q-Eulerian Polynomials Arising from Coxeter Groups

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We study the polynomials obtained by enumerating a finite Coxeter group by number of descents. The type A case gives rise to the familiar Eulerian polynomials, while the B and D cases provide two new q-analogues. Various recursion relations, generating functions and unimodality properties are derived, which generalize and unify earlier results of Dolgachev, Lunts, Stanley and Stembridge.

1. Introduction

In this work we study the polynomials, \( P(W; x) \), obtained by enumerating a (finite) Coxeter system \((W, S)\) with respect to the number of descents (see Section 2 for definitions). It is easy to see that it is enough to study these polynomials for irreducible Coxeter systems. While for type A these polynomials are known to coincide with classical Eulerian polynomials (see, e.g., [8, 24]), and thus have been extensively studied from a combinatorial point of view (see, e.g., [9]), only some results of a combinatorial nature are known for type B (see, [8, 15, 17]), and very little is known for type D (see [16, 25]). In this paper we show that essentially all of the classical results for Eulerian polynomials have analogues for these other polynomials, and that it is possible to define natural q-analogues of them which are also q-analogues of the Eulerian polynomials. Our results generalize and unify previous results of Dolgachev, Lunts, Stanley and Stembridge, and generalize the classical theory of Eulerian numbers and polynomials. As a by-product we obtain a combinatorial proof of a simple relation between these three types of polynomials and we are led to several conjectures about them.

The polynomials \( P(W; x) \) are not only of enumerative interest but also of geometric significance. In fact, it is known (see Theorem 2.3) that \( P(W; x) \) is also the generating function of the h-vector of the Coxeter complex associated to \((W, S)\). Since this Coxeter complex is isomorphic to the boundary complex of a simplicial convex polytope (see, e.g., [2, Proposition 2.3.9]) there follows from standard results in the theory of toric varieties (see, e.g., [14, Chapter 2] and [7, Theorem 10.8]) that this h-vector equals the sequence of even-dimensional Betti numbers of certain toric projective varieties (which have been studied, e.g., in [8] and (24)) associated to the polytope, and that \( P(W; x^2) \) is the Poincaré polynomial of these varieties. In the light of this, it would be interesting to have geometric proofs of all the results in this work.

The organization of the paper is as follows. In the next section we collect the definitions, notation and preliminary results that will be used in the rest of this work. In Section 3 we study the polynomials \( P(W; x) \) when \( W \) is a Coxeter group of type B. It turns out that one can define q-analogues of these polynomials that have the additional property of reducing, when \( q = 0 \), to the ordinary Eulerian polynomials. We show that many results on Eulerian polynomials generalize to these q-polynomials. In particular, we obtain recurrence relations, generating functions, a ‘q-Worpitzky identity’, unimodality and total positivity properties, and several combinatorial interpretations for these q-Eulerian polynomials. In Section 4 we study the polynomials \( P(W; x) \) when \( W \) is a Coxeter group of type D. Again, it is possible to define q-analogues of these polynomials which reduce, when \( q = 0 \), to the Eulerian polynomials. The main result
of this section is a simple relation between these \( q \)-Eulerian polynomials and those introduced in Section 3. This relation generalizes one found by Stembridge and allows us to obtain analogues of most (not all) of the results of Section 3. Finally, in Section 5, we discuss some conjectures and open problems, both of a geometric and combinatorial character, arising from our work.

2. Notation and Preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let \( \mathbf{P} \) denote \( \{1, 2, 3, \ldots\} \) and \( \mathbb{N} = \mathbf{P} \cup \{0\} \); for \( a \in \mathbb{N} \) we let \( [a] \) denote \( \{1, 2, \ldots, a\} \) (where \( [0] = \emptyset \)). The cardinality of a set \( A \) will be denoted by \( |A| \).

Given a sequence \( \sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), we say that a pair \( (i, j) \in [n] \times [n] \) is an inversion of \( \sigma \) if \( i < j \) and \( a_i > a_j \). We say that \( i \in [n - 1] \) is a descent of \( \sigma \) if \( a_i > a_{i+1} \). We denote by \( \text{inv}(\sigma) \) (respectively, \( d(\sigma) \)) the number of inversions (respectively, descents) of \( \sigma \).

Given a set \( T \) we will let \( S(T) \) be the set of all bijections \( \pi: T \rightarrow T \), and \( S_n = S([n]) \). For \( n \in \mathbf{P} \) and \( 1 \leq k \leq n \) the number

\[
A(n, k) \overset{\text{def}}{=} |\{ \pi \in S_n : d(\pi) = k - 1 \}|
\]

is called an Eulerian number, and the polynomial

\[
A_n(x) \overset{\text{def}}{=} \sum_{k=1}^{n} A(n, k) x^k
\]

is called the \( n \)th Eulerian polynomial. Eulerian numbers and polynomials have been widely studied (see, e.g., [9]), for convenience we will let \( A_0(x) \overset{\text{def}}{=} 1 \).

By a simplicial complex we will mean a collection of sets \( \Delta \) with the property that if \( A \in \Delta \) and \( B \subseteq A \), then \( B \in \Delta \). We call the elements of \( \Delta \) the faces of \( \Delta \). For \( S \in \Delta \), the dimension of \( S \) is \( |S| - 1 \). The dimension of \( \Delta \) is \( \max\{|A| - 1 : A \in \Delta\} \). Given a simplicial complex \( \Delta \) of dimension \( d - 1 \) we let \( f_{-1}(\Delta) \overset{\text{def}}{=} \{ A \in \Delta : |A| = i \} \), for \( i = 0, \ldots, d \), and call \( f(\Delta) \overset{\text{def}}{=} (f_{-1}(\Delta), f_0(\Delta), f_1(\Delta), \ldots, f_{d-1}(\Delta)) \) the \( f \)-vector of \( \Delta \). We then define the \( h \)-vector of \( \Delta \), \( h(\Delta) \overset{\text{def}}{=} (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta)) \), by letting

\[
\sum_{i=0}^{d} h_i(\Delta) x^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta) (x - 1)^{d-i}.
\]

(1)

Clearly, knowledge of the \( f \)-vector of \( \Delta \) is equivalent to the knowledge of its \( h \)-vector.

Given \( p \in \mathbf{P} \), we will sometimes use the basis of the vector space \( V_p \) of real polynomials of degree \( \leq p \) consisting of the polynomials \( \{x^i + p^{-i}\}_{i=0, \ldots, p} \). The reason why this basis is often used in enumerative combinatorics lies in the following result which is, essentially, a restatement of the binomial theorem (see, e.g., [21, p. 16]).

**Theorem 2.1.** Let \( A(x) \) be a real polynomial of degree \( d \). Then

\[
A(x) = \sum_{i=0}^{d} b_i \binom{x + d - i}{d}
\]

if and only if

\[
\sum_{n=0}^{\infty} A(n)x^n = \sum_{i=0}^{d} b_i x^i \frac{x^n}{(1-x)^{d+1}},
\]

as formal power series in \( \mathbb{R}[[x]] \).
A proof of the preceding result can be found in [21, Remark (d), p. 209].

A sequence \( \{a_0, a_1, \ldots, a_d\} \) (of real numbers) is log-concave if \( a_i^2 \geq a_{i-1}a_{i+1} \) for \( i = 1, \ldots, d-1 \). It is unimodal if there exists an index \( 0 \leq j \leq d \) such that \( a_i \leq a_{i+1} \) for \( i = 0, \ldots, j-1 \) and \( a_i \geq a_{i+1} \) for \( i = j, \ldots, d-1 \). It has no internal zeros if there are not three indices \( 0 \leq i < j < k \leq d \) such that \( a_i, a_k \neq 0 \) and \( a_j = 0 \). It is symmetric if \( a_i = a_{d-i} \) for \( i = 0, \ldots, \lfloor d/2 \rfloor \). A polynomial \( \sum_{i=0}^{d} a_i x^i \) is log-concave (respectively, unimodal, with no internal zeros, symmetric) if the sequence \( \{a_0, a_1, \ldots, a_d\} \) has the corresponding property. It is well known that if \( \sum_{i=0}^{d} a_i x^i \) is a polynomial with non-negative coefficients and with only real zeros, then the sequence \( \{a_0, a_1, \ldots, a_d\} \) is log-concave and unimodal, with no internal zeros (see, e.g., [3], or [6, Thm B, p. 270]). If \( p(x) \) is a symmetric unimodal polynomial then we let \( C(p) \overset{\text{def}}{=} (\deg(p) + \text{mult}(0; p))/2 \), where mult(0, p) is the multiplicity of 0 as a zero of p. We call \( C(p) \) the center of symmetry of \( p(x) \). So, for example, \( C(x^2 + 3x^3 + x^4) = 3 \) and \( C(1 + x) = \frac{1}{2} \).

We now recall a result that relates the basis \( \{x^i p^{-i}\}_{i=0, \ldots, \deg(p)} \) with polynomials having only real zeros.

**Theorem 2.2.** Let

\[
A(x) = \sum_{i=0}^{d} b_i \binom{x + d - i}{d}
\]

be a polynomial of degree \( d \). Suppose that \( A(x) \) has all its zeros in the interval \([-1, 0]\). Then the polynomial \( \sum_{i=0}^{d} b_i x^i \) has only real zeros.

**Theorem 2.2** is non-trivial and we refer the reader to [3, Thm 4.4.4], for its proof.

We will follow [11] for general Coxeter group notation and terminology. All Coxeter groups in this paper are assumed to be finite. Given a Coxeter system \( (W, S) \) and \( \sigma \in W \), we denote by \( l_w(\sigma) \) the length of \( \sigma \) in \( W \), with respect to \( S \), and we let

\[
\mathcal{D}_W(\sigma) \overset{\text{def}}{=} \{ s \in S \mid l_w(\sigma s) < l_w(\sigma) \},
\]

and

\[
d_W(\sigma) = |\mathcal{D}_W(\sigma)|.
\]

We call \( \mathcal{D}_W(\sigma) \) the descent set of \( \sigma \) and say that \( \sigma \) has \( d_w(\sigma) \) \( W \)-descents. If \( (W, S) \) is a Coxeter system of type \( A_n \), we will write simply \( l_A(\sigma) \), \( d_A(\sigma) \), etc., instead of \( l_w(\sigma) \), \( d_w(\sigma) \), etc., and similarly for type \( B_n \) and \( D_n \). Given a Coxeter system \( (W, S) \) we will denote by \( e \) the identity of \( W \). Recall that the Coxeter complex of a Coxeter system \( (W, S) \) is the simplicial complex \( \Delta(W, S) \) which is the nerve of the covering of \( W \) by left cosets of maximal parabolic subgroups. We denote by \( \mathcal{G}(\Delta(W, S)) \) the set of chambers of \( \Delta(W, S) \) (i.e. the set of all maximal faces of \( \Delta(W, S) \)).

