A Characterization of Particular Symmetric (0, 1) Matrices

D. Marušič*
Department of Mathematics
University of California
Santa Cruz, California 95064

R. Scapellato†
Dipartimento di Matematica
Via Università 12
43100 Parma, Italy

and

N. Zagaglia Salvi†
Dipartimento di Matematica
Politecnico di Milano
Piazza L. da Vinci 32
20133 Milano, Italy

Submitted by Richard A. Brualdi

ABSTRACT

A characterization of a class of symmetric (0, 1) matrices A such that AP is a symmetric matrix too, where P is a permutation matrix, is given, and an application to double coverings of graphs is considered.

*This work was partially supported by the Research Foundation of Slovenia and by the Politecnico di Milano.

†This work is part of the G.N.S.A.G.A. Research Program of C.N.R. and has been supported by the Italian M.P.I.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

1. INTRODUCTION

A square symmetric (0,1) matrix $A = [a_{ij}]$ such that $a_{ii} = 0$, $1 \leqslant i \leqslant n$, is called a *g-matrix*. Thus a g-matrix is the adjacency matrix of an undirected graph.

A g-matrix A is said to be *compatible with* a permutation matrix $P \neq I$ if AP is also a g-matrix. A *compatible g-matrix* is a g-matrix compatible with some nonidentity permutation matrix.

In this paper we consider the problem of the existence and construction of compatible g-matrices. In particular, in Theorem 3.1 we prove that a g-matrix A is compatible with a permutation matrix P representing an n-cycle if and only if $A = 0$. In Theorem 3.4 we characterize the class of g-matrices compatible with matrices corresponding to permutations partitioned in h disjoint cycles, where $h \geqslant 2$.

Moreover, in Theorem 4.3 we give an interesting application concerning the automorphism group of a subclass of double coverings of graphs.

2. PRELIMINARIES

In this section we prove a number of lemmas which will be used later on in the proofs of our main results in Section 3.

LEMMA 2.1. *Let H_j, $1 \leqslant j \leqslant n$, be the set of ordered pairs $\{(j+i, j-i), (j-i, j+i);\ 0 \leqslant i \leqslant n/2\}$ of integers* mod n, *with n even. Then*

(i) $|H_j| = n$,
(ii) $H_j = H_{j+n/2}$,
(iii) $H_{j_1} \cap H_{j_2} = \varnothing$ *for* $j_2 \notin \{j_1, j_1 + n/2\}$.

Proof. (i): We note that the relation $j + i \equiv j - i$ (mod n) implies $2i \equiv 0$ (mod n); therefore the components of the pair $(j+i, j-i)$ coincide if and only if $i = 0$ or $i = n/2$. Moreover, distinct values of i give rise to distinct elements of H_j. Namely, suppose that $i_1, i_2 \in \{0, 1, \dots, n/2\}$ are distinct and the pair $(j+i_1, j-i_1)$ coincides either with $(j+i_2, j-i_2)$ or with $(j-i_2, j+i_2)$. Then it follows that $j + i_1 \equiv j + i_2$ (mod n) in the first case or $j + i_1 \equiv j - i_2$ (mod n) in the second case, both of which are clearly impossible. Therefore $|H_j| = n$.

Part (ii) follows from the fact that $(j+i, j-i) = (j+n/2+(i-n/2), j+n/2-(i-n/2))$.

In order to prove (iii) assume that $H_{j_1} \cap H_{j_2} \neq \varnothing$ for some $j_2 \notin \{j_1, j_1 + n/2\}$. Then there exists an integer h, with $|h| < n/2$, such that $j_2 \equiv j_1 + h \pmod{n}$, and integers $i_1, i_2 \in \{0, 1, \ldots, n/2\}$ such that $(j_1 + i_1, j_1 - i_1)$ equals $(j_2 + i_2, j_2 - i_2)$ or $(j_2 - i_2, j_2 + i_2)$. In the first case $j_1 + i_1 \equiv j_2 + i_2 \pmod{n}$, which implies $i_2 \equiv i_1 - h \pmod{n}$. Moreover, $j_1 - i_1 \equiv j_2 - i_2 \pmod{n}$, which implies $i_2 \equiv i_1 + h \pmod{n}$. Therefore we obtain the equation $2h \equiv 0 \pmod{n}$, which contradicts the choice of h.

