $a,b\in\Pi$ with $m_{ab}\neq\infty$. Furthermore, suppose that there exists a group homomorphism $\phi_{\mathscr{C}}:W\to G_{\mathscr{C}}$ satisfying $\phi_{\mathscr{C}}(r_a)=\rho_a$ for all $a\in\Pi$. This homomorphism together with the $G_{\mathscr{C}}$-action on V give rise to a W-action on V: for each $w\in W$ and $x\in V$, define $wx\in V$ by $wx=\phi_{\mathscr{C}}(w)x$. It can be easily checked that this W-action preserves $(\,,\,)$. Denote the length function of W with respect to R by ℓ. Then we have:

Proposition 1.3. *(See [9, Lecture 1].) Let $G_{\mathscr{C}}$, W and R be as the above, and let $w\in W$ and $a\in\Pi$. Then $\ell(wr_a)\geqslant\ell(w)$ implies that $wa\in\mathrm{PLC}(\Pi)$.*

Corollary 1.4. *(See [9, Lecture 1].) $\phi_{\mathscr{C}}:W\to G_{\mathscr{C}}$ is an isomorphism.*

Proof. All we need to show is that $\phi_{\mathscr{C}}$ is injective. Let $w\in W$ such that $wa=a$ for all $a\in\Pi$. If $w\neq 1$ then $\ell(w)\geqslant 1$, and so we can write $w=w'r_a$ with $a\in\Pi$ and $\ell(w')=\ell(w)-1$. Since $\ell(w'r_a)>\ell(w')$ the above proposition yields that $w'a\in\mathrm{PLC}(\Pi)$; but then

$$a=wa=w'r_aa=w'(-a)=-w'a,$$

implying $0=a+w'a\in\mathrm{PLC}(\Pi)$, contradicting (C2) of the definition of a root basis. □

In particular, the above corollary yields that $(G_{\mathscr{C}},\{\rho_a\mid a\in\Pi\})$ is a Coxeter system isomorphic to (W,R). We call (W,R) the *abstract Coxeter system* associated to the Coxeter datum $\mathscr{C}$ and we call W a *Coxeter group* of rank $\#R$, where # denotes cardinality.

Definition 1.5. The *root system* of W in V is the set

$$\Phi = \{wa \mid w \in W \text{ and } a \in \Pi\}.$$

The set $\Phi^+ = \Phi \cap \mathrm{PLC}(\Pi)$ is called the set of *positive roots*, and the set $\Phi^- = -\Phi^+$ is called the set of *negative roots*.

From Proposition 1.3 and Corollary 1.4 we may readily deduce that:

Proposition 1.6. *(See [9, Lecture 3].)*

(i) *Let $w \in W$ and $a \in \Pi$. Then*

$$\ell(wr_a) = \begin{cases} \ell(w) - 1 & \text{if } wa \in \Phi^-, \\ \ell(w) + 1 & \text{if } wa \in \Phi^+. \end{cases}$$

(ii) *$\Phi = \Phi^+ \uplus \Phi^-$, where $\uplus$ denotes disjoint union.*

(iii) *W is finite if and only if Φ is finite.*

Let $T = \bigcup_{w \in W} wRw^{-1}$, and we call it the set of *reflections* in W. For $x \in \Phi$, let $\rho_x \in \mathrm{GL}(V)$ be defined by the rule: $\rho_x(v) = v - 2(v, x)x$, for all $v \in V$. Since $x \in \Phi$, it follows that $x = wa$ for some $w \in W$ and $a \in \Pi$. Direct calculations yield that $\rho_x = (\phi_{\mathscr{C}}(w))\rho_a(\phi_{\mathscr{C}}(w))^{-1} \in G_{\mathscr{C}}$. Now let $r_x \in W$ such that $\phi_{\mathscr{C}}(r_x) = \rho_x$. Then $r_x = wr_a w^{-1} \in T$, and we call it the reflection corresponding to x. It is readily checked that $r_x = r_{-x}$ for all $x \in \Phi$ and $T = \{r_x \mid x \in \Phi\}$. For each $t \in T$ we let α_t be the unique positive root with the property that $r_{\alpha_t} = t$. It is also easily checked that there is a bijection $T \leftrightarrow \Phi^+$ given by $t \to \alpha_t$ $(t \in T)$, and $x \to \phi_{\mathscr{C}}^{-1}(\rho_x)$ $(x \in \Phi^+)$. We call this bijection the *canonical bijection* between T and Φ^+.

For each $x \in \Phi^+$, as in [3], we define the *depth* of x relative to R, written $\mathrm{dp}(x)$, by requiring $\mathrm{dp}(x) = \min\{\ell(w) \mid w \in W \text{ and } wx \in \Phi^-\}$. For $x, y \in \Phi^+$, we say that x *precedes* y, written $x \prec y$ if and only if the following condition holds: there exists $w \in W$ such that $y = wx$ and $\mathrm{dp}(y) = \ell(w) + \mathrm{dp}(x)$. It is readily seen that precedence is a partial order on Φ^+, and $(\Phi^+, \prec)$ forms a *root poset* in the sense of [1]. The next result is taken from [3]:

Lemma 1.7. *(See [3, Lemma 1.7].) Let $r \in R$ and $\alpha \in \Phi^+ \setminus \{\alpha_r\}$. Then*

$$\mathrm{dp}(r\alpha) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } (\alpha, \alpha_r) > 0, \\ \mathrm{dp}(\alpha) & \text{if } (\alpha, \alpha_r) = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } (\alpha, \alpha_r) < 0. \end{cases}$$

Define functions $N : W \to \mathcal{P}(\Phi^+)$ and $\overline{N} : W \to \mathcal{P}(T)$ (where $\mathcal{P}$ denotes power set) by setting $N(w) = \{x \in \Phi^+ \mid wx \in \Phi^-\}$ and $\overline{N}(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$ for all $w \in W$. Standard arguments as those used in [11] yield that for each $w \in W$, $\ell(w) = \#N(w)$ and $\overline{N}(w) = \{r_x \mid x \in N(w)\}$. In particular, $N(r_a) = \{a\}$ for each $a \in \Pi$. Furthermore, $\ell(wv^{-1}) + \ell(v) = \ell(w)$, for some $w, v \in W$, if and only if $N(v) \subseteq N(w)$.

A subgroup W' of W is a *reflection subgroup* of W if $W' = \langle W' \cap T \rangle$ (W' is generated by the reflections that it contains). For any reflection subgroup W' of W, let

$$S(W') = \{t \in T \mid \overline{N}(t) \cap W' = \{t\}\}$$

and

$$\Delta(W') = \{x \in \Phi^+ \mid r_x \in S(W')\}.$$

It was shown by Dyer in [6] and Deodhar in [5] that $(W', S(W'))$ forms a Coxeter system:

Theorem 1.8 *(Dyer).*

(i) *Suppose that W' is a reflection subgroup of W. Then $(W', S(W'))$ forms a Coxeter system, and furthermore, $W' \cap T = \bigcup_{w \in W'} wS(W')w^{-1}$.*
(ii) *Suppose that W' is a reflection subgroup of W and suppose that $a, b \in \Delta(W')$ are distinct. Then*

$$(a,b) \in \{-\cos(\pi/n) \mid n \in \mathbb{N} \text{ and } n \geqslant 2\} \cup (-\infty, -1].$$

And conversely if Δ is a subset of Φ^+ satisfying the condition that

$$(a,b) \in \{-\cos(\pi/n) \mid n \in \mathbb{N} \text{ and } n \geqslant 2\} \cup (-\infty, -1]$$

for all $a, b \in \Delta$ with $a \neq b$, then $\Delta = \Delta(W')$ for some reflection subgroup W' of W. In fact, we have $W' = \langle\{r_a \mid a \in \Delta\}\rangle$.