Now let \( (W, S) \) be a finite Coxeter system and \( \Delta(W, S) \) be the associated Coxeter complex. The following result follows from Theorems 1.6, 2.1 and Proposition 1.2 of [1].

**Theorem 2.3.** Let \( (W, S) \) be a finite Coxeter system of rank \( n \) and let \( \Delta(W, S) \) be the associated Coxeter complex. Then

\[
h(\Delta(W, S)) = (d_0(W), \ldots, d_n(W))
\]

where \( d_i(W) \overset{\text{def}}{=} |\{ \sigma \in W \mid d_W(\sigma) = i \}| \), for \( i = 0, \ldots, n \).

It is well known (see, e.g., [2, Proposition 2.3.9]) that \( \Delta(W, S) \) is isomorphic to the boundary complex of a simplicial convex polytope and (see, e.g., [22, Theorem...
20) that the $h$-vector of a simplicial convex polytope is always a symmetric unimodal sequence: therefore Theorem 2.3 and Proposition 2.3.9 of [2] imply the following result which is, in a sense, 'dual' to Theorem 19 of [22].

**Theorem 2.4.** Let $(W, S)$ be a finite Coxeter system of rank $n$. Then the polynomial

$$
\sum_{\sigma \in W} x^{d_W(\sigma)}
$$

is symmetric and unimodal with center of symmetry at $\lfloor n/2 \rfloor$.

Given a (finite) Coxeter system $(W, S)$, we let

$$
P(W; x) \overset{\text{def}}{=} \sum_{\sigma \in W} x^{d_W(\sigma)}. \tag{2}
$$

3. **q-Eulerian Polynomials of Type B**

In this section we examine in detail a natural $q$-analogue of $P(W; x)$ in the case that $W$ is a Coxeter group of type $B_n$. We slightly abuse notation by denoting this group by $B_n$. We regard $B_n$ as the group of all signed permutations on $[n]$. More precisely, we view each element $\sigma \in B_n$ as a function $\sigma: [n] \to [-n, n]\setminus\{0\}$ such that $|\sigma| \in S_n$ (where $|\sigma|(i) \overset{\text{def}}{=} |\sigma(i)|$ for $i \in [n]$). We write $\sigma = a_1 \cdots a_n$ (or, sometimes, $\sigma = [a_1, \ldots, a_n]$) if $a_i = \sigma(i)$ for $i \in [n]$, and call this the one-line notation of $\sigma$. We call an element $\sigma \in B_n$ a signed cycle if $|\sigma|$ is a cycle. It is clear that each $\sigma \in B_n$ can be expressed as a product of disjoint signed cycles, and it will be convenient to have a canonical way of writing such a product. We say that $\sigma \in B_n$ is written in **standard disjoint cycle form** if:

(i) each cycle is written with its largest (in absolute value) element first;

(ii) the cycles are written in increasing order of the absolute values of their first elements.

For example, if $\sigma = [-3, 5, 1, -7, 2, 8, -6, -4]$ then the standard disjoint cycle form of $\sigma$ is $\sigma = (-3, 1)(5, 2)(8, -4, -7, -6)$.

As a set of Coxeter generators for $B_n$ we take $S \overset{\text{def}}{=} \{s_1, \ldots, s_n\}$, where $s_i \overset{\text{def}}{=} [1, 2, \ldots, i-1, i+1, i+2, \ldots, n]$ for $i \in [n-1]$, and $s_n \overset{\text{def}}{=} [-1, 2, \ldots, n]$.

Our first step is that of obtaining an explicit combinatorial description of the polynomial $P(B_n; x)$. In order to do this it is necessary first to obtain a simple combinatorial description of the length function of $B_n$. For $\sigma \in B_n$ we let

$$
\text{inv}_B(\sigma) \overset{\text{def}}{=} \text{inv}(\sigma(1), \ldots, \sigma(n)) - \sum_{\{j \in [n]: \sigma(j) < 0\}} \sigma(j). \tag{3}
$$

For example, if $\sigma = [-3, 2, 7, -1, -6, 5, -4]$ then $\text{inv}_B(\sigma) = 12 + (3 + 1 + 6 + 4) = 26$. Note that $\text{inv}_B(\sigma) = \text{inv}(\sigma)$ iff $\sigma \in S_n$.

**Proposition 3.1.** Let $n \in \mathbb{P}$. Then

$$
l_B(\sigma) = \text{inv}_B(\sigma), \tag{4}
$$

for all $\sigma \in B_n$.

**Proof.** We prove first that

$$
\text{inv}_B(\sigma) \leq l_B(\sigma) \tag{5}
$$
for all \( \sigma \in B_n \). Let \( v \in B_n \). Now, it is easy to see that \( \text{inv}(us \sigma) = \text{inv}(v) - v(1) + \text{sgn}(v(1)) \), and it therefore follows from (3) that

\[
\text{inv}_B(us \sigma) = \text{inv}_B(v) + \text{sgn}(v(1)).
\]  

(6)

On the other hand, for \( i \in [n-1] \) we clearly have that

\[
\text{inv}_B(us \sigma) = \begin{cases} 
\text{inv}_B(v) + 1, & \text{if } v(i) < v(i + 1), \\
\text{inv}_B(v) - 1, & \text{if } v(i) > v(i + 1).
\end{cases}
\]  

(7)

Since \( \text{inv}_B(e) = l_B(e) = 0 \), (6) and (7) prove (5), as claimed.

We now prove (4) by induction on \( \text{inv}_B(\sigma) \). If \( \text{inv}_B(\sigma) = 0 \) then \( \sigma = 12 \cdots n = e \) and (4) clearly holds. So let \( t \in \mathbb{N} \) and \( v \in B_n \) be such that \( \text{inv}_B(v) = t + 1 \). Then \( v \neq e \) and hence there exists \( s \in S \) such that \( \text{inv}_B(us \sigma) = t \) (otherwise (6) and (7) would imply that \( 0 < v(1) < v(2) < \cdots < v(n) \) and hence that \( v = e \)). This, by the induction hypothesis, implies that \( l_B(us \sigma) = t \) and hence that \( l_B(v) \leq t + 1 \). Therefore \( l_B(v) \leq \text{inv}_B(v) \) and this, by (5), concludes the induction step and hence the proof.

Other (more complicated) combinatorial descriptions of the length function of \( B_n \) appear in [19, §5, eq. (6)], [16, §2] and [15, Prop. 2.1].

As a consequence of Proposition 3.1 we obtain the following simple combinatorial description of the number of descents of an element of \( B_n \).

**Corollary 3.2.** Let \( n \in \mathbb{P} \) and let \( v \in B_n \). Then

\[
d_B(v) = |\{ i \in [n] \mid v(i - 1) > v(i) \}|,
\]  

(8)

where \( v(0) \overset{\text{def}}{=} 0 \).

**Proof.** Fix \( n \in \mathbb{P} \) and \( v \in B_n \). By our definitions and Proposition 3.1 we have that

\[
d_B(v) \overset{\text{def}}{=} |\{ s \in S \mid l_B(us \sigma) < l_B(v) \}| = |\{ s \in S \mid \text{inv}_B(us \sigma) < \text{inv}_B(v) \}|,
\]

and the thesis follows from (6) and (7). \( \square \)

Given \( \sigma \in B_n \) we let

\[
N(\sigma) \overset{\text{def}}{=} |\{ i \in [n] \mid \sigma(i) < 0 \}|,
\]  

(9)

and \( P(\sigma) \overset{\text{def}}{=} n - N(\sigma) \). We note the following consequence of (9) and Proposition 3.1, which was also observed (independently) by V. Reiner in [15] (see Proposition 2.2).

**Proposition 3.3.** For \( n \in \mathbb{P} \) we have that

\[
\sum_{\sigma \in B_n} x^{l_B(\sigma)} q^{N(\sigma)} = \prod_{i=1}^{n} (1 + x + \cdots + x^{i-1})(1 + qx^i).
\]

**Proof.** We proceed by induction on \( n \in \mathbb{P} \). Let \( \sigma \in B_{n-1} \) and let \( \sigma_t \overset{\text{def}}{=} \sigma \cup \{ n \} \).
for \( i \in [n] \). Hence we conclude that

\[
\sum_{w \in B_n} x^{l_B(w)}q^{N(w)} = \sum_{\sigma \in B_n} \left( \sum_{i=1}^{n} x^{l_B(\sigma_i)}q^{N(\sigma_i)} + \sum_{i=1}^{n} x^{l_B(\sigma_{n-i})}q^{N(\sigma_{n-i})} \right)
\]

\[
= \sum_{\sigma \in B_{n-1}} \sum_{i=1}^{n} x^{l_B(\sigma)}q^{N(\sigma)}(x^{n-i} + qx^{n+i-1})
\]

\[
= \sum_{\sigma \in B_{n-1}} x^{l_B(\sigma)}q^{N(\sigma)}(1 + qx^n)(1 + x + \cdots + x^{n-1}),
\]

and the thesis follows by induction. \(\square\)

Corollary 3.2 enables us to define \(q\)-analogues of the polynomials \(P(B_n; x)\) which have the additional property of reducing, when \(q=0\), to the Eulerian polynomials. For \(n \in \mathbb{P}\) we let

\[
B_n(x; q) \overset{\text{def}}{=} \sum_{k=0}^{n} B_{n,k}(q)x^k \overset{\text{def}}{=} \sum_{\sigma \in B_n} q^{N(\sigma)}x^{l_B(\sigma)}.
\] (10)

Our goal is to generalize to the \(B_{n,k}(q)\)'s the theory of the ordinary Eulerian numbers. Many of the results in this section are new even in the case \(q = 1\).

The next result generalizes several well known formulas for ordinary Eulerian polynomials (see, e.g., [6, Exercise 3, p. 292; Theorem F, p. 245; Theorem D, p. 243; eq. [5], p. 244] as well as Corollary 2 of [8] and Proposition 7.1(b) of [25]).