A similar argument takes care of the second case. This proves Lemma 2.1. ■

LEMMA 2.2. *Let K_j, $1 \leqslant j \leqslant n$, be the set of ordered pairs $\{(j+i, j-i-1), (j-i-1, j+i);\ 0 \leqslant i \leqslant n/2-1\}$ of integers* mod n, *with n even. Then*

(i) $|K_j| = n$,
(ii) $K_j = K_{j+n/2}$,
(iii) $K_{j_1} \cap K_{j_2} = \varnothing$ *for* $j_2 \notin \{j_1, j_1 + n/2\}$.

Proof. (i): The relation $j + i \equiv j - i - 1 \pmod{n}$ implies $2i \equiv -1 \pmod{n}$; this is impossible for n even. So all the elements of K_j have distinct components.

Moreover, distinct values of i give rise to distinct elements of K_j. Namely, suppose that $i_1, i_2 \in \{0, 1, \ldots, n/2 - 1\}$ are distinct and that the pair $(j + i_1, j - i_1 - 1)$ coincides either with $(j + i_2, j - i_2 - 1)$ or with $(j - i_2 - 1, j + i_2)$. Then it follows that $i_1 \equiv i_2 \pmod{n}$ in the first case or $i_1 + i_2 \equiv -1 \pmod{n}$ in the second case, both of which are clearly impossible in view of the choice of i_1 and i_2. Therefore $|K_j| = n$.

Part (ii) follows from the fact that $(j+i, j-i-1) = (j + n/2 + (i - n/2), j + n/2 - (i - n/2) - 1)$.

(iii): Suppose that $K_{j_1} \cap K_{j_2} \neq \varnothing$ for $j_2 \notin \{j_1, j_1 + n/2\}$. Then there exists an integer h, with $|h| < n/2$, such that $j_2 \equiv j_1 + h \pmod{n}$, and integers i_1, i_2 such that $(j_1 + i_1, j_1 - i_1 - 1)$ coincides with $(j_2 + i_2, j_2 - i_2 - 1)$ or with $(j_2 - i_2 - 1, j_2 + i_2)$.

In the first case $i_1 \equiv i_2 + h \pmod{n}$ and $i_2 \equiv i_1 + h \pmod{n}$. Hence $2h \equiv 0 \pmod{n}$, contradicting the choice of h.

In the second case $i_1 + i_2 \equiv h - 1 \pmod{n}$ and $i_1 + i_2 \equiv -h - 1 \pmod{n}$. Hence $2h \equiv 0 \pmod{h}$, which contradicts the choice of h. ■

LEMMA 2.3. *Let L_j, $1 \leqslant j \leqslant n$, be the set of ordered pairs* $\{(j+i, j-i),(j-i, j+i);\ 0 \leqslant i \leqslant (n-1)/2\}$ *of integers* mod n, *where n is odd. Then*

(i) $|L_j| = n$,
(ii) $L_{j_1} \cap L_{j_2} = \varnothing$ *for* $j_2 \neq j_1$.

Proof. (i): We note that $j+i \equiv j-i \pmod n$ implies $i=0$. Therefore the only element of L_j with equal components is (j, j). Moreover, distinct values of i give rise to distinct elements of L_j. Namely, if i_1 and i_2 are distinct elements of $\{0,1,\dots,(n-1)/2\}$, then $(j+i_1, j-i_1)$ coincides either with $(j+i_2, j-i_2)$ or with $(j-i_2, j+i_2)$, both of which are impossible. Therefore $|L_j| = n$.