Proof. (i) [6, Theorem 3.3].
(ii) [6, Theorem 4.4]. $\square$

Suppose that W' is a reflection subgroup of W and suppose that $(\,,\,)'$ is the restriction of $(\,,\,)$ on the subspace of V spanned by $\Delta(W')$. Then $\mathscr{C}' = (\operatorname{span}(\Delta(W')), \Delta(W'), (\,,\,)')$ is a Coxeter datum with $(W', S(W'))$ being the associated abstract Coxeter system. Consequently the notion of a root system applies to $\mathscr{C}'$. We let $\Phi(W')$, $\Phi^+(W')$ and $\Phi^-(W')$ be, respectively, the set of roots, positive roots and negative roots for the datum $\mathscr{C}'$. Then it follows from Definition 1.5 that $\Phi(W') = W'\Delta(W')$, $\Phi^+(W') = \Phi(W') \cap \operatorname{PLC}(\Delta(W'))$ and $\Phi^-(W') = -\Phi^+(W')$. Note that Theorem 1.8 (i) yields that

$$\Phi(W') = \{x \in \Phi \mid r_x \in W'\}.$$

We call $S(W')$ the set of *canonical generators* of W', and we call $\Delta(W')$ the set of *canonical roots* of $\Phi(W')$ (note that $\Delta(W')$ forms a root basis for the Coxeter datum $\mathscr{C}'$). In this paper a reflection subgroup W' is called a *dihedral reflection subgroup* if $\#S(W') = 2$.

A subset Φ' of Φ is called a *root subsystem* if $r_y x \in \Phi'$ whenever x, y are both in Φ'. It is easily seen that there is a bijective correspondence between reflection subgroups W' of W and root subsystems Φ' of Φ given by $W' \mapsto \Phi(W')$ and $\Phi' \mapsto \langle\{r_x \mid x \in \Phi'\}\rangle$.

Theorem 1.8 (ii) yields that if $a, b \in \Phi^+$ then $\{a, b\}$ forms the set of canonical roots for the dihedral reflection subgroup $\langle\{r_a, r_b\}\rangle$ generated by r_a and r_b if and only if $(a,b) = -\cos(\pi/n)$ for some integer $n \geqslant 2$ or else $(a,b) \leqslant -1$. Observe that in either of these cases, $\{a, b\}$ is linearly independent. In the former case a similar calculation as in Proposition 1.2 (i) yields that $(r_a r_b)^n$ acts trivially on V, furthermore, the dihedral reflection subgroup $\langle\{r_a, r_b\}\rangle$ is finite. In the latter case, let $\theta = \cosh^{-1}(-(a,b))$, and for each integer i, we employ the following notation throughout this paper:

$$c_i = \begin{cases} \frac{\sinh(i\theta)}{\sinh\theta}, & \text{if } \theta \neq 0; \\ i, & \text{if } \theta = 0. \end{cases} \tag{1.1}$$

Then similar calculations as in Proposition 1.2 (ii) yield that for each i,

$$\begin{cases} (r_a r_b)^i a = c_{2i+1} a + c_{2i} b; \\ r_b (r_a r_b)^i a = c_{2i+1} a + c_{2i+2} b; \\ (r_b r_a)^i b = c_{2i} a + c_{2i+1} b; \\ r_a (r_b r_a)^i b = c_{2i+2} a + c_{2i+1} b. \end{cases} \tag{1.2}$$

It is well known (and can be easily deduced from (1.2)) that

$$\Phi\big(\langle\{r_a, r_b\}\rangle\big) = \{c_i a + c_{i\pm1} b \mid i \in \mathbb{Z}\}. \tag{1.3}$$

Since $c_i > 0$ for all $i > 0$, it follows from (1.2) and the fact that $\{a, b\}$ is linearly independent that $r_a r_b$ has infinite order, and consequently $\langle\{r_a, r_b\}\rangle$ is an infinite dihedral reflection subgroup of W. Observe that $c_i \neq c_j$ whenever $i \neq j$, hence (1.2) yields that a and b are not conjugate to each other under the action of $\langle\{r_a, r_b\}\rangle$, and consequently $\langle\{r_a, r_b\}\rangle$ has two orbits on $\Phi(\langle\{r_a, r_b\}\rangle)$, one containing a and the other containing b. The root $c_i a + c_{i\pm1} b$ lies in the former orbit if and only if i is odd, and it lies in the latter orbit if and only if i is even.

For the rest of this section we assume that $a, b \in \Phi^+$ with $(a, b) \leqslant -1$ and we keep all the notation of the preceding paragraph.

Proposition 1.9. *Suppose that W' is a reflection subgroup of the dihedral reflection subgroup $\langle\{r_a, r_b\}\rangle$. Then $\#S(W') \leqslant 2$.*

Proof. Suppose for a contradiction that there are at least three canonical generators x, y and z for the subsystem Φ'. Then from (1.3) we know that there are three integers m, n and p with $x = c_m a + c_{m\pm1} b$, $y = c_n a + c_{n\pm1} b$ and $z = c_p a + c_{p\pm1} b$. If either

$$\begin{cases} x = c_m a + c_{m+1} b, \\ y = c_n a + c_{n+1} b \end{cases} \quad \text{or} \quad \begin{cases} x = c_m a + c_{m-1} b, \\ y = c_n a + c_{n-1} b, \end{cases}$$

then either $(x, y) = \cosh((m-n)\theta) \geqslant 1$ (if $\theta \neq 0$), or else $(x, y) = 1$ (if $\theta = 0$), resulting in a contradiction to Theorem 1.8 (ii). Without loss of generality, we may assume that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n-1} b$. Now if $z = c_p a + c_{p+1} b$, then a short calculation yields that, again, either $(x, z) = \cosh((m-p)\theta) \geqslant 1$ (if $\theta \neq 0$), or else $(x, z) = 1$ (if $\theta = 0$), a contradiction to Theorem 1.8 (ii); on the other hand if $z = c_p a + c_{p-1} b$ then, as before, either $(z, y) = \cosh((n-p)\theta) \geqslant 1$ (if $\theta \neq 0$), or else $(z, y) = 1$ (if $\theta = 0$), again a contradiction to Theorem 1.8 (ii). □

We close this section with an explicit calculation of the canonical roots for an arbitrary dihedral reflection subgroup of $\langle\{r_a, r_b\}\rangle$. These technical results will be used in Section 3. Let $\theta = \cosh^{-1}(-(a, b))$, as before.

Suppose that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n-1} b$ are positive roots in $\Phi(\langle\{r_a, r_b\}\rangle)$ (that is, m is a non-negative integer and n is a positive integer). Then either $(x, y) = -\cosh((m+n)\theta) \leqslant -1$ (when $\theta \neq 0$), or else $(x, y) = -1$ (when $\theta = 0$), and hence it follows from Theorem 1.8 (ii) that $\{x, y\} = \Delta(\langle\{r_x, r_y\}\rangle)$.

Suppose that $x = c_m a + c_{m+1} b$ and $y = c_n a + c_{n+1} b$ are roots in $\Phi(\langle\{r_a, r_b\}\rangle)$ (with $n < m \in \mathbb{Z}$). Put $d = m - n$. Proposition 1.2 (ii) yields that

$$\Phi\big(\langle\{r_x, r_y\}\rangle\big) = \{c_{kd-m} a + c_{kd-m-1} b, c_{kd+m} a + c_{kd+m+1} b \mid k \in \mathbb{Z}\}. \tag{1.4}$$

Let α, β be the canonical roots for this root subsystem. Then we claim that $\alpha = c_i a + c_{i-1} b$ and $\beta = c_j a + c_{j+1} b$ for some positive integer i and non-negative integer j. Indeed, (1.3) yields that the only other possibilities are either

$$\begin{cases} \alpha = c_i a + c_{i+1} b, \\ \beta = c_j a + c_{j+1} b \end{cases} \quad \text{or} \quad \begin{cases} \alpha = c_i a + c_{i-1} b, \\ \beta = c_j a + c_{j-1} b, \end{cases}$$

and in either of these two cases, either $(\alpha, \beta) = \cosh((i-j)\theta) \geqslant 1$, or else $(\alpha, \beta) = 1$, both contradicting Theorem 1.8 (ii). Therefore our claim holds, and in view of (1.4) we have

$$\begin{cases} \alpha = c_{k_1(m-n)-m}a + c_{k_1(m-n)-m-1}b, \\ \beta = c_{k_2(m-n)+m}a + c_{k_2(m-n)+m+1}b, \end{cases} \tag{1.5}$$

for some integers k_1 and k_2. In fact, k_1 and k_2 satisfy the condition that $k_1(m-n)-m$ is the smallest positive integer of this form and $k_2(m-n)+m$ is the smallest non-negative integer of this form.