**Theorem 3.4.** For \(n \in \mathbb{N}\), let \(B_n(x; q)\) be the polynomial defined by (10). Then we have that:

(i)

\[
B_n(x; q) = (1 + ((1 + q)n - 1)x)B_{n-1}(x; q) + (1 + q)(x - x^2)\frac{\partial}{\partial x}(B_{n-1}(x; q));
\] (11)

for \(n \in \mathbb{P}\), where \(B_0(x; q) \overset{\text{def}}{=} 1\);

(ii)

\[
\sum_{i \geq 0} ((1 + q)i + 1)^n x^i = \frac{B_n(x; q)}{(1 - x)^{n+1}};
\] (12)

as formal power series in \(\mathbb{Z}[q][[x]]\);

(iii)

\[
((1 + q)x + 1)^n = \sum_{k=0}^{n} B_{n,k}(q)\binom{x + n - k}{n};
\] (13)

(iv)

\[
\sum_{n \geq 0} B_n(x; q) \frac{t^n}{n!} = \frac{(1 - x)e^{(1 - x)(1 + q)}}{1 - xe^{(1 - x)(1 + q)}};
\] (14)

as formal power series in \(\mathbb{Z}[x; q][[t]]\).
\textbf{Proof.} (i) Let $\sigma \in B_{n-1}$, $i \in [n]$ and $\sigma_i, \sigma_{i-1}$ have the same meaning as in the proof of Proposition 3.3. Then it follows from Corollary 3.2 that
\begin{align*}
d_B(\sigma_{\pm i}) = \begin{cases} 
d_B(\sigma), & \text{if } \sigma(i-1) > \sigma(i), \\
d_B(\sigma) + 1, & \text{otherwise}, \end{cases}
\end{align*}
for $i \in [n-1]$, and
\begin{align*}
d_B(\sigma_n) = d_B(\sigma), \quad d_B(\sigma_{-n}) = d_B(\sigma) + 1.
\end{align*}
Therefore,
\begin{align*}
B_n(x; q) = \sum_{i=1}^{n} \sum_{\sigma \in B_{n-1}} (x^{d_B(\sigma)} q^{N(\sigma)}) + x^{d_B(\sigma_{-1})} q^{N(\sigma_{-1})})
= (1 + xq)B_{n-1}(x, q) + (1 + q) \sum_{\sigma \in B_{n-1}} q^{N(\sigma)} \sum_{i=1}^{n-1} x^{d_B(\sigma_i)},
\end{align*}
but
\begin{align*}
\sum_{\sigma \in B_{n-1}} q^{N(\sigma)} \sum_{i=1}^{n-1} x^{d_B(\sigma_i)} &= \sum_{\sigma \in B_{n-1}} q^{N(\sigma)} (d_B(\sigma)x^{d_B(\sigma)} + (n - 1 - d_B(\sigma))x^{d_B(\sigma)+1})
= (n-1)xB_{n-1}(x, q) + (1-x) \sum_{\sigma \in B_{n-1}} d_B(\sigma)q^{N(\sigma)}x^{d_B(\sigma)},
\end{align*}
and (11) follows.

(ii) It follows from well known properties of rational generating functions (see, e.g., [21, Corollary 4.3.1, p. 208]) that there exist polynomials $W_n(x, q)$ of degree $\leq n$ in $x$ such that
\begin{align}
\sum_{i=0}^{n} ((1 + q)i + 1)^n x^i = \frac{W_n(x, q)}{(1-x)^{n+1}},
\end{align}
for $n \in \mathbb{N}$, as formal power series in $\mathbb{Z}[q][[x]]$. Therefore we have that
\begin{align*}
\frac{W_n(x, q)}{(1-x)^{n+1}} = (1 + q)x \partial_x \left[ \frac{W_{n-1}(x, q)}{(1-x)^n} \right] + \frac{W_{n-1}(x, q)}{(1-x)^n},
\end{align*}
which yields that
\begin{align}
W_n(x, q) = (1 + q)x \left[ (1-x) \partial_x (W_{n-1}(x, q)) + nW_{n-1}(x, q) \right]
+ (1-x)W_{n-1}(x, q).
\end{align}
However, it follows immediately from (16) that $W_0(x, q) = 1$, so (ii) follows from (17), (16), and (i).

(iii) This follows immediately from (ii) and Theorem 2.1.

(iv) By (ii) we have that
\begin{align*}
\sum_{n \geq 0} B_n(x, q) \frac{t^n}{n!} = \sum_{i \geq 0} x^i \sum_{n \geq 0} ((1 + q)i + 1)^n (1-x)^{n+1} \frac{t^n}{n!}
= (1-x)e^{(1-x)(1+q)} \sum_{i \geq 0} x^i e^{(1-x)(1+q)},
\end{align*}
and (14) follows. \hfill \Box

Note that by equating the coefficients of $q^k$ on both sides of (11) we obtain, by (10), that
\begin{align}
B_{n,k}(q) = [1 + (1+q)k]B_{n-1,k}(q) + [(1+q)(n-k+1) - 1]B_{n-1,k-1}(q)
\end{align}
for all $0 \leq k \leq n, n \in \mathbb{P}$. 

LEMMA 3.5. For \( n \in \mathbb{P} \) we have that

\[
\sum_{\sigma \in B_n} x^{d(\sigma)} q^{N(\sigma)} = (1 + q)^n A_n(x). \]

PROOF. Let \( T \overset{\text{def}}{=} \{ \sigma \in B_n : d(\sigma) = 0 \} \). Note that, since an element \( \nu \in T \) is uniquely determined by the set \( \{ \nu(i) : \nu(i) > 0 \text{ and } i \in [n] \} \), we have that

\[
\sum_{\nu \in T} q^{N(\nu)} = (1 + q)^n. \tag{19}
\]

Furthermore, if \( u \in S_n \) and \( \sigma \in T \) then

\[
d(\sigma u) = d(u) \tag{20}
\]

(since \( \sigma(i) < \sigma(j) \) iff \( i < j \), for all \( i, j \in [n] \)). This implies that if \( u, v \in S_n \) with \( u \neq v \), then \( Tu \cap Tv = \emptyset \) (since if \( \sigma_1 u = \sigma_2 v \) for some \( \sigma_1, \sigma_2 \in T \) then \( \sigma_1 uv^{-1} = \sigma_2 \) and hence \( 0 = d(\sigma_2) = d(\sigma_1 uv^{-1}) = d(uv^{-1}) \), which implies that \( uv^{-1} = e \)). Since \( |Tu| = |T| = 2^n \) for all \( u \in S_n \) this shows that \( B_n = \bigcup_{u \in S_n} Tu \). Therefore, using (19) and (20) we obtain that

\[
\sum_{\sigma \in B_n} x^{d(\sigma)} q^{N(\sigma)} = \sum_{u \in S_n} \sum_{\sigma \in T} x^{d(\sigma u)} q^{N(\sigma u)}
\]

\[
= \sum_{u \in S_n} x^{d(u)} \sum_{\sigma \in T} q^{N(\sigma)}
\]

\[
= (1 + q)^n \frac{A_n(x)}{x},
\]

as desired.

The following recurrence generalizes a well known recurrence for ordinary Eulerian polynomials (see, e.g., [9]).

THEOREM 3.6. The polynomials \( B_n(x; q) \) defined by (10) satisfy the recurrence relation

\[
B_n(x; q) = \sum_{i=1}^{n} \binom{n-1}{i-1} B_{i-1}(x; q)(1 + q)^{n-i+1}x B_{n-i}(x; 0)
\]

\[
+ (1 - x)B_{n-1}(x; q), \tag{21}
\]

for \( n \in \mathbb{P} \), with the initial condition \( B_0(x; q) = 1 \).

PROOF. Let \( \sigma \in B_{n-1} \) and \( \sigma_i, \sigma_{-i} \), for \( i \in [n] \), have the same meaning as in the proof of Proposition 3.3. Then it follows from Corollary 3.2 that

\[
d_B(\sigma_+ \sigma) = d_B(\sigma(1), \ldots, \sigma(i - 1)) + d(\sigma(i), \ldots, \sigma(n - 1)) + 1 \tag{22}
\]

(where \( d_B(\emptyset) \overset{\text{def}}{=} 0 \)), for \( i \in [n - 1] \). Using this, (15), and Lemma 3.5, we therefore obtain, reasoning as in the proof of (11), that

\[
B_n(x; q) = (1 + xq)B_{n-1}(x; q) + (1 + q) \sum_{i=1}^{n-1} \sum_{\sigma \in B_{n-i}} x^{d(\sigma)} q^{N(\sigma)}.
\]
But
\[
\sum_{i=1}^{n-1} \sum_{\sigma \in B_{n-1}} x^{d_\sigma(\sigma)} q^{N(\sigma)} = \sum_{i=1}^{n-1} (n-1) \sum_{\sigma \in B_{n-1}} \sum_{\tau \in B_{n-1}} x^{d_\sigma(\sigma)+d(\tau)+1} q^{N(\sigma)+N(\tau)}
= \sum_{i=1}^{n-1} (n-1) B_{i-1}(x; q)(1+q)^{n-i} A_{n-i}(x)
\]
and the thesis follows.

A consequence of Theorem 3.4 is the following strengthening of Theorem 2.4 for Coxeter groups of type $B_n$ and $A_n$. The result follows immediately from Theorems 3.4(iii) and 2.2.

**Corollary 3.7.** For each $q \geq 0$ and $n \in \mathbb{N}$, the polynomial $B_n(x, q) \in \mathbb{R}[x]$ defined by (10) has only real zeros. In particular, the sequence $(B_{n,0}(q), B_{n,1}(q), \ldots, B_{n,n}(q))$ is log-concave and unimodal.

Note that for $q = 0$ the preceding corollary reduces to a well known result for Eulerian polynomials (see, e.g., [6, p. 292, Ex. 3]).

Using a well known result from the theory of total positivity (see, e.g., [12, Chap. 8, §2] or [3, Theorem 2.2.4]) we see that the preceding result implies the following determinantal inequalities.

**Corollary 3.8.** For $n, i, r \in \mathbb{P}$ and $q \geq 0$ we have that
\[
\det \begin{bmatrix}
B_{n,i}(q) & B_{n,i+1}(q) & \cdots & B_{n,i+r}(q) \\
B_{n,i-1}(q) & B_{n,i}(q) & \cdots & B_{n,i+r-1}(q) \\
\vdots & \vdots & \ddots & \vdots \\
B_{n,i-r}(q) & B_{n,i-r+1}(q) & \cdots & B_{n,i}(q)
\end{bmatrix} \geq 0 \quad (23)
\]
(where $B_{n,k}(q) \equiv 0$ if $k < 0$).