(ii): Suppose that $L_{j_1} \cap L_{j_2} \neq \varnothing$ for $j_2 \neq j_1$. Then there exists an integer h, with $|h| < (n-1)/2$, and distinct elements $i_1, i_2 \in [0,(n-1)/2]$ such that $j_2 = j_1 + h \pmod n$ and (j_1+i_1, j_1-i_1) coincides either with (j_2+i_2, j_2-i_2) or with (j_2-i_2, j_2+i_2).

In the first case we have $i_1 - i_2 \equiv h \pmod n$ and $i_1 - i_2 \equiv -h \pmod n$; it follows that $2h \equiv 0 \pmod n$, which is impossible. A similar argument holds in the second case. ■

We let Π_q, q a positive integer, denote the matrix representing the cyclic permutation $(12\dots q)$. A matrix M of order q is said to be *retrocirculant* [2] if $M = \Pi_q M \Pi_q$.

Let (r, s) be the greatest common divisor of r and s. The following lemma is proved in [5].

LEMMA 2.4. *Let M be an $r \times s$ matrix. The relation $M = \Pi_r M \Pi_s$ is satisfied if and only if M is partitioned into $rs/(r,s)^2$ equal retrocirculant blocks of order (r,s).*

3. THE CHARACTERIZATION OF g-MATRICES

In this section, we consider the problem of the existence and construction of compatible g-matrices.

First we study the case when G is compatible with an n-cycle.

THEOREM 3.1. *Let A be a g-matrix of order n, and let $\Pi = \Pi_n$. Then $A\Pi$ is a g-matrix if and only if $A = 0$.*

Proof. It is obvious that if $A=0$, then $A\Pi=0$ is a g-matrix. Conversely, assume that

$$A\Pi = 0 \tag{1}$$

is a g-matrix; we shall prove that $A=0$.

Let $A=[a_{ij}]$ and $B=[b_{ij}]$. It follows from (1) that

$$b_{ij}=a_{i\,j-1} \tag{2}$$

and therefore

$$a_{ij}=b_{i\,j+1}, \tag{3}$$

where the subscripts are taken mod n. The symmetry of A and B together with (2) and (3) implies the equalities

$$\begin{aligned} a_{jj} &= b_{j\,j+1}=b_{j+1\,j}=a_{j+1\,j-1}=a_{j-1\,j+1}=\cdots=a_{j+i\,j-i} \\ &= a_{j-i\,j+i}=b_{j-i\,j+i+1}=b_{j+i+1\,j-i} \end{aligned} \tag{4}$$

for $0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$.

There are two cases to be considered.

Case 1: n is even. Then the indices corresponding to the entries of A in the sequence (4) coincide for $i=0$ and $i=n/2$. We have the sequence of equalities

$$0=a_{jj}=a_{j+1\,j-1}=a_{j-1\,j+1}=\cdots=a_{j+n/2\,j-n/2} \tag{5}$$

for $1 \leqslant j \leqslant n$. The set of ordered pairs corresponding to the indices of the sequence (5) coincides with the set H_j in the statement of Lemma 2.1. Therefore, by Lemma 2.1, there are $n/2$ disjoint sets H_j and hence $n \times n/2$ entries of A in distinct positions equal to 0. By the symmetry of A and B and by (2) and (3) we also obtain the following sequence of equalities:

$$\begin{aligned} 0 &= b_{jj}=a_{j\,j-1}=a_{j-1\,j}=b_{j-1\,j+1}=\cdots=a_{j+i\,j-i-1} \\ &= a_{j-i-1\,j+i}=b_{j-i-1\,j+i+1}=b_{j+i+1\,j-i-1} \end{aligned} \tag{6}$$

for $1 \leqslant j \leqslant n$. The indices corresponding to the elements of B in (6) coincide for $i = 0$ and $i = (n-2)/2$. Therefore the entries of A satisfy the following sequence of equalities:

$$0 = a_{jj-1} = a_{j-1j} = a_{j+1j-2} = \cdots = a_{j+n/2-1j-n/2} = a_{j-n/2j+n/2-1} \quad (7)$$

for $1 \leqslant j \leqslant n$. The set of ordered pairs corresponding to the indices of the elements in the sequence (7) coincides with the set K_j in the statement of Lemma 2.2. By Lemma 2.2 there are $n \times n/2$ elements of A, in distinct positions, equal to 0.