Suppose that $x = c_{m+1}a + c_m b$ and $y = c_{n+1}a + c_n b$ are roots in $\Phi(\langle\{r_a, r_b\}\rangle)$ (with $n, m \in \mathbb{Z}$). Put $d = m - n$. Interchanging the roles of a and b in the preceding paragraph, we see that

$$\Phi\big(\big\langle\{r_x, r_y\}\big\rangle\big) = \{c_{ld+m+1}a + c_{ld+m}b, c_{ld-m-1}a + c_{ld-m}b \mid k \in \mathbb{Z}\}. \tag{1.6}$$

Let α', β' be the canonical roots for this root subsystem. Exactly the same reasoning as in the preceding paragraph yields that

$$\begin{cases} \alpha' = c_{l_1(m-n)+m+1}a + c_{l_1(m-n)+m}b, \\ \beta' = c_{l_2(m-n)-m-1}a + c_{l_2(m-n)-m}b, \end{cases} \tag{1.7}$$

for some integers l_1 and l_2. Indeed l_1 and l_2 satisfy the conditions that $l_1(m-n)+m$ is the smallest non-negative integer of this form and $l_2(m-n)-m$ is the smallest positive integer of this form.

2. Canonical coefficients

For a Coxeter datum $\mathscr{C} = (V, \Pi, (\,,\,))$, since Π may be linearly dependent, the expression of a root in Φ as a linear combination of elements of Π may not be unique. Thus the concept of the coefficient of an element of Π in any given root in Φ is potentially ambiguous. This section gives a canonical way of expressing a root in Φ as a linear combination of elements from Π. This canonical expression follows from a standard construction similar to the one considered in [10].

Given a Coxeter datum $\mathscr{C} = (V, \Pi, (\,,\,))$, let E be a vector space over $\mathbb{R}$ with basis $\Pi_E = \{e_a \mid a \in \Pi\}$ in bijective correspondence with Π and let $(\,,\,)_E$ be the unique bilinear form on E satisfying

$$(e_a, e_b)_E = (a, b), \quad \text{for all } a, b \in \Pi.$$

Then $\mathscr{C}_E = (E, \Pi_E, (\,,\,)_E)$ is a Coxeter datum. Moreover, $\mathscr{C}_E$ and $\mathscr{C}$ are associated to the same abstract Coxeter system (W, R). Corollary 1.4 yields that $\phi_{\mathscr{C}_E} : W \to G_{\mathscr{C}_E} = \langle\{\rho_{e_a} \mid a \in \Pi\}\rangle$ is an isomorphism. Furthermore, W acts faithfully on E via $r_a y = \rho_{e_a} y$ for all $a \in \Pi$ and $y \in E$.

Let $f : E \to V$ be the unique linear map satisfying $f(e_a) = a$, for all $a \in \Pi$. It is readily checked that $(f(x), f(y)) = (x, y)_E$, for all $x, y \in E$. Now for all $a \in \Pi$ and $y \in E$,

$$\begin{aligned} r_a\big(f(y)\big) = \rho_a\big(f(y)\big) &= f(y) - 2\big(f(y), a\big)a = f(y) - 2\big(f(y), f(e_a)\big)f(e_a) \\ &= f\big(y - 2(y, e_a)_E e_a\big) \\ &= f(\rho_{e_a} y) \\ &= f(r_a y). \end{aligned}$$

Then it follows that $wf(y) = f(wy)$, for all $w \in W$ and all $y \in E$, since W is generated by $\{r_a \mid a \in \Pi\}$. Let Φ_E denote the root system associated to the datum $\mathscr{C}_E$, and let Φ_E^+ (respectively, Φ_E^-) denote the corresponding set of positive roots (respectively, negative roots). Then a similar reasoning as that of Proposition 2.9 of [10] enables us to have:

Proposition 2.1. *The restriction of f defines a W-equivariant bijection $\Phi_E \to \Phi$.*

Proof. Since $f(we_a) = wa$ for all $w \in W$ and $a \in \Pi$, it follows that $f(\Phi_E) = \Phi$. Proposition 1.6 applied to $\mathscr{C}_E$ yields that, $we_a \in \Phi_E^+$ if and only if $\ell(wr_a) = \ell(w) + 1$, and this happens if and only if

$wa \in \Phi^+$, so $f(\Phi_E^+) = \Phi^+$. We are done if we can show that the restriction of f on Φ_E^+ is injective. Suppose that there are $x, y \in \Phi_E^+$ with $f(x) = f(y)$. Then $\phi_{\mathscr{C}}\phi_{\mathscr{C}_E}^{-1}(\rho_x) = \rho_{f(x)} = \rho_{f(y)} = \phi_{\mathscr{C}}\phi_{\mathscr{C}_E}^{-1}(\rho_y)$. Since $\phi_{\mathscr{C}}$ is an isomorphism, it follows that $\phi_{\mathscr{C}_E}^{-1}(\rho_x) = \phi_{\mathscr{C}_E}^{-1}(\rho_y)$, that is, x and y correspond to the same reflection in W. Since $x, y \in \Phi_E^+$, it follows that $x = y$, as required. □

Since Π_E is linearly independent, it follows that each root $y \in \Phi_E$ can be written uniquely as $\sum_{a \in \Pi} \lambda_a e_a$; we say that λ_a is the *coefficient* of e_a in y and it is denoted by $\mathrm{coeff}_{e_a}(y)$. We use this fact together with the W-equivariant bijection $f : \Phi_E \leftrightarrow \Phi$ to give a canonical expression of a root in Φ in terms of Π:

Definition 2.2. Suppose that $x \in \Phi$. For each $a \in \Pi$, define the *canonical coefficient* of a in x, written $\mathrm{coeff}_a(x)$, by requiring that $\mathrm{coeff}_a(x) = \mathrm{coeff}_{e_a}(f^{-1}(x))$. The *support*, written $\mathrm{supp}(x)$, is the set of $a \in \Pi$ with $\mathrm{coeff}_a(x) \neq 0$.

3. The dominance hierarchy

Definition 3.1.

(i) For x and $y \in \Phi$, we say that x *dominates* y with respect to W if $\{w \in W \mid wx \in \Phi^-\} \subseteq \{w \in W \mid wy \in \Phi^-\}$. If x dominates y with respect to W then we write $x \trianglerighteq y$.
(ii) For each $x \in \Phi^+$, set $D(x) = \{y \in \Phi^+ \mid y \neq x \text{ and } x \trianglerighteq y\}$, and if $x \in \Phi^+$ and $D(x) = \emptyset$ then x is called *elementary*. For each $n \in \mathbb{N}$, define $D_n = \{x \in \Phi^+ \mid \#D(x) = n\}$.

Note that D_0 here is the same set as $\mathscr{E}$ of [3] and [4]. In [3] and [4] dominance is only defined on Φ^+, and it is found in [3] that dominance is a partial order on Φ^+. Here we have generalized the notion of dominance to the whole of Φ, as was considered in, for example, [10]. It can be readily seen that this generalized dominance is a partial order on Φ. Observe that it is clear from the above definition that

$$\Phi^+ = \biguplus_{n \in \mathbb{N}} D_n.$$

The set D_0 has been properly investigated in [3] and [4]: if W is finite then $D_0 = \Phi^+$ (that is, if W is finite, then there is no non-trivial dominance among its roots), whereas if W is an infinite Coxeter group of finite rank, then $\#D_0 < \infty$ and furthermore, we can explicitly compute D_0. Observe that in the latter case $\biguplus_{n \in \mathbb{N}, n \geqslant 1} D_n$ will be an infinite set. One major result of this paper (Theorem 3.8 below) is that if R is finite then D_n is finite for all natural numbers n. We also give upper and lower bounds on $\#D_n$ (Corollary 3.9 and Corollary 3.21 below). But first we need a few elementary results:

Lemma 3.2.

(i) *If x and $y \in \Phi^+$, then $x \trianglerighteq y$ if and only if $(x, y) \geqslant 1$ and $\mathrm{dp}(x) \geqslant \mathrm{dp}(y)$ (with equality on depth if and only if $x = y$).*
(ii) *Dominance is W-invariant: if $x \trianglerighteq y$ then $wx \trianglerighteq wy$ for any $w \in W$.*
(iii) *Suppose that $x, y \in \Phi$, and $x \trianglerighteq y$. Then $-y \trianglerighteq -x$.*
(iv) *Suppose that $x \in \Phi^+$ and $y \in \Phi^-$. Then $x \trianglerighteq y$ if and only if $(x, y) \geqslant 1$.*
(v) *Let $x, y \in \Phi$. Then there is dominance between x and y if and only if $(x, y) \geqslant 1$.*

Proof. (i) Essentially the same reasoning as in [3, Lemma 2.3] applies.