It would be interesting to have a combinatorial, or geometric (for $q = 0, 1$), interpretation of the numbers on the l.h.s. of (23), in the spirit of [10] and [5]. Let us here only mention that this problem is open even for $q = 0$ and $r = 1$.

The presence of the parameter $q$ enables us to obtain several other refinements and generalizations of the unimodality of $P(B_n; x)$ and $A_n(x)$. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, let
\[
A_{n,k}(x) \overset{\text{def}}{=} \sum_{\sigma \in B_n : N(\sigma) = k} x^{d_\sigma(\sigma)}. \quad (24)
\]

**Corollary 3.9.** For $n \in \mathbb{N}$, and $0 \leq k \leq n$, the polynomial $A_{n,k}(x)$ defined by (24) has only real zeros. In particular, it is log-concave and unimodal.

**Proof.** Taking the coefficient of $q^k$ on both sides of (12) yields, by (24), that
\[
\sum_{i=0}^{n} \binom{n}{k} i^k (i+1)^{n-k} x^i = \frac{A_{n,k}(x)}{(1-x)^{n+1}},
\]
for $k = 0, \ldots, n$, as formal power series in $\mathbb{Z}[[x]]$. The thesis then follows from Theorems 2.1 and 2.2.

Note that, in general, the polynomials $A_{n,k}(x)$ are not symmetric. Therefore, it is not
possible to deduce the unimodality of $P(B_n; x)$ from (24) and the preceding corollary. However, this is possible if we pair the $A_{n,k}(x)$'s together correctly. To do this we need the following symmetry property, which generalizes the fact that $P(B_n; x)$ is a symmetric polynomial with center of symmetry at $[n/2]$

**Proposition 3.10.** For $n \in \mathbb{N}$, and $0 \leq k \leq n$ we have that

$$B_{n,k}(q) = q^{n}B_{n,n-k}(1/q).$$

**Proof.** For $v \in B_n$, let $-v \overset{\text{def}}{=} [-v(1), \ldots, -v(n)]$. Then it follows immediately from Corollary 3.2, and from (9), that

$$d_B(v) + d_B(-v) = n, \quad N(v) + N(-v) = n,$$

for all $v \in B_n$. The thesis then follows from (10). \hfill \square

The next result, together with (24), provides a refinement of Theorem 2.4 for Coxeter groups of type $B_n$.

**Theorem 3.11.** Let $n \in \mathbb{N}$, and $0 \leq k \leq [n/2]$. Then the polynomial

$$A_{n,k}(x) + A_{n,n-k}(x)$$

is symmetric and unimodal with center of symmetry at $[n/2]$, and has only real zeros. In particular, it is log-concave with no internal zeros.

**Proof.** Note first that Proposition 3.10 is equivalent, by (24), to the statement that

$$A_{n,k}(x) = x^n A_{n,n-k}(1/x),$$

for all $0 \leq k \leq n$. This immediately implies that $A_{n,k}(x) + A_{n,n-k}(x)$ is a symmetric polynomial with center of symmetry at $[n/2]$. Furthermore, since $A_n(x)$ is a symmetric polynomial with center of symmetry at $[(n+1)/2]$ (see, e.g., [6, p. 241, eq. [5c] and p. 292, Ex. 3]) we deduce from (26) and (24) that

$$A_{n,n}(x) = x^n A_{n,0}(1/x) = x^{n+1} A_n(1/x) = A_n(x) = x A_{n,0}(x).$$

(27)

Now, from (25) we deduce that

$$\sum_{i=0}^{n} \binom{n}{k} i^k (i+1)^k [(i+1)^{n-2k} + i^{n-2k}] x^i = \frac{A_{n,k}(x) + A_{n,n-k}(x)}{(1-x)^{n+1}},$$

(28)

and

$$\sum_{i=0}^{2k} \binom{2k}{k} i^k (i+1)^k x^i = \frac{A_{2k,k}(x)}{(1-x)^{2k+1}},$$

(29)

and

$$\sum_{i=0}^{n} [(i+1)^{n-2k} + i^{n-2k}] x^i = \frac{A_{n-k,0}(x) + A_{n-k,n-k}(x)}{(1-x)^{n-2k+1}}.$$

(30)
By (27) and Corollary 3.9, the numerators on the r.h.s. of (29) and (30) are polynomials with (non-negative coefficients and) only real zeros. The thesis now follows from Theorem 2.15 of [26].

There is another aspect of the theory of Eulerian polynomials which generalizes to the polynomials $B_n(x; q)$ and which provides yet another refinement of the unimodality of $P(B_n; x)$ and $A_n(x)$.

To state this, we will need to use some terminology from the theory of symmetric functions. In particular, we denote by $\mathcal{P}$ the set of all integer partitions and, for $\lambda \in \mathcal{P}$, by $s_{\lambda}$ the Schur function corresponding to $\lambda$ and by $f^\lambda$ the number of standard tableaux of shape $\lambda$ (see [13, Ch. I], for details). Also, given an integral domain $R$, we denote by $\overset{\lambda}{A}_R$ the subring of $R[[x_1, x_2, \ldots]]$ consisting of all $p \in R[[x_1, x_2, \ldots]]$ such that $p(x_1, x_2, \ldots) = p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ for all bijections $\sigma: \mathcal{P} \to \mathcal{P}$.

**Theorem 3.12.** For each $\lambda \in \mathcal{P}$, define a polynomial $T_\lambda(x; q)$ by

$$\sum_\lambda T_\lambda(x; q)s_\lambda \overset{\text{def}}{=} \frac{\sum_{k \geq 0} (1 + xq)^k s_{(k)}}{1 - \sum_{k \geq 2} (x + x^2 + \cdots + x^{k-1})(1 + q)^k s_{(k)}},$$

as formal power series in $\overset{\lambda}{A}_{Z[x,q]}$. Then:

(i) $T_\lambda(x; 0)$ and $T_\lambda(x; 1)$ are symmetric unimodal polynomials with centers of symmetry at $(|\lambda| - 1)/2$ and $|\lambda|/2$, respectively;

(ii) $(xq)^{\lambda_1} T_{\lambda}(x, q) = T_\lambda(x; q)$ for all $\lambda \in \mathcal{P}$;

(iii) $\sum_{\lambda \vdash n} f^\lambda T_\lambda(x; q) = B_n(x; q)$, for all $n \in \mathcal{P}$.

**Proof.** (i) is an immediate consequence of Proposition 1 and Theorem 2 of [4]. For $T_\lambda(x; 0)$ this is proved in Section 3 of [4]. For $T_\lambda(x; 1)$ take $a_\lambda = b_\lambda = c_\lambda = s_\lambda$ for $\lambda \in \mathcal{P}$, and let $S \overset{\text{def}}{=} \{(k): k \geq 2\}$, $r_\mu(x) \overset{\text{def}}{=} 2^{\mu_1}(x + x^2 + \cdots + x^{\mu_1 - 1})$, for $\mu \in S$, $T \overset{\text{def}}{=} \{(k): k \geq 0\}$, $u(\nu)(x) = (1 + x)^{\nu_1}$ for $\nu \in T$, in the notation of Theorem 2 of [4].

To prove (ii) just substitute $x$ with $x^{-1}$, $q$ with $q^{-1}$ and $x_i$ with $x_\sigma_i$, for $i \in \mathcal{P}$, in (31) to obtain

$$\sum_\lambda (xq)^{\lambda_1} T_{\lambda}(x, q)s_\lambda \overset{\text{def}}{=} \frac{\sum_{k \geq 0} (xq)^k \left(\frac{1}{xq}\right)^k s_{(k)}}{1 - \sum_{k \geq 2} x^k(x^{-1} + x^2 + \cdots + x^{-k+1})q^k \left(\frac{1}{xq}\right)^k s_{(k)}} = \sum_\lambda T_\lambda(x; q)s_\lambda,$$

and (ii) follows.

To prove (iii) define a ring homomorphism, $R: A_{\mathbb{Q}[x]} \to \mathbb{Q}[x][[t]]$, by

$$R(p) \overset{\text{def}}{=} \sum_{n \geq 0} [x_1 \cdots x_n](p) \frac{t^n}{n!},$$

where $p \in A$ and $[x_1 \cdots x_n](p)$ denotes the coefficient of $x_1 \cdots x_n$ in $p$ (where $\prod_{i=1}^n x_i \overset{\text{def}}{=} 1$ if $n = 0$). It is then well known that

$$R(s_\lambda) = f^\lambda \frac{t^{\lambda_1}}{|\lambda|!},$$

(32)
for all $\lambda \in \mathcal{P}$ (see, e.g., [4, §4], for details). Now applying $R$ to both sides of (31) and using (32) we obtain

$$\sum_{\lambda \in \mathcal{P}} f^\lambda T_\lambda(x; q) \frac{t^{|\lambda|}}{|\lambda|!} = \frac{e^{(1+q)r}}{1 - \frac{1}{1-x}(xe^{(1+q)r} - x - e^{(1+q)r} + 1)}$$

$$= \frac{(1-x)e^{(1+q)r}}{e^{(1+q)r}x - xe^{(1+q)r}}$$

$$= \sum_{n \geq 0} B_n(x; q) \frac{t^n}{n!}$$

by part (iv) of Theorem 3.4. Equating the coefficients of $t^n$ yields (iii), as desired. □

For $q = 0$ the preceding result was first proved by Stanley (see [22, Prop. 12]) using results from the theory of representations of the symmetric group and from algebraic geometry. Note that the preceding theorem immediately implies Theorem 2.4 for Coxeter groups of types $B_n$ and $A_n$ as well as Proposition 3.10.

We now give a combinatorial interpretation to the polynomials $T_\lambda(x; q)$. By an extended signed tableau we mean an array $T \overset{\text{def}}{=} (T_{i,j}), i, j \in \mathbb{P}$ such that:

(i) $T_{i,j} \in \mathbb{Z} \cup \{+\infty, -\infty\}$ for all $(i, j) \in \mathbb{P} \times \mathbb{P}$;
(ii) $T_{i,j} = 0$ for all but finitely many $(i, j) \in \mathbb{P} \times \mathbb{P}$;
(iii) $|T_{i,j}| \leq |T_{i,j+1}|$ for all $(i, j) \in \mathbb{P} \times \mathbb{P}$ such that $T_{i,j+1} \neq 0$;
(iv) $|T_{i,j}| < |T_{i+1,j}|$ for all $(i, j) \in \mathbb{P} \times \mathbb{P}$ such that $T_{i+1,j} \neq 0$.