We next prove that

$$H_{j_1} \cap K_{j_2} = \varnothing \quad (8)$$

for $j_1, j_2 \in \{1, 2, \ldots, n\}$. Assume the contrary, and let $j_2 = j_1 + h$. There exist $i_1 \in [0, n/2]$ and $i_2 \in [0, (n-2)/2]$ such that $(j_1 + i_1, j_1 - i_1) = (j_2 + i_2, j_2 - i_2 - 1)$ or $(j_2 - i_2 - 1, j_2 + i_2)$. In the first case $i_1 \equiv i_2 + h \pmod{n}$ and $i_1 \equiv i_2 - h + 1 \pmod{n}$, and therefore $2h \equiv 1 \pmod{n}$, a contradiction. A similar argument is used in the second case. This proves (8), which implies that n^2 entries of A in distinct positions are equal to 0, that is, $A = 0$.

Case 2: n is odd. The relation $j + i \equiv j - i \pmod{n}$—that is, $2i \equiv 0 \pmod{n}$—is only satisfied for $i = 0$, and the relation $j - i \equiv j + i + 1 \pmod{n}$ is only satisfied for $i = (n-1)/2$. Then in the sequence (4) the subscripts of the entries of A only coincide for $i = 0$, while the subscripts of the entries of B only coincide for $i = (n-1)/2$. So we have the sequence of equalities

$$\begin{aligned} 0 = a_{jj} = a_{j+1j-1} = a_{j-1j+1} = \cdots &= a_{j+(n-1)/2j-(n-1)/2} \\ &= a_{j-(n-1)/2j+(n-1)/2}. \end{aligned} \quad (9)$$

The set of ordered pairs corresponding to the indices of the sequence (9) coincides with the set L_j in the statement of Lemma 2.3. Therefore we conclude by Lemma 2.3 that A has n^2 entries in distinct positions equal to 0, that is, $A = 0$. ■

Let R be a permutation matrix which represents an n-cycle. Then there exists a permutation matrix Q such that $R = Q_T \Pi_n Q$. The following result is an immediate consequence of Theorem 3.1.

COROLLARY 3.2. *Let A be a g-matrix, and R be a permutation matrix representing an n-cycle. Then AR is a g-matrix if and only if $A = 0$.*

We define $\Pi_{r_1, r_2, \dots, r_h}$ to be the block-diagonal matrix with blocks $\Pi_{r_1}, \Pi_{r_2}, \dots, \Pi_{r_h}$. Let R be a permutation matrix of order n corresponding to some permutation decomposable into h cycles of lengths $r_1, r_2, \dots, r_h$. Then there exists a permutation matrix Q such that $R = Q_T \Pi_{r_1, r_2, \dots, r_h} Q$.

The proof of the next proposition is straightforward.

PROPOSITION 3.3. *Let A be a g-matrix, $P = \Pi_{r_1, r_2, \dots, r_h}$, and $R = Q_T P Q$ for some permutation matrix Q. Then AP is a g-matrix if and only if $Q_T A Q R$ is a g-matrix.* ■

THEOREM 3.4. *Let $P = \Pi_{r_1, r_2, \dots, r_h}$, and A be a g-matrix of order $n = r_1 + r_2 + \cdots + r_h$. Then AP is g-matrix if and only if A is partitioned into blocks C_{ij}, $i, j \in \{1, 2, \dots, h\}$, such that*

(a) $C_{ii} = 0$;

(b) *if $i < j$, then C_{ij} is a $(0,1)$ matrix of size $r_i \times r_j$ partitioned into $r_i r_j / (r_i, r_j)^2$ blocks all equal to some retrocirculant matrix of order (r_i, r_j);*

(c) *if $i > j$, then $C_{ij} = (C_{ji})_T$.*

Proof. The proof is by induction on h.