(ii) Clear from the definition of dominance.

(iii) Suppose for a contradiction that there exists $w \in W$ such that $w(-y) \in \Phi^-$ and $w(-x) \in \Phi^+$. Then $w(y) \in \Phi^+$ yet $w(x) \in \Phi^-$, contradicting the assumption that $x \trianglerighteq y$.

(iv) Suppose that $x \trianglerighteq y$. Since dominance is W-invariant, it follows that $r_y x \trianglerighteq r_y y \in \Phi^+$ and hence $r_y x \in \Phi^+$. Now part (i) yields that $(r_y x, r_y y) \geqslant 1$. Since $(\,,\,)$ is W-invariant, it follows that $(x, y) \geqslant 1$.

Conversely, suppose that $x \in \Phi^+$ and $y \in \Phi^-$ with $(x, y) \geqslant 1$. Then clearly $r_y x = x - 2(x, y)y \in \Phi^+$. Thus $r_y x$ and $r_y y = -y$ are both positive. Then it follows from part (i) that there is dominance between $r_y x$ and $r_y y$. Since dominance is W-invariant, it follows that there is dominance between x and y. Finally, given that $x \in \Phi^+$ and $y \in \Phi^-$, it is clear that $x \trianglerighteq y$.

(v) Suppose that $x, y \in \Phi^-$. Then part (i) yields that there is dominance between $-x$ and $-y$ if and only if $(-x, -y) = (x, y) \geqslant 1$. This combined with part (i) and part (iv) above yields the desired result. □

The following is a simple result that we use repeatedly in this paper:

Lemma 3.3. *Let $x, y \in \Phi$ be distinct with $x \trianglerighteq y$ and $y \in D_0$. Then*:

(i) $r_y x \in \Phi^+$;
(ii) $(r_y x, x) \leqslant -1$ *and* $(r_y x, y) \leqslant -1$, *and in particular, $r_y x$ cannot dominate either x or y.*

Proof. (i) Suppose for a contradiction that $r_y x \in \Phi^-$. Lemma 3.2 (ii) then yields that $r_y x \trianglerighteq r_y y = -y$. Now Lemma 3.2 (iii) yields that $y \trianglerighteq -r_y x \in \Phi^+$. Since $y \in D_0$, this forces $-r_y x = y$, contradicting $x \neq y$.

(ii) Since $x \trianglerighteq y$, it follows from Lemma 3.2 (v) that $(x, y) \geqslant 1$. Then $(r_y x, y) = (x, -y) \leqslant -1$ and hence there is no dominance between $r_y x$ and y. Also $(r_y x, x) = (x, x) - 2(x, y)^2 \leqslant -1$, and thus there is no dominance between x and $r_y x$ either. □

Suppose that $x, y \in \Phi$ with $x \trianglerighteq y$. It is worthwhile investigating the connection between this dominance and the canonical generators of the root subsystem $\Phi(\langle\{r_x, r_y\}\rangle)$.

Proposition 3.4. *Suppose that $x, y \in \Phi$ are distinct with $x \trianglerighteq y$. Let a, b be the canonical roots for the root subsystem $\Phi(\langle\{r_x, r_y\}\rangle)$. Then there exists $w \in \langle\{r_x, r_y\}\rangle$ such that either*

$$\begin{cases} wx = a, \\ wy = -b \end{cases} \quad \text{or else} \quad \begin{cases} wx = b, \\ wy = -a. \end{cases}$$

In particular, $(a, b) = -(x, y)$.

Proof. By Theorem 1.8 (ii) we know that

$$(a, b) \in (-\infty, -1] \cup \left\{-\cos(\pi/n) \;\middle|\; n \in \mathbb{N} \text{ and } n \geqslant 2\right\}.$$

Suppose for a contradiction that $(a, b) = -\cos(\pi/n)$ for some integer $n \geqslant 2$. Write $\theta = \pi/n$, and Proposition 1.2 (i) yields that

$$\Phi\left(\left\langle\{r_a, r_b\}\right\rangle\right) = \left\{\frac{\sin(m+1)\theta}{\sin\theta}a + \frac{\sin m\theta}{\sin\theta}b \;\middle|\; m \in \mathbb{N} \text{ and } 0 \leqslant m < 2n\right\}.$$

So there are distinct integers m_1 and m_2 (strictly less than $2n$) with

$$x = \frac{\sin(m_1+1)\theta}{\sin\theta}a + \frac{\sin m_1\theta}{\sin\theta}b \quad \text{and} \quad y = \frac{\sin(m_2+1)\theta}{\sin\theta}a + \frac{\sin m_2\theta}{\sin\theta}b.$$

But then $(x, y) = \cos((m_1 - m_2)\pi/n) < 1$, contradicting Lemma 3.2 (v). Thus $(a, b) \leqslant -1$ and so Lemma 3.2 (v) yields that $a \trianglerighteq -b$ and $b \trianglerighteq -a$. It then follows readily that there are two dominance chains in the root subsystem $\Phi(\langle\{r_a, r_b\}\rangle)$, namely:

$$\cdots \trianglerighteq r_a r_b r_a(b) \trianglerighteq r_a r_b(a) \trianglerighteq r_a(b) \trianglerighteq a \trianglerighteq -b \trianglerighteq r_b(-a) \trianglerighteq r_b r_a(-b) \trianglerighteq \cdots \tag{3.1}$$

and

$$\cdots \trianglerighteq r_b r_a r_b(a) \trianglerighteq r_b r_a(b) \trianglerighteq r_b(a) \trianglerighteq b \trianglerighteq -a \trianglerighteq r_a(-b) \trianglerighteq r_a r_b(-a) \trianglerighteq \cdots. \tag{3.2}$$

Observe that each element of $\Phi(\langle\{r_a, r_b\}\rangle)$ lies in exactly one of the above chains, and the negative of any element of one of these chains lies in the other. Thus $x', y' \in \Phi(\langle\{r_a, r_b\}\rangle)$ are in the same chain if and only if $(x', y') \geqslant 1$ and in different chains if and only if $(x', y') \leqslant -1$.

From (3.1) we see that the roots dominated by a are all negative, and from (3.2) we see that the roots dominated by b are all negative. Clearly we may choose $w \in \langle\{r_a, r_b\}\rangle$ such that either $wx = a$ or $wx = b$, and since $wx \trianglerighteq wy$, it follows that either

$$wx = a \quad \text{and} \quad wy \in \Phi\big(\big\langle\{r_a, r_b\}\big\rangle\big) \cap \Phi^- \tag{3.3}$$

or

$$wx = b \quad \text{and} \quad wy \in \Phi\big(\big\langle\{r_a, r_b\}\big\rangle\big) \cap \Phi^-. \tag{3.4}$$

Suppose that $wx = a$. Then $(a, -wy) = (wx, -wy) = -(x, y) \leqslant -1$. Since $-wy \in \Phi(\langle\{r_x, r_y\}\rangle) \cap \Phi^+$ and $\langle\{r_a, r_{wy}\}\rangle = \langle\{r_x, r_y\}\rangle$, it follows from Theorem 1.8 (ii) that $\{a, -wy\}$ is the set of canonical roots for $\Phi(\langle\{r_x, r_y\}\rangle)$, which then forces that $-wy = b$. Similarly, in the case $wx = b$, we may conclude that $wy = -a$. $\square$

Lemma 3.5. *Suppose that $x, y \in \Phi$ are distinct with $x \trianglerighteq y$. Let a and b be the canonical roots for $\Phi(\langle\{r_x, r_y\}\rangle)$. Then either*

$$\begin{cases} x = c_m a + c_{m+1} b, \\ y = c_{m-1} a + c_m b \end{cases} \quad \text{or} \quad \begin{cases} x = c_m a + c_{m-1} b, \\ y = c_{m-1} a + c_{m-2} b, \end{cases}$$

for some integer m, where c_i is as defined in (1.1) for each integer i.