The shape of an extended signed tableau $T$ is the integer partition $sh(T) \overset{\text{def}}{=} (\lambda_1, \lambda_2, \ldots)$, where

$$\lambda_i \overset{\text{def}}{=} |\{j \in \mathbb{P} : T_{i,j} \neq 0\}|,$$

for $i \in \mathbb{P}$. Given $T$ as above and $\alpha \in \mathbb{Z} \cup \{+\infty, -\infty\}$, $\alpha \neq 0$, we let $m_\alpha(T) \overset{\text{def}}{=} |\{(i, j) \in \mathbb{P} \times \mathbb{P} : T_{i,j} = \alpha\}|$, and $N(T) \overset{\text{def}}{=} \sum_{\alpha \neq 0} m_\alpha(T)$. We then say that $T$ is admissible if $m_{+\infty}(T) + m_{-\infty}(T) > 0$ implies $m_i(T) + m_{-i}(T) > 0$ for all $i \in \mathbb{P}$. For example,

$$-1 \quad -2 \quad 2$$

$$2 \quad +\infty \quad -\infty$$

$$-\infty$$

is an admissible extended signed tableau, while

$$-1 \quad 1 \quad 2 \quad +\infty$$

$$2 \quad -2$$

$$-4$$

is an extended signed tableau, but is not admissible.

A rooted extended signed tableau is a pair $(T, R)$ in which $T$ is an extended signed tableau and $R \subseteq \{(i, j) \in \mathbb{P} \times \mathbb{P} : T_{i,j} \neq 0, \pm \infty\}$. Given such a pair we define its index to be

$$\text{ind}(T, R) \overset{\text{def}}{=} \sum_{(i,j) \in R} |(a, b) \in \mathbb{P} \times \mathbb{P} : |T_{a,b}| = |T_{i,j}| \text{ and } b < j|.$$

We will usually represent (and think of) a rooted extended signed tableau $T$ as an extended signed tableau in which some of the entries have been 'barred'. For example,

$$-1 \quad 2 \quad \text{barred} \quad -\infty \quad +\infty$$

$$\text{barred} \quad -3 \quad -3$$

$$+\infty$$
is a rooted extended signed tableau of index $0 + 0 + 2 = 2$.

**Theorem 3.13.** For $\lambda \in \mathcal{P}$, let $T_\lambda(x; q)$ be the polynomial defined by (31). Then

$$T_\lambda(x; q) = \sum_{(T, R)} x^{\text{ind}(T, R) + \text{m}_+(T)} q^{N(T)},$$

(33)

where the sum is over all rooted admissible extended signed tableaux $(T, R)$ of shape $\lambda$ such that, for all $1 \leq i < +\infty$, exactly one entry equal to $\pm i$ is rooted, except the leftmost one.

**Proof.** For $\lambda \in \mathcal{P}$, denote by $T'_\lambda(x; q)$ the polynomial on the r.h.s. of (33) and let

$$\tilde{T}_\lambda(x; q) \overset{\text{def}}{=} \sum_{(T, R)} x^{\text{ind}(T, R)} q^{N(T)},$$

(34)

where $(T, R)$ ranges over all rooted tableaux appearing in the sum on the r.h.s. of (33) for which $m_+(T) + m_-(T) = 0$. Then it is clear that, for all $\lambda \in \mathcal{P}$,

$$T'_\lambda(x; q) = \sum_{\mu} \tilde{T}_\mu(x; q)(1 + xq)^{1/|\lambda\setminus\mu|},$$

(35)

where the sum is over all partitions $\mu \subseteq \lambda$ such that $\lambda \setminus \mu$ is a horizontal strip. Multiplying both sides of (35) by $s_\lambda$ and using a well known result for the multiplication of Schur functions (see, e.g., [13, eq. (5.16), p. 42]) we obtain that

$$\sum_{\lambda \in \mathcal{P}} T'_\lambda(x; q)s_\lambda = \sum_{\mu} \tilde{T}_\mu(x; q) \sum_{k=0} (1 + xq)^{1/|\lambda\setminus\mu|} s_\mu s_{(k)}$$

$$= \sum_{\mu} \tilde{T}_\mu(x; q)s_\mu \sum_{k=0} (1 + xq)^{1/|\lambda\setminus\mu|} s_{(k)},$$

(36)

On the other hand, the definition (34) of the $\tilde{T}_\lambda(x; q)$'s implies that $\tilde{T}_{(0)}(x; q) = 1$ and, for all $\lambda \in \mathcal{P} \setminus \{0\}$,

$$\tilde{T}_\lambda(x; q) = \sum_{\mu} \tilde{T}_\mu(x; q)(x + x^2 + \cdots + x^{|\lambda\setminus\mu|-1})(1 + q)^{1/|\lambda\setminus\mu|},$$

(37)

where the sum is over all partitions $\mu \subseteq \lambda$ such that $\lambda \setminus \mu$ is a horizontal strip and $|\lambda\setminus\mu| \geq 2$. Now multiplying both sides of (37) by $s_\lambda$ and summing over all $\lambda \in \mathcal{P}$ yields that

$$\sum_{\lambda} \tilde{T}_\lambda(x; q)s_\lambda = 1 + \sum_{\mu} \sum_{k \geq 2} (x + x^2 + \cdots + x^{k-1})(1 + q)^k \tilde{T}_\mu(x; q)s_\mu s_{(k)}$$

$$= 1 + \sum_{\mu} \tilde{T}_\mu(x; q)s_\mu \sum_{k \geq 2} (x + x^2 + \cdots + x^{k-1})(1 + q)^k s_{(k)},$$

(38)

where we have again used equation (5.16) on page 42 of [13]. Now rewriting (38) as

$$\sum_{\lambda} \tilde{T}_\lambda(x; q)s_\lambda = \frac{1}{1 - \sum_{k \geq 2} (x + x^2 + \cdots + x^{k-1})(1 + q)^k s_{(k)}},$$

and comparing with (36) and (31) yields the desired result. \qed

It can be shown that for $q = 0$ the preceding theorem is equivalent to a result of J. Stembridge (see [24, Thm 4.2]).

We now generalize a classical result of Frobenius for Eulerian polynomials; namely, we give a combinatorial interpretation to the coefficients of $B_\nu(x, q)$ when expressed in powers of $(x - 1)$. Let $S \subseteq \mathcal{P}$: a signed partition of $S$ is a collection $\pi = \{B_1, \ldots, B_k\}$ of
subsets of \(-S \cup S\) such that \(\{B_1, \ldots, B_k, -B_1, \ldots, -B_k\}\) is a partition of \(-S \cup S\) (where \(-\{a_1, \ldots, a_r\} \defeq \{-a_1, \ldots, -a_r\}\), for \(a_1, \ldots, a_r \in \mathbb{Z}\)). We call \(B_1, \ldots, B_k\) the blocks of \(\pi\) and say that \(\pi\) has \(k\) blocks. We also let

\[
P(\pi) \defeq \left| \left\{ x \in \bigcup_{i=1}^{k} B_i : x > 0 \right\} \right|.
\]

A partial signed partition of \(S\) is a signed partition of some subset of \(S\). For example, \(\pi \defeq \{-1, 3\}, \{-4\}, \{5, 7\}\) is a partial signed partition of \([7]\) and \(P(\pi) = 3\). We denote by \(B_{\Pi}(S; k)\) (respectively, \(B_{\Pi}(S; k)\)) the set of all signed partitions (respectively, partial signed partitions) of \(S\) having \(k\) blocks, and let

\[
S_B(n, k; q) \defeq \sum_{\pi \in B_{\Pi}(\{n\}, k)} q^{n - P(\pi)}. \tag{39}
\]

**Theorem 3.14.** For \(n \in \mathbb{N}\), let \(B_n(x; q)\) be the polynomial defined by (10). Then

\[
B_n(x; q) = \sum_{k=0}^{n} k! S_B(n, k; q)(x - 1)^{n-k}. \tag{40}
\]

**Proof.** We proceed by induction on \(n \in \mathbb{N}\), the thesis being clearly true if \(n = 0\). Therefore fix \(n \in \mathbb{P}\) and let \(\pi \in B_{\Pi}(\{n\}, k)\). If \(\{n\}\) (respectively, \(-\{n\}\)) is a block of \(\pi\), then removing it from \(\pi\) yields a partial signed partition of \([n-1]\) into \(k-1\) blocks. If \(n\) (respectively, \(-n\)) is in a block of \(\pi\) of size \(\geq 2\), then removing \(n\) (respectively, \(-n\)) from this block yields a partial signed partition of \([n-1]\) into \(k\) blocks. Finally, if neither \(n\) nor \(-n\) appear in \(\pi\), then \(\pi \in B_{\Pi}(\{n-1\}, k)\). Therefore we conclude from (39) that

\[
S_B(n, k; q) = (1 + q)S_B(n-1, k-1; q) + (k(1 + q) + q)S_B(n-1, k; q), \tag{41}
\]

for \(0 \leq k \leq n\). Multiplying both sides of (41) by \(k! (x - 1)^{n-k}\) and summing for \(k = 0, \ldots, n\) yields, by our induction hypothesis and part (i) of Theorem 3.4,

\[
\sum_{k=0}^{n} k! S_B(n, k; q)(x - 1)^{n-k} = (1 + q) \left[ (1 - x) \frac{\partial}{\partial x} (B_{n-1}(x; q)) + nB_{n-1}(x; q) \right]
\]

\[
+ (1 + q)(x - 1) \left[ (n-1)B_{n-1}(x; q) + (1 - x) \right]
\]

\[
\times \frac{\partial}{\partial x} (B_{n-1}(x; q)) + q(x - 1)B_{n-1}(x; q)
\]

\[
= B_n(x; q),
\]

which completes the induction step and hence the proof. \(\square\)

Note that it follows easily from (39) that

\[
S_B(n, k; q) = \sum_{i=k}^{n} \binom{n}{i} q^{i-k}(1 + q)^i S(i, k). \tag{42}
\]

In particular, this shows that \(S_B(n, k; 0) = S(n, k)\), so that Theorem 3.14 does reduce,
when \( q = 0 \), to Frobenius’ theorem (see, e.g., [6, Thm E, p. 244]). When \( q = 1 \), Theorem 3.14 reduces to Corollary 1 of [8], and to Proposition 7.1(a) of [25]. It is also possible to deduce Theorem 3.14 algebraically from (42) and from part (iv) of Theorem 3.4.