If $h = 1$, the result follows from Theorem 3.1.

Suppose now that h is greater than 1 and that the theorem holds for any integer smaller than h. Let A be a g-matrix partitioned into blocks in the following way:

$$A = \left[\begin{array}{cccc|c} & & & & M_1 \\ & & & & M_2 \\ & & A_{h-1} & & \vdots \\ & & & & M_{h-1} \\ \hline N_1 & N_2 & \cdots & N_{h-1} & M_h \end{array}\right],$$

where A_{h-1} is a g-matrix of order $n - r_h$, M_i is an $r_i \times r_h$ matrix, and $N_i = (M_i)_T$, where $1 \leqslant i \leqslant h$. Then we have

$$B = A\Pi_{r_1, r_2, \dots, r_h} = \left[\begin{array}{cccc|c} & & & & M_1\Pi_{r_h} \\ & & & & M_2\Pi_{r_h} \\ & & A_{h-1}\Pi_{r_1, r_2, \dots, r_{h-1}} & & \vdots \\ & & & & M_{h-1}\Pi_{r_h} \\ \hline N_1\Pi_{r_1} & N_2\Pi_{r_2} & \cdots & N_{h-1}\Pi_{r_{h-1}} & M_h\Pi_{r_h} \end{array}\right].$$

It is clear that B is g-matrix if and only if each of the following conditions is satisfied:

(i) $A_{h-1}\Pi_{r_1, r_2, \dots, r_{h-1}}$ is g-matrix;
(ii) $M_h \Pi_{r_h}$ is a g-matrix;
(iii) $(M_i \Pi_{r_h})_T = N_i \Pi_{r_i}$, that is, $M_i = \Pi_{r_i} M_i \Pi_{r_h}$.

The condition (i) is equivalent to saying that A_{h-1} satisfies the theorem, which is true in view of the induction hypothesis. By Theorem 3.1 the condition (ii) is equivalent to $M_h = 0$. The condition (iii) holds if and only if M_i is a $(0,1)$ matrix satisfying Lemma 2.4 for each $i \in \{1, 2, \dots, h-1\}$.

This completes the proof of Theorem 3.4. ∎

4. STABLE GRAPHS

The characterizations obtained in Theorems 3.1 and 3.4 have an interesting application for stable graphs. We first introduce some terminology.

Let G be an undirected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and adjacency matrix A. Consider the double covering $B(G)$ of G, that is, the conjunction of G by K_2, whose adjacency matrix is

$$\mathscr{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}.$$

Note that $B(G)$ is a bipartite graph with vertex set $V \cup V'$, where $V' = \{v'_1, v'_2, \dots, v'_n\}$ and with edges of the form $v_i v'_j$, where $v_i v_j$ is an edge of G. In [1] it was proved that $B(G)$ is connected if and only if G is connected and is not bipartite.

We shall hereafter assume that G is a connected graph which is not bipartite. Let Aut G be the automorphism group of G.

If P is the permutation matrix corresponding to an automorphism of G, it is easy to see that the matrices

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$$

represent automorphisms of $B(G)$. Hence $\operatorname{Aut} G \times Z_2$ is isomorphic to a subgroup of Aut $B(G)$.

We say that a graph G is stable if $\operatorname{Aut} B(G) = \operatorname{Aut} G \times Z_2$. An example of a stable graph is given by the triangle $G = K_3$.