Proof. Proposition 3.4 yields that $(a, b) \leqslant -1$. Since a, b are the canonical roots of $\Phi(\langle\{r_x, r_y\}\rangle)$, it follows from Eq. (1.3) that $x = c_m a + c_{m\pm1} b$ and $y = c_n a + c_{n\pm1} b$, for some integers m and n. Let $\theta = \cosh^{-1}(-(a, b))$. If either

$$\begin{cases} x = c_m a + c_{m+1} b, \\ y = c_n a + c_{n-1} b \end{cases} \quad \text{or} \quad \begin{cases} x = c_m a + c_{m-1} b, \\ y = c_n a + c_{n+1} b, \end{cases}$$

then either $(x, y) = -\cosh((n + m)\theta) \leqslant -1$ (when $\theta \neq 0$), or else $(x, y) = -1$ (when $\theta = 0$), contradicting $x \trianglerighteq y$. Therefore there are only two possibilities, namely:

$$\begin{cases} x = c_m a + c_{m+1} b, \\ y = c_n a + c_{n+1} b \end{cases} \tag{3.5}$$

or

$$\begin{cases} x = c_m a + c_{m-1} b, \\ y = c_n a + c_{n-1} b. \end{cases} \tag{3.6}$$

First suppose that (3.5) is the case. Since a and b are the canonical roots for $\Phi(\langle\{r_a, r_b\}\rangle) = \Phi(\langle\{r_x, r_y\}\rangle)$, it follows from Eq. (1.5) that there are integers k_1 and k_2 such that

$$1 = k_1(m-n) - m \quad \text{and} \quad 0 = k_2(m-n) + m.$$

But then $k_1 + k_2 = \frac{1}{m-n} \in \mathbb{Z}$. Clearly this is only possible when $m - n = \pm 1$. On the other hand, since $x \trianglerighteq y$, it is readily seen that $m > n$, giving us $x = c_m a + c_{m+1} b$ and $y = c_{m-1} a + c_m b$. On the other hand, if (3.6) is the case, then by taking Eq. (1.7) into consideration, a similar reasoning as above yields that $x = c_m a + c_{m-1} b$ and $y = c_{m-1} a + c_{m-2} b$. $\square$

Remark 3.6. Let x and y be as in Proposition 3.4 and Lemma 3.5 above. Then in fact x and y are consecutive terms in precisely one of the dominance chains (3.1) or (3.2).

Now we are ready for the first key result of this paper:

Theorem 3.7. *$D_1 \subseteq \{r_a b \mid a, b \in D_0\}$. Furthermore, if $\#R < \infty$ then $\#D_1 \leqslant (\#D_0)^2 - \#D_0$.*

Proof. Suppose that $x \in D_1$ and let $D(x) = \{y\}$. Clearly $y \in D_0$. By Lemma 3.3 (i), we know that $r_y x \in \Phi^+$. Thus to prove Theorem 3.7, it suffices to show that $r_y x \in D_0$.

Suppose for a contradiction that $r_y x \in \Phi^+ \setminus D_0$. Then there exists $z \in \Phi^+ \setminus \{r_y x\}$ with $r_y x \trianglerighteq z$. Since dominance is W-invariant, it follows that $x \trianglerighteq r_y z$. If $r_y z = y$ then $z \in \Phi^-$, contradicting our choice for z. Then the fact $D(x) = \{y\}$ implies that $r_y z \in \Phi^-$ and in particular, $(z, y) > 0$. Since $r_y x \trianglerighteq z$ and $x \trianglerighteq y$, it follows from Lemma 3.2 (i) that $(r_y x, z) \geqslant 1$ and $(x, y) \geqslant 1$. Then

$$\begin{aligned} 1 \leqslant (r_y x, z) &= \big(x - 2(x, y)y, z\big) \\ &= (x, z) - 2(x, y)(y, z), \end{aligned}$$

implying that $1 \leqslant (x, z)$. Hence Lemma 3.2 (v) yields that either $x \trianglerighteq z$ or else $z \trianglerighteq x$. In the latter case $r_y x \trianglerighteq z \trianglerighteq x$, contradicting Lemma 3.3 (ii). On the other hand, if $x \trianglerighteq z$, then our construction forces $z = y$. But then $r_y x \trianglerighteq y$, again contradicting Lemma 3.3 (ii). Thus $r_y x \in D_0$, as required. Since $x \in D_1$ was arbitrary, it follows that $D_1 \subseteq \{r_a b \mid a, b \in D_0\}$.

Finally, since D_1 does not contain elements of the form $r_a a$, where $a \in D_0$, it follows that

$$D_1 \subseteq \{r_a b \mid a, b \in D_0\} \setminus -D_0. \tag{3.7}$$

In the case that $\#R < \infty$, Theorem 2.8 of [3] yields that $\#D_0 < \infty$, and so it follows from (3.7) that $\#D_1 \leqslant (\#D_0)^2 - \#D_0$. $\square$

The above treatment of D_1 can be generalized to D_n for arbitrary $n \in \mathbb{N}$. Indeed we have:

Theorem 3.8. *For $n \in \mathbb{N}$,*

$$D_n \subseteq \left\{ r_a b \;\middle|\; a \in D_0,\ b \in \biguplus_{m \leqslant n-1} D_m \right\}.$$

Proof. The case $n = 1$ has been covered by Theorem 3.7, so we may assume that $n > 1$.

Let $x \in D_n$, and suppose that $D(x) = \{y_1, y_2, \ldots, y_n\}$, with y_n being minimal with respect to dominance. Clearly $y_n \in D_0$ and so Lemma 3.3 (i) yields that $r_{y_n} x \in \Phi^+$. Hence either $r_{y_n} x \in D_0$ or else $r_{y_n} x \in \Phi^+ \setminus D_0$.

If $r_{y_n}x \in D_0$, then

$$x \in \{r_a b \mid a, b \in D_0\} \subseteq \left\{r_a b \;\middle|\; a \in D_0,\ b \in \biguplus_{m \leqslant n-1} D_m\right\},$$

and the desired result clearly follows, given the arbitrary choice of x.

If $r_{y_n}x \in \Phi^+ \setminus D_0$, let $z \in D(r_{y_n}x)$. We claim that there are at most $(n-1)$ possible values for z. Observe that this claim implies the following:

$$r_{y_n}x \in \biguplus_{m \leqslant n-1} D_m,$$

and it follows immediately that $D_n \subseteq \{r_a b \mid a \in D_0,\ b \in \biguplus_{m \leqslant n-1} D_m\}$, since $x \in D_n$ was arbitrary.

Thus all it remains to do is to prove the above claim. Since $r_{y_n}x \trianglerighteq z$, Lemma 3.2 (ii) yields that $x \trianglerighteq r_{y_n}z$. Thus either $r_{y_n}z \in \Phi^+$ and in which case $r_{y_n}z = y_i$, for $1 \leqslant i \leqslant n-1$; or else $r_{y_n}z \in \Phi^-$. If $r_{y_n}z \in \Phi^-$ then clearly $(y_n, z) > 0$. Since $r_{y_n}x \trianglerighteq z$ and $x \trianglerighteq y_n$, Lemma 3.2 (v) yields that $(r_{y_n}x, z) \geqslant 1$ and $(x, y_n) \geqslant 1$. Then

$$\begin{aligned} 1 \leqslant (r_{y_n}x, z) &= \bigl(x - 2(x, y_n)y_n, z\bigr) \\ &= (x, z) - 2(x, y_n)(y_n, z), \end{aligned}$$

and hence it follows that $(x, z) \geqslant 1$. Similar to the proof of Theorem 3.7, we can conclude that $x \trianglerighteq z$ and so $z \in \{y_1, \ldots, y_n\}$. Since $x \trianglerighteq z$ as well as $r_{y_n}x \trianglerighteq z$, Lemma 3.3 (ii) yields that $z \in \{y_1, \ldots, y_{n-1}\}$. Summing up, if $z \in D(r_{y_n}x)$, then

$$z \in \bigl\{r_{y_n}(y_i) \bigm| r_{y_n}(y_i) \in \Phi^+,\ i \in \{1, \ldots, n-1\}\bigr\} \cup \bigl\{y_i \bigm| r_{y_n}(y_i) \in \Phi^-,\ i \in \{1, \ldots, n-1\}\bigr\},$$

and this is clearly a disjoint union of size $n-1$. Thus $r_{y_n}x \in D_m$, for some $m \leqslant n-1$ and the claim is proved. □

Note that for each positive integer n, Theorem 3.8 immediately yields the following upper bound for the size of the corresponding D_n.