It is well known that exceedances and descents are equidistributed on \( S_n \) and that this distribution is given by the Eulerian polynomials (see, e.g., [21, Prop. 1.3.12]). We now show how this result generalizes to the \( B_n(x; q) \)'s.

Given \( \sigma \in B_n \) and \( i \in [n] \), we say that \( i \) is an exceedance (respectively, weak exceedance) of \( \sigma \) if either \( \sigma(i) = -i \) (respectively, \( \sigma(i) = i \) or \( \sigma(\sigma(i)) > \sigma(i) \) (clearly, these two conditions are mutually exclusive). We say that \( i \) is an ascent of \( \sigma \) if \( \sigma(i) > \sigma(i - 1) \) (where \( \sigma(0) \equiv 0 \)). We denote by \( e(\sigma) \) (respectively, \( \we(\sigma) \), \( a(\sigma) \)) the number of exceedances (respectively, weak exceedances, ascents) of \( \sigma \). It is clear that if \( \sigma \in S_n \), then these notions of exceedance and weak exceedance coincide with the usual ones (see, e.g., [21, p. 23]). As an example, if \( \sigma = [3, -2, 5, 7, -1, -4, -6, -8, 9] \) then \( a(\sigma) = 4, \we(\sigma) = 5 \) and \( e(\sigma) = 6 \). We then have the following result.

**Theorem 3.15.** There is a bijection \( \tau : B_n \to B_n \) such that

\[
\alpha(\tau(\sigma)) = \we(\sigma),
\]

and

\[
\N(\tau(\sigma)) = N(\sigma)
\]

for all \( \sigma \in B_n \).

**Proof.** Let \( \sigma \in B_n \) and let \((a_{11} \cdots a_{1n})(a_{21} \cdots a_{2n}) \cdots (a_{s1} \cdots a_{sn})\) be its standard disjoint cycle form (i.e., each cycle has its largest (in absolute value) element first, and the cycles are written in increasing order of the absolute values of their first elements). Let \( \tau(\sigma) \) be the word obtained by deleting all parentheses in the standard disjoint cycle form of \( \sigma \) (we consider this word as the one-line notation of \( \tau(\sigma) \)). Clearly, \( \tau(\sigma) \in B_n \) and the map \( \tau : B_n \to B_n \) is a bijection such that \( \N(\tau(\sigma)) = N(\sigma) \) for all \( \sigma \in B_n \). Fix \( 1 \leq j \leq s \) and let \( \zeta_j \) be the \( j \)th cycle in the standard disjoint cycle form of \( \sigma \) (so \( \zeta_j = (a_{j1} \cdots a_{jj}) \)). Now, if \( |\zeta_j| \geq 2 \), then

\[
\we(\zeta_j) = |\{ 1 \leq r \leq j : a_{jr} < a_{jr+1} \}|
\]

(43)

(where \( a_{jr+1} \equiv a_{jr+1} \)). But (since \( \sigma \) is in standard form) \( a_{j+1,r} < a_{j+1} \) iff \( a_{j+1,r} > 0 \), and this happens iff \( a_{j-1,r-1} < a_{j,r} \) (where \( a_{0,0} \equiv 0 \)). Therefore, from (43) we conclude that

\[
\we(\zeta_j) = |\{ 1 \leq r \leq j-1 : a_{j,r} < a_{j+1,r+1} \}|
\]

(44)

(where \( a_{j,0} \equiv a_{j-1,j-1} \)). On the other hand, if \( |\zeta_j| = 1 \) then \( \we(\zeta_j) = 1 \) if and only if \( a_{j,j} > 0 \), and this happens iff \( a_{j-1,j-1} < a_{j,j} \), so that (44) is still valid in this case. Therefore, from (44), we conclude that

\[
\we(\sigma) = \sum_{j=1}^{\zeta_j} \we(\zeta_j) = \alpha(\tau(\sigma)),
\]

as desired. \( \square \)

Note that, when restricted to \( S_n \), the bijection \( \tau \) reduces to the ‘fundamental transformation’ of [9, p. 13] (see also [21, p. 17]).

**Corollary 3.16.** For \( n \in \mathbb{P} \) and \( 0 \leq k \leq n \) we have that

\[
\B_n^k(q) = \sum_{\sigma \in B_n; \alpha(\sigma) = k} q^{P(\sigma)} = \sum_{\sigma \in B_n; \we(\sigma) = k} q^{P(\sigma)} = \sum_{\sigma \in B_n; e(\sigma) = k} q^{P(\sigma) + e(\sigma) - \we(\sigma)}.
\]
PROOF. It follows immediately from our definitions that \( a(v) = d(-v) \) for all \( v \in B_n \) (where \(-v\) has the same meaning as in the proof of Proposition 3.10). Also, letting
\[
\alpha(v)(i) = \begin{cases} 
-v(i), & \text{if } |v(i)| = i, \\
v(i), & \text{otherwise},
\end{cases}
\]
for \( i \in [n] \) and \( v \in B_n \), defines an involution \( \alpha: B_n \rightarrow B_n \) such that \( e(v) = we(\alpha(v)) \) and \( N(\alpha(v)) = N(v) + we(v) - e(v) \) for all \( v \in B_n \). The thesis now follows from (10) and Theorem 3.15.

We should mention that a different notion of exceedance for elements of \( B_n \) (also equidistributed with \( B \)-descents) has been defined in [23]. Hence one can obtain yet another combinatorial interpretation for the polynomials \( B_{n,k}(q) \) using this notion.

4. \textbf{q-EULERIAN POLYNOMIALS OF TYPE D}

In this section we examine in detail a natural \( q \)-analogue of \( P(W; x) \) in the case that \( W \) is a Coxeter group of type \( D_n \). We will again slightly abuse notation and denote this group by \( D_n \). Recall that
\[
D_n \overset{\text{def}}{=} \{ \sigma \in B_n; N(\sigma) \equiv 0 \pmod{2} \},
\]
so that \( D_n \) is a subgroup of \( B_n \) of index 2. As a set of Coxeter generators for \( D_n \) we take \( S' = \{ s_1, \ldots, s_{n-1}, s_n' \} \), where \( s_n' = [-2, -1, 3, \ldots, n] \) (and \( s_1, \ldots, s_{n-1} \) have the same meaning as in Section 3).

As we have done for \( B_n \), we start by obtaining an explicit combinatorial description of the polynomials \( P(D_n; x) \). To do this, we first need a simple combinatorial description of the length function of \( D_n \). For convenience, we let
\[
\text{inv} D(\sigma) = \text{inv} B(\sigma) - N(\sigma) \quad (45)
\]
for all \( \sigma \in B_n \).

PROPOSITION 4.1. Let \( n \in \mathbb{P} \). Then
\[
l_D(\sigma) = \text{inv} D(\sigma), \quad (46)
\]
for all \( \sigma \in D_n \).

PROOF. We will prove first that
\[
\text{inv} D(\sigma) \leq l_D(\sigma) \quad (47)
\]
for all \( \sigma \in D_n \). Let \( v \in D_n \). Now, if \( i \in [n-1] \) than it follows from (7) and (45) that
\[
\text{inv} D(\sigma v) = \begin{cases} 
\text{inv} D(v) + 1, & \text{if } v(i) < v(i+1), \\
\text{inv} D(v) - 1, & \text{if } v(i) > v(i+1).
\end{cases}
\]
(48)

On the other hand, since \( s_n' = s_n s_1 s_n \) we obtain from (6) and (7) that
\[
\text{inv} B(\sigma v) = \begin{cases} 
\text{inv} B(v) + \text{sgn}(v(1)) + \text{sgn}(v(2)) + 1, & \text{if } -v(1) < v(2), \\
\text{inv} B(v) + \text{sgn}(v(1)) + \text{sgn}(v(2)) - 1, & \text{if } -v(1) > v(2),
\end{cases}
\]
(49)

and hence we conclude from (45) that
\[
\text{inv} D(\sigma v) = \begin{cases} 
\text{inv} D(v) + 1, & \text{if } v(1) + v(2) > 0, \\
\text{inv} D(v) - 1, & \text{if } v(1) + v(2) < 0.
\end{cases}
\]
(50)
Since \( \text{inv}_D(e) = l_D(e) = 0 \), (48) and (50) imply (47), as claimed.

We now prove (46) by induction on \( \text{inv}_D(\sigma) \). If \( \text{inv}_D(\sigma) = 0 \) then \( \text{inv}(\sigma) = 0 \) and \( \sum_{(j) \leq n; \sigma(j) < 0} (\sigma(j) + 1) = 0 \). Therefore \( \sigma(1) < \cdots < \sigma(n) \) and \( N(\sigma) \leq 1 \) and since \( \sigma \in D_n \), this implies that \( \sigma = e \) and (46) holds in this case. Now let \( t \in \mathbb{N} \) and \( v \in D_n \) be such that \( \text{inv}_D(v) = t + 1 \). Then \( v \neq e \) and we claim that there exists \( s \in S' \) such that \( \text{inv}_D(us) = t \). In fact, if \( \text{inv}_D(us) = t + 2 \) for all \( s \in S' \), then (48) and (50) would imply that \( v(1) < v(2) < \cdots < v(n) \) and \( v(1) + v(2) > 0 \). But this implies that \( v(1) > 0 \) and hence that \( v = e \), which proves our claim. So let \( s \in S' \) be such that \( \text{inv}_D(us) = t \). Then by our induction hypothesis \( l_D(us) = t \) and hence \( l_D(v) \leq t + 1 \). Therefore \( l_D(v) \leq \text{inv}_D(v) \) and this, by (47), concludes the induction step and hence the proof.

Other (more complicated) combinatorial descriptions of the length function of \( D_n \) appear in [19, §6, p. 180] and in [16, §2].

**Corollary 4.2.** Let \( n \in \mathbb{N} \) and \( v \in D_n \). Then

\[
d_D(v) = |\{ i \in [n] : v(i - 1) > v(i) \}|,
\]

where \( v(0) \overset{\text{def}}{=} -v(2) \) (and \( v(n + 1) \overset{\text{def}}{=} 0 \)).

