The Green Correspondence and the Brauer Lift

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The process of reduction modulo a prime $p$ is one of the most important tools in the representation theory of finite groups. The Brauer lift, a basic result, asserts that this process, in an appropriate sense, is surjective. The usual proof of this theorem depends on Brauer's characterization of characters. However, recently we showed [1] that the Green correspondence, the fundamental theorem of modular representation theory, implies the characterization of characters. We shall here demonstrate how Green's theorem also readily and directly implies the Brauer lift.

Specifically, let $R$ be a complete discrete valuation ring with residue class field $k$ of prime characteristic $p$. Let $G$ be a finite group; all modules for $RG$ and $kG$ will be assumed to be finitely generated and, in the case of $RG$, free as $R$-modules. If $M$ is an $RG$-module then $M^*$ is the $kG$-module which is the reduction modulo $p$ of $M$, that is, $M^* = M / \pi M$, where $\pi$ is a generator of the prime ideal of $R$. If $U$ is a $kG$-module then $U$ is said to lift if $U$ is isomorphic with $M^*$ for some $RG$-module $M$. We say that $U$ lifts virtually if there are $RG$-modules $M$ and $N$ such that $U \oplus M^*$ and $N^*$ have the same composition factors (counting multiplicity). In particular, if $U$ and $V$ are $kG$-modules with the same composition factors (counting multiplicities) then $U$ lifts virtually if, and only if, $V$ lifts virtually. Hence, if each composition factor of $U$ lifts virtually then so does $U$. Moreover, it is easy to see that if $V$ and $U \oplus V$ lift virtually then so does $U$. We can now state Brauer's result.

**Theorem.** If $U$ is a $kG$-module then $U$ lifts virtually.

**Proof.** We proceed by induction on the order of $G$. We know [1], as a consequence of the Green correspondence, that there exist $kG$-modules $P$, $P'$, and $L'$ such that

$$U \oplus L \oplus P \cong L' \oplus P',$$

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where $P$ and $P'$ are projective and $L$ and $L'$ are each the direct sum of modules induced from $p$-local subgroups of $G$. The modules $P$ and $P'$ lift since they are projective. Moreover, if every $p$-local subgroup of $G$ is proper then $L$ and $L'$ lift virtually. (Indeed, if $Q$ is a nonidentity $p$-subgroup of $G$, $W$ is a $kN_c(Q)$-module and $W \oplus M^*$ and $N^*$ have the same composition factors, for $RN_c(Q)$-modules $M$ and $N$, then the induced modules $W^G \oplus (M^G)^*$ and $(N^G)^*$ are similarly related.) Therefore, the theorem follows in this case. On the other hand, suppose that $Q$ is a nonidentity $p$-subgroup of $G$ and that $Q$ is normal. Since it suffices to prove the theorem in the case of simple $kG$-modules, we let $S$ be such a module. But $S$ is just a $k[G/Q]$-module viewed as a $kG$-module and $G/Q$ is of order smaller than the order of $G$ so $S$ lifts virtually by induction and the theorem is established in all cases.

Reference