Corollary 3.9. *Suppose that $\#R < \infty$. Then $\#D_n < \infty$ for all $n \in \mathbb{N}$. Indeed*

$$\#D_n \leqslant (\#D_0)^{n+1} - (\#D_0)^n.$$

Proof. Clearly $D_i \cap D_j = \emptyset$ whenever $i \neq j$, so Theorem 3.8 yields that $D_n \subseteq \{r_a b \mid a \in D_0,\ b \in \biguplus_{m \leqslant n-1} D_m\} \setminus (\biguplus_{m<n} D_m)$, and the desired result then follows from a simple induction on n. □

Having shown that $\#D_n < \infty$ for all $n \in \mathbb{N}$ if $\#R < \infty$, it is not immediately clear, at this stage, that for each $n \in \mathbb{N}$, the corresponding $D_n \neq \emptyset$. Lemma 3.10 to Corollary 3.21 below will, amongst other things, establish that $D_n \neq \emptyset$ for each $n \in \mathbb{N}$ if W is an infinite Coxeter group of finite rank.

Lemma 3.10. *For $n \in \mathbb{N}$,*

$$\bigl\{wa \bigm| a \in D_0,\ w \in W,\ \ell(w) < n\bigr\} \cap D_n = \emptyset.$$

Proof. Suppose for a contradiction that there exist some $n \in \mathbb{N}$ and $x = wa \in D_n$ such that $a \in D_0$ and $w \in W$ with $\ell(w) < n$. Suppose that $D(x) = \{y_1, \dots, y_n\}$. Since dominance is W-invariant, it follows that $a = w^{-1}x$ dominates all of $w^{-1}y_1, w^{-1}y_2, \dots, w^{-1}y_n$. Note that $a \notin \{w^{-1}y_1, \dots, w^{-1}y_n\}$. Since a is elementary, it follows that $w^{-1}y_1, \dots, w^{-1}y_n \in \Phi^-$, that is, $y_1, \dots, y_n \in N(w^{-1})$, but this contradicts the fact that $\#N(w^{-1}) = \ell(w^{-1}) = \ell(w) < n$. □

Lemma 3.11.

$$RD_0 \subseteq -D_0 \uplus D_0 \uplus D_1.$$

Proof. Suppose that $r \in R$ and $x \in D_0$ are arbitrary. If $rx \in \Phi^+$, then Lemma 3.10 above yields that $rx \in D_0 \uplus D_1$. On the other hand, if $rx \in \Phi^-$, then $x \in \Pi$, which in turn implies that $r = r_x$ and $rx = -x \in -\Pi \subseteq -D_0$. □

Generalizing Lemma 3.11, we have:

Lemma 3.12. *For all $n \geqslant 1$,*

$$RD_n \subseteq D_{n-1} \uplus D_n \uplus D_{n+1}.$$

Proof. Suppose that $n \geqslant 1$, and let $x \in D_n$, and $z \in \Pi$ be arbitrary. Since $x \neq z$, it follows that $r_z x \in \Phi^+$.

Suppose for a contradiction that $r_z x \in D_m$ for some $m \geqslant n+2$. Let $D(r_z x) = \{y_1, \dots, y_m\}$. Then $x \trianglerighteq r_z y_1, \dots, r_z y_m$. Since $x \in D_n$, and $m \geqslant n+2$, it follows that there are $1 \leqslant i < j \leqslant m$ with $r_z y_i \in \Phi^-$ and $r_z y_j \in \Phi^-$. But this is impossible, since r_z could only make one positive root negative. Therefore we may conclude that $r_z x \notin D_m$ where $m \geqslant n+2$. A similar argument also shows that $r_z x \notin D_{m'}$ where $m' \leqslant n-2$, and we are done. □

Lemma 3.13. *Suppose that x, y are in Φ^+ with $y \prec x$. Let $w \in W$ be such that $x = wy$ and $dp(x) = dp(y) + \ell(w)$. Then $y \in D_m$ implies that $x \in D_n$ for some $n \geqslant m$. Furthermore, $wD(y) \subseteq D(x)$.*

Proof. It is enough to show that the desired result holds in the case that $w = r_a$ for some $a \in \Pi$. The more general proof then follows from an induction on $\ell(w)$.

Since $x = r_a y$ and $y \prec x$, Lemma 1.7 yields that $(a, y) < 0$, and so Lemma 3.2 (v) yields that $a \notin D(y)$. Let $D(y) = \{z_1, z_2, \dots, z_m\}$. Then the fact $a \in \Pi$ implies $r_a D(y) \subset \Phi^+$. Since dominance is W-invariant, it follows that $x \trianglerighteq r_a z_i$ for all $i \in \{1, 2 \dots, m\}$. Therefore $\{r_a z_1, r_a z_2, \dots, r_a z_m\} \subseteq D(x)$, whence $x \in D_n$ for some integer $n \geqslant m$, and $r_a D(y) \subseteq D(x)$. □

The next proposition, somewhat an analogue to Lemma 1.7, has many applications, among which, we can deduce, for arbitrary positive root x, the integer n for which $x \in D_n$. Furthermore, it enables us to compute $D(x)$ explicitly as well as to obtain an algorithm to compute all the D_n's systematically.

Proposition 3.14. *Suppose that $x \in D_n$ with $n \geqslant 1$, and $a \in \Pi$. Then*

(i) $r_a x \in D_{n-1}$ *if and only if* $(x, a) \geqslant 1$;
(ii) $r_a x \in D_{n+1}$ *if and only if* $(x, a) \leqslant -1$;
(iii) $r_a x \in D_n$ *if and only if* $(x, a) \in (-1, 1)$.

Proof. (i) Suppose that $x \in D_n$ and $a \in \Pi$ such that $r_a x \in D_{n-1}$. Let $D(x) = \{z_1, z_2, \dots, z_n\}$. Since dominance is W-invariant, it follows that $r_a x \trianglerighteq r_a z_i$ for all $i \in \{1, 2, \dots, n\}$. Thus at least one of $r_a z_1, \dots, r_a z_n$ must be negative. Without loss of generality, we may assume that $r_a z_1 \in \Phi^-$. Since $a \in \Pi$, it follows that $a = z_1$. Therefore $x \trianglerighteq a$, and Lemma 3.2 (v) then yields that $(x, a) \geqslant 1$.

Conversely, suppose that $x \in D_n$ and $a \in \Pi$ such that $(x, a) \geqslant 1$. Then Lemma 3.2 (i) yields that $x \trianglerighteq a$; furthermore, Lemma 1.7 yields that $r_a x \prec x$. Hence Lemma 3.13 yields that

$$r_a D(r_a x) \subseteq D(x). \tag{3.8}$$

Now suppose for a contradiction that $r_a x \notin D_{n-1}$. Then Lemma 3.12 yields that $r_a x \in D_n \uplus D_{n+1}$. From (3.8) it is clear that $r_a x \notin D_{n+1}$. But if $r_a x \in D_n$, then (3.8) yields that $r_a D(r_a x) = D(x)$. Observe that $a \in D(x)$ and $a \notin r_a D(r_a x)$, producing a contradiction as desired.

(ii) Replace x by $r_a x$ in (i) above then we may obtain the desired result.

(iii) Follows from (i), (ii) and Lemma 3.12. □

Definition 3.15. For each $x \in \Phi^+$, define

$$S(x) = \{ w \in W \mid \ell(w) = dp(x) - 1 \text{ and } w^{-1}x \in \Pi \},$$
$$T(x) = \{ w \in W \mid \ell(w) = dp(x) \text{ and } w^{-1}x \in \Phi^- \}.$$

In other words, for $x \in \Phi^+$, $S(x)$ (respectively, $T(x)$) consists of all $w \in W$ of minimal length with $w^{-1}x \in \Pi$ (respectively, $w^{-1}x \in -\Pi$). Note that for each $w \in S(x)$, there exist some $w' \in T(x)$ and $a \in \Pi$ such that $w' = wr_a$ with $\ell(w') = \ell(w) + 1$.