**Proof.** By definition and Proposition 4.1 we have that

\[
d_D(v) = |\{ s \in S' : l_D(us) < l_D(v) \}| = |\{ s \in S' : \text{inv}_D(us) < \text{inv}_D(v) \}|,
\]

and the thesis follows from (48) and (50).

Corollary 4.2 enables us again to define \( q \)-analogues of the polynomials \( P(D_n; x) \) which have the additional property of reducing, when \( q = 0 \), to the Eulerian polynomials. For \( n \in \mathbb{N} \) we let

\[
D_n(x; q) \overset{\text{def}}{=} \sum_{k=0}^{n} D_{n,k}(q)x^k \overset{\text{def}}{=} \sum_{v \in D_n} q^{N(v(2), \ldots, v(n))}x^{d_D(v)}.
\]

It may at first sight seem that the most natural \( q \)-analogue of \( P(D_n; x) \) would be given by the joint distribution of \( d_D(v) \) and \( N(v) \) over \( D_n \). However, the polynomials \( D_n(x; q) \) carry more combinatorial information.

**Proposition 4.3.** For \( n \in \mathbb{N} \), let \( D_n(x; q) \) be the polynomial defined by (52). Then

\[
\sum_{v \in D_n} x^{d_D(v)}q^{N(v)} = \frac{1}{2}[(1 + q)D_n(x; q) + (1 - q)D_n(x; -q)].
\]

**Proof.** Note first that, for any \( v \in B_n \),

\[
N(v) = \begin{cases} N(v(2), \ldots, v(n)), & \text{if } v(1) > 0, \\ N(v(2), \ldots, v(n)) + 1, & \text{if } v(1) < 0. \end{cases}
\]

It therefore follows from (52) that

\[
(1 + q)D_n(x; q) = \sum_{v \in D_n} x^{d_D(v)}(q^{N(v)} + q^{N(v) + \text{sgn}(v(1)))}).
\]

Since \( N(v) \equiv 0 \pmod{2} \) for all \( v \in D_n \), (53) follows.

We note the following consequence of Proposition 4.1.
PROPOSITION 4.4. For \( n \in \mathbb{P} \), we have that
\[
\sum_{\sigma \in D_n} x^{l_0(\sigma)} q^{N(\sigma_2, \ldots, \sigma(n))} = \prod_{j=2}^{n} (1 + x + \cdots + x^{j-1})(1 + qx^{j-1}).
\]

PROOF. Given \( v \in B_n \), let \( \Gamma(v) \equiv [\!-v(1), v(2), \ldots, v(n)] \). Then \( \Gamma \) is clearly a bijection from \( D_n \) to \( B_n \setminus D_n \), and
\[
N(\Gamma(v)) = N(v) + \text{sgn}(v(1)),
\]
for all \( v \in B_n \). Then it follows from (46), (45) and (3) that
\[
l_D(\Gamma(v)) = \text{inv}_D(\Gamma(v)) - n(\Gamma(v)) = \text{inv}_B(v) + \text{sgn}(v(1)) - (N(v) + \text{sgn}(v(1))) = l_D(v).
\]
Therefore,
\[
\sum_{\sigma \in B_n} x^{l_0(\sigma)} q^{N(\sigma)} = \sum_{\sigma \in D_n} \left( x^{l_0(\sigma)} q^{N(\sigma)} + x^{l_0(\Gamma(\sigma))} q^{N(\Gamma(\sigma))} \right)
= \sum_{\sigma \in D_n} x^{l_0(\sigma)} (q^{N(\sigma)} + q^{N(\Gamma(\sigma))})
= \sum_{\sigma \in D_n} x^{l_0(\sigma)} q^{N(\sigma_2, \ldots, \sigma(n))}(1 + q).
\]
On the other hand, it follows from (45) that
\[
\sum_{\sigma \in B_n} x^{l_0(\sigma)} q^{N(\sigma)} = \sum_{\sigma \in B_n} x^{l_0(\sigma)} (q/x)^{N(\sigma)},
\]
and so the thesis follows from (56), (57) and Proposition 3.3. \( \square \)

We now wish to develop a theory for the polynomials \( D_{n,k}(q) \) parallel to that developed in the preceding section for the \( B_{n,k}(q) \)'s.

PROPOSITION 4.5. For \( n \geq 2 \) and \( 0 \leq k \leq n \), we have that
\[
D_{n,k}(q) = q^{n-k} D_{n,n-k}(1/q).
\]

PROOF. For \( v \in D_n \), let
\[
\Delta(v) \equiv \begin{cases} -v, & \text{if } n \text{ is even,} \\ -\Gamma(v), & \text{if } n \text{ is odd}, \end{cases}
\]
where \( \Gamma: D_n \to B_n \setminus D_n \) has the same meaning as in the proof of Proposition 4.4. Then it is clear that \( \Delta: D_n \to D_n \) is a bijection and that \( N(v_2, \ldots, v(n)) + N(\Delta(v)_2, \ldots, \Delta(v)(n)) = n - 1 \). Furthermore, it follows easily from Corollary 4.2 that
\[
d_D(v) = d_D(\Gamma(v)),
\]
for all \( v \in B_n \). This implies that \( d_D(v) + d_D(\Delta(v)) = n \), and (58) follows from (52). \( \square \)
We now derive a recurrence relation satisfied by the polynomials \( D_n(x; q) \). In order to do this, however, we also need to consider the polynomials

\[
\tilde{D}_n(x; q) \overset{\text{def}}{=} \sum_{v \in B_n} x^{d_\sigma(v)} q^{N(v)},
\]

for \( n \in \mathbb{N} \). Note that

\[
\tilde{D}_n(x; q) = (1 + q) D_n(x; q),
\]

for \( n \geq 2 \).

We can now derive a recurrence relation for the polynomials \( D_n(x; q) \) which is the analogue of Theorem 3.6.

**Theorem 4.6.** The polynomials \( D_n(x; q) \) defined by (52) satisfy the recurrence relation

\[
D_n(x; q) = \sum_{i=1}^{n} \binom{n-1}{i-1} \tilde{D}_{i-1}(x; q)(1 + q)^{n-i} x B_{n-i}(x; 0)
\]

\[
+ (1 - x) D_{n-1}(x; q),
\]

for \( n \geq 3 \), with the initial conditions \( D_0(x; q) \overset{\text{def}}{=} 1 \), \( D_1(x; q) \overset{\text{def}}{=} (1 + x)(1 + qx) \).

**Proof.** Let \( \sigma \in B_{n-1} \) and \( \sigma_i \), \( \sigma_{i-1} \), for \( i \in [n] \), have the same meaning as in the proof of Proposition 3.3. Then it follows from Corollary 4.2 that

\[
d_P(\sigma_{i+1}) = d_P(\sigma_1), \ldots, \sigma(i - 1)) + d(\sigma(i), \ldots, \sigma(n - 1)) + 1,
\]

for \( i \in [n-1]\setminus\{2\} \) (where \( d_P(\emptyset) \overset{\text{def}}{=} 0 \)), and that \( d_P(\sigma_2) = d(\sigma(2), \ldots, \sigma(n - 1)) + 1 \), \( d_P(\sigma_{n-2}) = d(\sigma(2), \ldots, \sigma(n - 1)) + 2 \), \( d_P(\sigma_n) = d_P(\sigma) \), and \( d_P(\sigma_{n-1}) = d_P(\sigma) + 1 \).

Using this and Lemma 3.5 we obtain that

\[
\sum_{v \in B_n} x^{d_\sigma(v)} q^{N(v)} = \sum_{i=1}^{n} \sum_{\sigma \in B_{n-1}} (x^{d_\sigma(\sigma_1)} q^{N(\sigma_1)} + x^{d_\sigma(\sigma_{i-1})} q^{N(\sigma_{i-1})})
\]

\[
= (1 + q) \sum_{\sigma \in [n-1]\setminus\{2\}} \sum_{\sigma \in B_{n-1}} x^{d_\sigma(\sigma)} q^{N(\sigma)}
\]

\[
+ (1 + xq) \sum_{\sigma \in B_{n-1}} (x^{d_\sigma(\sigma_2)} q^{N(\sigma)} + x^{d_\sigma(\sigma)} q^{N(\sigma)})
\]

\[
= (1 + q) \sum_{i \in [n-1]\setminus\{2\}} \binom{n-1}{i-1} \sum_{\sigma \in B_{n-1}} \sum_{\tau \in B_{n-i}} x^{d_\sigma(\sigma) + d(\tau) + 1} q^{N(\sigma) + N(\tau)}
\]

\[
+ (1 + xq) \left( \tilde{D}_{n-1}(x; q) + \sum_{\sigma \in B_{n-1}} x^{d(\sigma_2), \ldots, \sigma(n-1) + 1} q^{N(\sigma)} \right)
\]

\[
= (1 + q) \sum_{i \in [n-1]\setminus\{2\}} \binom{n-1}{i-1} \tilde{D}_{i-1}(x; q)(1 + q)^{n-i} A_{n-i}(x)
\]

\[
+ (1 + xq) (\tilde{D}_{n-1}(x; q) + (n - 1)(1 + q)(1 + q)^{n-2} A_{n-2}(x)),
\]

and the thesis follows. \( \square \)

Note that, by Theorems 3.6 and 4.6, the polynomials \( B_n(x; q) \) and \( \tilde{D}_n(x; q) \) satisfy the same recurrence relation, but with different initial conditions. Despite this similarity, though, it seems to be difficult to derive a recurrence relation analogous to part (1) of Theorem 3.4 for the \( D_n(x; q) \)'s.
However, we can derive analogues of most of the results in Section 3 thanks to the following consequence of Corollary 4.2.