Let $x \in V(G)$. By $N(x)$ we denote the neighborhood of x. We say that G is *vertex-determining* (v-d) if $N(v) \neq N(u)$ for any two distinct vertices u and v of G [3].

PROPOSITION 4.1. *If G is stable, then it is vertex-determining.*

Proof. Let G be a stable graph, and let $x, y \in V(G)$ have the same neighborhoods. Then the permutation of $V(B(G))$ which interchanges x and y and fixes everything else is an automorphism of $B(G)$ which does not correspond to an element of $\operatorname{Aut} G \times Z_2$. ∎

PROPOSITION 4.2. *A v-d graph G of order n and adjacency matrix A is stable if and only if there does not exist a pair of distinct permutation matrices Q and R such that QA = AR, corresponding to permutations of V(G) of the same order and not belonging to* Aut *G.*

Proof. Let

$$\mathscr{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$$

be the adjacency matrix of $B(G)$. Since $B(G)$ is bipartite, it follows that an automorphism of $B(G)$ corresponds to a permutation matrix $\mathscr{P}$ partitioned into one of the two forms

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix}. \tag{10}$$

It is easy to see that the condition $\mathscr{P}\mathscr{A} = \mathscr{A}\mathscr{P}$ implies the equalities $QA = AR$ and $AQ = RA$, which are equivalent. Hence $\mathscr{P}$ corresponds to an automorphism of $B(G)$ if and only if

$$QA = AR. \tag{11}$$

Now, G is stable if and only if there does not exist an automorphism $\mathscr{P}$ partitioned into one of the two forms (10), for $Q \neq R$. Namely, if G is stable, then of course $Q = R$ in (10).

Conversely, if G is not stable, then there exists some permutation matrix $\mathscr{P}$ such that $Q \neq R$ in (10).

Assume that one of the two permutations of $V(G)$ corresponding to Q and R belongs to Aut G. Then $AQ = AR$, that is, $AQR_T = A$, where $QR_T \neq I$. It is easy to see that this relation implies that there are at least two coincident columns in A. So there exist at least two distinct vertices of G with equal neighborhoods, a contradiction. Therefore the permutations corresponding to Q and R do not belong to Aut G. Finally, suppose that the orders q and r of Q and R are different, say $q < r$. Then (11) implies that $A = AR^q$. By the preceding arguments this is not possible. Therefore $q = r$. This concludes the proof of Proposition 4.2. ■

THEOREM 4.3. *Let G be a v-d graph with a noncompatible adjacency matrix. Then G is stable.*

Proof. Let $\mathscr{P}$ be an automorphism of $B(G)$ partitioned into one of the forms (10). Then the relation (11) holds, and therefore

$$AS = B, \tag{12}$$

where $S = RQ_T$ and $B = QAQ_T$.

Since A is not compatible, (12) implies $S = I$. Hence $Q = R$, and by (11) R corresponds to an automorphism of G. ■

It is natural to ask whether the converse of Theorem 4.3 holds; however, it appears difficult to construct a nonstable graph G with a compatible adjacency matrix. So we conjecture that the sufficient condition for the stability of a graph in the statement of Theorem 4.3 is also a necessary one.

REFERENCES

1 R. A. Brualdi, F. Harary, and Z. Miller, Bigraphs versus digraphs via matrices, *J. Graph Theory* 4:51–73 (1980).
2 P. J. Davis, *Circulant Matrices*, Wiley-Interscience, 1979.
3 L. Porcu, Sul raddoppio di un grafo, *Istit. Lombardo Accad. Sci. Lett. Rend. A* 110:453–480 (1976).
4 D. Sumner, The nucleus of a point determining graph, *Discrete Math.* 14:91–97 (1976).
5 N. Zagaglia Salvi, Una particolare classe di matrici torneo, *Istit. Lombardo Accad. Sci. Lett. Rend. A* 117:127–134 (1983).

Received 14 December 1987; final manuscript accepted 30 September 1988