Proposition 3.16. *Suppose that $x \in \Phi^+$ and let $w \in S(x)$ be arbitrarily chosen. Then $x \in D_n$ where $n = \#\{b \in N(w^{-1}) \mid (x, b) \geqslant 1\}$. In particular, the integer n is independent of the choice of $w \in S(x)$.*

Proof. Let $x \in \Phi^+$ and write $x = wa$ where $w \in S(x)$ and $a \in \Pi$. Let $w = r_{a_1} \cdots r_{a_l}$ be such that $l = \ell(w)$ and $a_1, a_2, \ldots, a_l \in \Pi$. Observe that for each $i \in \{2, \ldots, l\}$,

$$\begin{aligned} w^{-1}(r_{a_1} r_{a_2} \cdots r_{a_{i-2}}) a_{i-1} &= r_{a_l} \cdots r_{a_1} r_{a_1} \cdots r_{a_{i-2}} a_{i-1} \\ &= r_{a_l} \cdots r_{a_i} r_{a_{i-1}} a_{i-1} \\ &= -r_{a_l} \cdots r_{a_i} a_{i-1}. \end{aligned} \tag{3.9}$$

Under our assumptions

$$\ell(r_{a_l} r_{a_{l-1}} \cdots r_{a_i} r_{a_{i-1}}) = \ell(r_{a_l} \cdots r_{a_i}) + 1 \quad \text{and} \quad \ell(r_{a_1} r_{a_2} \cdots r_{a_{i-2}} r_{a_{i-1}}) = \ell(r_{a_1} r_{a_2} \cdots r_{a_{i-2}}) + 1,$$

hence Proposition 1.6 (i) yields that $r_{a_l} \cdots r_{a_i} a_{i-1} \in \Phi^+$ and $r_{a_1} r_{a_2} \cdots r_{a_{i-2}} a_{i-1} \in \Phi^+$. Thus (3.9) yields that

$$(r_{a_1} r_{a_2} \cdots r_{a_{i-2}}) a_{i-1} \in N(w^{-1}). \tag{3.10}$$

Now by Proposition 3.14, we can immediately deduce that $x \in D_n$ where

$$\begin{aligned} n &= \#\{ i \mid (a_{i-1}, r_{a_i} r_{a_{i+1}} \cdots r_{a_l} a) \leqslant -1 \} \\ &= \#\{ i \mid (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}), r_{a_1} \cdots r_{a_l}(a)) \leqslant -1 \} \\ &= \#\{ i \mid (r_{a_1} \cdots r_{a_{i-1}}(a_{i-1}), x) \leqslant -1 \} \\ &= \#\{ i \mid (-r_{a_1} \cdots r_{a_{i-2}}(a_{i-1}), x) \leqslant -1 \} \\ &= \#\{ b \in N(w^{-1}) \mid (-b, x) \leqslant -1 \} \\ &= \#\{ b \in N(w^{-1}) \mid (b, x) \geqslant 1 \}. \end{aligned}$$

Lemma 3.2 (v) then yields that either $x \trianglerighteq b$ or $b \trianglerighteq x$. Since all such b are in $N(w^{-1})$ where $w \in S(x)$, it follows that $w^{-1}x \in \Pi$ and $w^{-1}b \in \Phi^-$. Thus b cannot dominate x. So we may conclude that $x \in D_n$, where

$$n = \#\{b \in N(w^{-1}) \mid x \trianglerighteq b\}, \tag{3.11}$$

for all $w \in S(x)$. But (3.11) says precisely that $D(x) \subseteq N(w^{-1})$ and

$$\begin{aligned} D(x) &= \{b \in N(w^{-1}) \mid x \trianglerighteq b\} \\ &= \{b \in N(w^{-1}) \mid (x, b) \geqslant 1\}. \qquad \square \end{aligned}$$

From the above proof we immediately have:

Corollary 3.17. *Let $x \in \Phi^+$. Then $D(x) \subseteq \bigcap_{w \in S(x)} N(w^{-1})$.*

It turns out that we can also say something about the roots in $\bigcap_{w \in S(x)} N(w^{-1}) \setminus D(x)$. Indeed in the next two lemmas we deduce that if $b \in \bigcap_{w \in S(x)} N(w^{-1})$, then $(x, b) > 0$.

Lemma 3.18. *Suppose that $x \in \Phi^+$, $w \in T(x)$ and $b \in N(w^{-1})$. Then $(b, x) > 0$.*

Proof. If $\mathrm{dp}(x) = 1$ then $x \in \Pi$, whence $T(x) = \{r_x\}$ and $x = b$, and so $(b, x) = 1$ as required. Thus we may assume that $\mathrm{dp}(x) > 1$ and proceed by an induction on $\mathrm{dp}(x)$. Let $a \in \Pi \cap N(w^{-1})$. Then

$$\ell(r_a w) = \ell(w^{-1} r_a) = \ell(w^{-1}) - 1 = \ell(w) - 1.$$

Now since $(r_a w)^{-1}(r_a x) = w^{-1}x \in \Phi^-$, it follows that

$$\mathrm{dp}(r_a x) \leqslant \ell(r_a w) < \ell(w) = \mathrm{dp}(x),$$

and hence Lemma 1.7 yields that $(a, x) > 0$. If $b = a$ then we are done, thus we may assume that $b \neq a$ (in particular, $r_a b \in \Phi^+$) and let $w' = r_a w$. Observe that then $w' \in T(r_a x)$. Since $b \in N(w^{-1})$, it follows that $r_a b \in N(w'^{-1})$ and so the inductive hypothesis yields that $(r_a b, r_a x) > 0$. Finally since $(,)$ is W-invariant, it follows that $(b, x) > 0$ as required. $\square$

Lemma 3.19. *Suppose that $x \in \Phi^+$, $w \in S(x)$ and $b \in N(w^{-1})$. Then $(b, x) > 0$.*

Proof. Follows from Lemma 3.18 and the fact that for each $w \in S(x)$ there is a $w' \in T(x)$ such that $N(w^{-1}) \subset N(w'^{-1})$. $\square$

Lemma 3.20. *For $n \in \mathbb{N}$, if $D_n = \emptyset$, then $D_m = \emptyset$ for all $m \in \mathbb{N}$ such that $m > n$.*

Proof. Suppose for a contradiction that there exists $n \in \mathbb{N}$ such that $D_n = \emptyset$ and yet $D_{n+1} \neq \emptyset$. Let $x \in D_{n+1}$. Then Lemma 3.12 yields that $r_a x \in D_{n+1} \uplus D_{n+2}$, for all $a \in \Pi$. Furthermore, Lemma 3.13 yields that if $a \in \Pi$ such that $r_a x \prec x$ then $r_a x \in D_{n+1}$ still. Write $x = wb$, where $b \in \Pi$, and $w \in S(x)$. Suppose that $w = r_{a_1} r_{a_2} \cdots r_{a_l}$ with $\ell(w) = l$ and $a_1, a_2, \ldots, a_l \in \Pi$. Then $r_{a_i} \cdots r_{a_2} r_{a_1} x \in D_{n+1}$, for all $i \in \{1, \ldots, l\}$, and in particular, $b = r_{a_l} \cdots r_{a_1} x \in D_{n+1}$, contradicting the fact that $b \in \Pi \subset D_0$. $\square$

Corollary 3.21. *Let W be an infinite Coxeter group with $\#R < \infty$. Then for each non-negative integer n, the corresponding D_n is non-empty.*

Proof. It is clear from the definition of the D_n's that $\Phi^+ = \biguplus_{n \geqslant 0} D_n$. Since W is an infinite Coxeter group, Proposition 1.6 (iii) yields that $\#\Phi^+ = \infty$. On the other hand, since $\#R < \infty$, Theorem 3.8 yields that for each non-negative integer n, $\#D_n < \infty$. Thus the desired result follows from Lemma 3.20. □

The following is a generalization of Proposition 3.14:

Proposition 3.22. *Suppose that $x \in D_n$ with $n > 0$, and let $a \in \Phi^+$. Then*

(i) $\#D(r_a x) < n$ *if* $(x, a) \geqslant 1$;
(ii) $\#D(r_a x) > n$ *if* $(x, a) \leqslant -1$.

Proof. (i) If $\mathrm{dp}(a) = 1$ then this is just Proposition 3.14. Hence we may assume that $\mathrm{dp}(a) > 1$, and proceed by an induction on $\mathrm{dp}(a)$.

Write $a = r_b c$ where $b \in \Pi$ and $c \in \Phi^+$. Then $r_a = r_b r_c r_b$. Furthermore, suppose that

$$\mathrm{dp}(a) = \mathrm{dp}(c) + 1. \tag{3.12}$$

Now since $(x, a) = (x, r_b c) = (r_b x, c) \geqslant 1$, it follows from the inductive hypothesis that

$$\#D\big(r_c(r_b x)\big) < \#D(r_b x). \tag{3.13}$$

Then we have three possibilities to consider:

1) $(b, x) \geqslant 1$;
2) $(b, x) \leqslant -1$;
3) $(b, x) \in (-1, 1)$.

If 1) is the case, then Proposition 3.14 yields that $r_b x \in D_{n-1}$ and hence

$$\begin{aligned}
\#D(r_a x) &= \#D\big(r_b(r_c r_b x)\big) \\
&\leqslant \#D\big(r_c(r_b x)\big) + 1 \quad \text{(follows from Lemma 3.12)} \\
&\leqslant \#D(r_b x) \quad \text{(follows from (3.13))} \\
&= n - 1,
\end{aligned}$$

as required.

If 2) is the case, then Proposition 3.14 yields that $r_b x \in D_{n+1}$, and $(b, r_c(r_b x)) = (b, r_b x - 2(r_b x, c)c) = (b, r_b x) - 2(x, a)(b, c)$. Observe that Lemma 1.7 and (3.12) together yield that $(b, c) < 0$ and since by assumption $(x, a) \geqslant 1$, it follows that

$$\big(b, r_c(r_b x)\big) > (b, r_b x) \geqslant 1. \tag{3.14}$$

Then

$$\begin{aligned}
\#D(r_a x) &= \#D\big(r_b(r_c r_b x)\big) \\
&= \#D(r_c r_b x) - 1 \quad \text{(by (3.14) above and Proposition 3.14)} \\
&\leqslant \#D(r_b x) - 2 \quad \text{(by (3.13))} \\
&\leqslant n - 1 \quad \text{(since } r_b x \in D_{n+1} \text{ in case 2))}
\end{aligned}$$

as required.

If 3) is the case, then we are done unless $\#D(r_c(r_b x)) = n-1$ together with $(b, r_c r_b x) \leqslant -1$. But this is impossible, since

$$\begin{aligned}(b, r_c r_b x) &= (b, r_b x) - 2(r_b x, c)(b, c)\\ &= -(b, x) - 2\underbrace{(a, x)}_{\geqslant 1}\underbrace{(b, c)}_{<0}\\ &> -1.\end{aligned}$$

Thus $\#D(r_a x) = \#D(r_b r_c r_b x) < n$ in this case too. This completes the proof of (i).

(ii) Replace x by $r_a x$, then apply (i) above. □

Lemma 3.23. *Suppose that $x \in D_n$ with $n \geqslant 1$. Then there exists some $y \in D_{n-1}$ with $y \prec x$.*

Proof. Suppose that the contrary is true. Let $x \in D_n$ such that there is no root in D_{n-1} preceding x. Write $x = wa$, where $a \in \Pi$, and $w \in S(x)$. Let $w = r_{a_1} r_{a_2} \cdots r_{a_l}$ for some $a_1, \ldots, a_l \in \Pi$ with $\ell(w) = l$. Then $a = r_{a_l} \cdots r_{a_1} x$. Observe that then

$$a \prec r_{a_{l-1}} \cdots r_{a_1} x \prec r_{a_{l-2}} \cdots r_{a_1} x \prec \cdots \prec r_{a_1} x \prec x. \tag{3.15}$$

The assumption that x is not preceded by any root in D_{n-1}, together with Proposition 3.14 yield that all the roots in (3.15), including a, are in D_n, contradicting the fact the $a \in \Pi \subseteq D_0$. □

Next we give an algorithm to systematically compute all the D_n's for an arbitrary Coxeter group W of finite rank:

Proposition 3.24. *Suppose that W is a Coxeter group of finite rank. For $n \geqslant 1$, there is an algorithm to compute D_n provided that D_{n-1} is known.*

Proof. We outline such an algorithm:

1) Set $D = \emptyset$.
2) Enumerate all the elements of D_{n-1} in some order, that is, write $D_{n-1} = \{x_1, \ldots, x_m\}$, where $m = \#D_{n-1}$.
3) Starting with x_1, apply all the reflections r_a where $a \in \Pi$, to x_1, one at a time. If $(a, x_1) \leqslant -1$, then add $r_a x_1$ to D if it is not already in D.
4) Repeat 3) to $x_2, \ldots, x_m$.
5) Enumerate all the elements of the modified set D in some order, that is, write $D = \{x'_1, x'_2, \ldots, x'_{\#D}\}$.
6) Starting with x'_1, apply all the reflections r_a where $a \in \Pi$, to x'_1, one at a time. If $(a, x'_1) \in (-1, 0)$ and $r_a x'_1 \notin D$, then add $r_a x'_1$ to D.
7) Repeat 6) to $x'_2, \ldots, x'_{\#D}$.
8) Repeat steps 5) to 7) above.
9) Repeat 8) until no new elements can be added to D.
10) Set $D_n = D$.

Next we show that the above algorithm will be able to produce all elements of D_n within a finite number of iterations.

Let $x \in D_n$ $(n \geqslant 1)$ be arbitrary. Lemma 3.23 yields that there exists a $y \in D_{n-1}$ with $y \prec x$. Write $x = wy$ for some $w \in W$ with $\ell(w) = \mathrm{dp}(x) - \mathrm{dp}(y)$. Let $w = r_{a_1} r_{a_2} \cdots r_{a_l}$ where $a_1, \ldots, a_l \in \Pi$ and $\ell(w) = l$. Then

$$y \prec r_{a_l} y \prec r_{a_{l-1}} r_{a_l} y \prec \cdots \prec r_{a_1} r_{a_2} \cdots r_{a_l} y = x.$$

Since $x \in D_n$ and $y \in D_{n-1}$, it follows from Lemma 3.13 that

$$r_{a_l} y, r_{a_{l-1}} r_{a_l} y, \ldots, r_{a_2} r_{a_3} \cdots r_{a_l} y \in D_{n-1} \uplus D_n.$$

Therefore there exists $i \in \{1, 2, \ldots, l\}$ such that

$$\begin{aligned} y &\in D_{n-1}, \\ r_{a_l} y &\in D_{n-1}, \\ &\vdots \\ r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y &\in D_{n-1} \end{aligned}$$

and

$$\begin{aligned} r_{a_i}(r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y) &\in D_n, \\ r_{a_{i-1}} r_{a_i}(r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y) &\in D_n, \\ &\vdots \\ r_{a_1} r_{a_2} \cdots r_{a_l} y = x &\in D_n. \end{aligned}$$

Since $r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y \in D_{n-1}$, it follows that $r_{a_i} r_{a_{i+1}} r_{a_{i+2}} \cdots r_{a_l} y$ is an element of D_n obtainable by going through steps 3) and 4) above. This in turn implies that $r_{a_{i-1}} r_{a_i} \cdots r_{a_l} y$ is an element obtainable by going through steps 5) to 7). It then follows that $r_{a_{i-2}} r_{a_{i-1}} r_{a_i} \cdots r_{a_l} y$ and so on are all obtainable by (repeated) application of step 8). In particular, $x = r_{a_1} \cdots r_{a_l} y$ can be obtained after $(i-2)$ iterations of step 8). Thus x can be obtained by going through steps 1) to 8), with step 8) repeated finitely many times. Since $x \in D_n$ was arbitrary, it follows that every element of D_n can be obtained from the above algorithm in this manner with step 8) repeated finitely many times.

Finally, W is of finite rank, so $\#D_n < \infty$ and $\#D_{n-1} < \infty$. Therefore step 9) will only be repeated a finite number of times and hence the algorithm will terminate completing the proof. □

Corollary 3.25. *If $\#R < \infty$, then we may compute D_n, for all $n \in \mathbb{N}$.*

Proof. [4] gives a complete description of D_0 when $\#R < \infty$. Now combine [4] and Proposition 3.24, the result follows immediately. □

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