**Theorem 4.7.** For \( n \in \mathbb{N} \), we have that

\[
D_n(x; q) = \frac{1}{q(1-x)} [(q - x)B_n(x; q) - x(q - 1)(1 + q)^n B_n(x; 0)] - nx(1 + q)^{n-1} B_{n-1}(x; 0).
\]

\[(63)\]

**Proof.** By our definitions and Lemma 3.5 we know that

\[
B_n(x; q) = \sum_{\sigma \in B_n} x^{d_\sigma(q)} q^{N_\sigma},
\]

and

\[
(1 + q)^n B_n(x; 0) = \sum_{\sigma \in B_n} x^{d_\sigma(q)} q^{N_\sigma},
\]

for \( n \geq 2 \). Therefore, letting \( B_n^+ \overset{\text{def}}{=} \{ v \in B_n : v(1) > 0 \} \) and \( B_n^- \overset{\text{def}}{=} B_n \setminus B_n^+ \), we have, by Corollary 3.2, that

\[
B_n(x; q) - (1 + q)^n B_n(x; 0) = (x - 1) \sum_{\sigma \in B_n^-} x^{d_\sigma(q)} q^{N_\sigma},
\]

and

\[
B_n(x; q) - x(1 + q)^n B_n(x; 0) = (1 - x) \sum_{\sigma \in B_n^+} x^{d_\sigma(q)} q^{N_\sigma}.
\]

Hence we may rewrite (63) as

\[
D_n(x; q) = \frac{1}{q} \left[ x \sum_{v \in B_n^+} x^{d_v(q)} q^{N(v)} + q \sum_{v \in B_n^-} x^{d_v(q)} q^{N(v)} \right] - nx \sum_{\sigma \in B_{n-1}} x^{d_\sigma(q)} q^{N_\sigma},
\]

and therefore, using (52), as

\[
\sum_{v \in B_n} x^{d_v(q)} q^{N(v) + \ldots + N(n)} = \sum_{v \in D_n} x^{d_v(q)} q^{N(v) + \ldots + N(n)} + n \sum_{\sigma \in B_{n-1}} x^{d_\sigma(q) + 1} q^{N_\sigma}.
\]

\[(64)\]

We will prove the theorem by giving a bijection that will establish (64) combinatorially. More precisely, we will construct two bijections

\[
\Phi : \{ v \in B_n : v(1) > \text{sgn}(v(1))v(2) \} \to \bigcup_{j=1}^n B([n] \setminus \{ j \})
\]

\[(65)\]

and

\[
\Theta : \{ v \in B_n : v(1) < \text{sgn}(v(1))v(2) \} \to D_n,
\]

\[(66)\]

such that \( d(\Phi(v)) = d_B(v) - 1 \), \( N(\Phi(v)) = N(v(2), \ldots, v(n)) \), \( d_\rho(\Theta(v)) = d_B(v) \) and \( N(\Theta(v)(2), \ldots, \Theta(v)(n)) = N(v(2), \ldots, v(n)) \). We construct \( \Phi \) first. Given \( v \in B_n \) with \( v(1) > \text{sgn}(v(1))v(2) \), we let

\[
\Phi(v) \overset{\text{def}}{=} (v(2), \ldots, v(n)).
\]

Note that, since \( v(1) > \text{sgn}(v(1))v(2) \), either \( v(1) < 0 \) or \( v(1) > v(2) \) (but not both) and therefore

\[
d(\Phi(v)) = d(v(2), \ldots, v(n)) = d_B(v) - 1.
\]
Furthermore, if \( \sigma_1, \ldots, \sigma_{n-1} \in B([n] \setminus j) \) for some \( j \in [n] \), then we let 
\( \varphi(\sigma_1, \ldots, \sigma_{n-1}) \overset{\text{def}}{=} [\text{sgn}(j - \sigma_3), \sigma_1, \ldots, \sigma_{n-1}] \). It is then easy to see that \( \Phi \) and \( \varphi \) are inverses of each other, so that \( \Phi \) is a bijection.

To construct \( \Theta \) let \( v \in B_n \) be such that \( v(1) < \text{sgn}(v(1))v(2) \) and let

\[
\Theta(v) \overset{\text{def}}{=} \begin{cases} v, & \text{if } v \in D_n, \\ \{-v(1), v(2), \ldots, v(n)\}, & \text{otherwise}. \end{cases}
\]

(67)

Note that, since \( v(1) < \text{sgn}(v(1))v(2) \), \( d_\sigma(v) = d_\sigma(v) \) (for if \( v(1) > 0 \), then \( 0 < v(1) < v(2) \) and hence \( v(1) + v(2) > 0 \); while if \( v(1) < 0 \) then \( v(1) < v(2) \) and hence \( v(1) + v(2) < 0 \). Therefore, from (67) and Corollary 4.2 we conclude that \( d_\sigma(\Theta(v)) = d_\sigma(v) = d_\sigma(v) \). Now let, for \( u \in D_n \),

\[
\vartheta(u) \overset{\text{def}}{=} \begin{cases} u, & \text{if } u(1) < \text{sgn}(u(1))u(2), \\ \{-u(1), u(2), \ldots, u(n)\}, & \text{otherwise}. \end{cases}
\]

It is then clear that \( \Theta \) and \( \vartheta \) are inverses of each other, so that \( \Theta \) is a bijection, and this concludes the proof.

For \( q = 1 \) the preceding theorem yields the following simple relation, which was first observed by J. Stembridge (see [25], Lemma 9.1), and which also follows from Theorem 4.2 of [16].

**Corollary 4.8.** For \( n \in \mathbb{N} \) and \( 0 \leq k \leq n \), we have that

\[
B_{n,k}(1) = D_{n,k}(1) + n2^{n-1}A(n-1, k-1).
\]

(68)

**Corollary 4.9.** For \( n \in \mathbb{N} \), let \( D_n(x; q) \) be the polynomial defined by (52). Then

\[
\sum_{n=0} D_n(x; q) \frac{t^n}{n!} = \frac{(q-x)e^{t(1-x)} + x(1 - q - q^t + qx)e^{t+q(1-x)}}{q(1 - xe^{1+q(1-x)})}
\]

(69)

as formal power series in \( \mathbb{Z}[x, q][][[t]] \).

**Proof.** From (63), and part (iv) of Theorem 3.4, we obtain that

\[
\sum_{n=0} D_n(x; q) \frac{t^n}{n!} = \frac{(q-x)e^{t(1-x)} - xt\left(1-xe^{1+q(1-x)}\right)}{q(1 - xe^{1+q(1-x)})} + \frac{x(q-1)e^{t(1+q(1-x))}}{q(1 - xe^{(1+q)(1-x)})}
\]

and (69) follows.

Other consequences of Theorem 4.7 are the following two results which we state, for the sake of simplicity, only in the case \( q = 1 \). We denote by \( B_n(x) \) the \( n \)th Bernoulli polynomial and by \( B(n) \) the \( n \)th Bernoulli number, for \( n \in \mathbb{N}, (16, p. 48) \).

**Theorem 4.10.** For \( n \geq 2 \) we have that

\[
\sum_{i=0} [(2i+1)^n - 2^{n-1}(B_n(i+1) - B(n))]x^i = \frac{P(D_n; x)}{(1 - x)^{n+1}},
\]

(70)

as formal power series in \( \mathbb{Z}[x][][[x]] \).
PROOF. It follows from Theorems 4.7 and 3.4 that
\[
\frac{D_n(x; 1)}{(1 - x)^{n+1}} = \frac{B_n(x; 1)}{(1 - x)^{n+1}} - 2n^{-1} n B_{n-1}(x; 0)
\]
\[
= \sum_{i \geq 0} \left( (2i + 1)^n - 2n^{-1} n \sum_{j=0}^{n-1} j^{n-1} \right) x^i,
\]
and the thesis follows from well known properties of Bernoulli polynomials (see, e.g., [6, p. 154, Ex. 4]). \qed

COROLLARY 4.11. For \( n \geq 2 \) we have that
\[
\sum_{k=0}^{n} D_{n,k}(1) \binom{x + n - k}{n} = (2x + 1)^n - 2n^{-1}(B_n(x + 1) - B(n)).
\]

Note that the coefficient of \( x^i \) in the l.h.s. of (70) does not in general, have only real zeros as a polynomial in \( i \) (take, e.g., \( n = 4 \)). Therefore we cannot use Theorem 2.2 to deduce, from (70), the analogue of Corollary 3.7 for \( P(D_n; x) \).

We note the following interesting consequence of the preceding result and Theorem 3.4 which gives, essentially, the expansion of Bernoulli polynomials when expressed with respect to the basis \( \{ (x + n - 1)! \}_{i=0, \ldots, n} \).

COROLLARY 4.12. For \( n \geq 2 \) we have that
\[
2n^{-1}(B_n(x + 1) - B(n)) = \sum_{k=0}^{n} (B_{n,k}(1) - D_{n,k}(1)) \binom{x + n - k}{n}.
\]

We now give a combinatorial interpretation to the coefficients of \( P(D_n; x) \) when expressed in powers of \( (x - 1) \). Let \( \pi \) be a partial signed partition of \([n]\) and \( B \in \pi \). We say that \( B \) is special if \( P(B) = 1 \).

THEOREM 4.13. For \( n \geq 2 \) we have that
\[
P(D_n; x) = (x - 1)^n + \sum_{k=1}^{n} (k - 1)! S_D(n, k)(x - 1)^{n-k},
\]
where \( S_D(n, k) \) equals the number of partial signed partitions of \([n]\) into \( k \) blocks that have one rooted non-special block.

PROOF. From Theorems 4.7, 3.14 and (42) we obtain that
\[
D_n(x; 1) = B_n(x; 1) - 2n^{-1} n [(x - 1)B_{n-1}(x; 0) + B_{n-1}(x; 0)]
\]
\[
= \sum_{k=0}^{n} k! S_B(n, k; 1)(x - 1)^{n-k} - 2n^{-1} n \sum_{k=1}^{n} (k - 1)! S(n, k)(x - 1)^{n-k}
\]
\[
= (x - 1)^n + \sum_{k=1}^{n} (k - 1)! [kS_B(n, k; 1) - 2n^{-1} nS(n, k)](x - 1)^{n-k}.
\]  
(71)

Now let \( i \in [n] \), \( \pi = \{B_1, \ldots, B_k\} \), be a partition of \([n]\) into \( k \) blocks and \( S \subseteq [n] \setminus \{i\} \).
We may assume that \( i \in B_1 \). We can then construct a partial signed partition \( \pi' \) of \( \{B_1, \ldots, B_k\} \) of \( [n] \) into \( k \) blocks having one rooted special block by letting

\[
B_i' = (B_i \setminus S) \cup (- (B_i \cap S)),
\]

for \( i = 2, \ldots, k \),

\[
B_1' = \{i\} \cup (- (B_1 \cap S)),
\]

and then rooting \( B_1' \). It is easy to see that this construction is a bijection. This shows that there are \( n2^{n-1}S(n;k) \) partial signed partitions of \( [n] \) into \( k \) blocks having one rooted special block, and since \( kS_B(n;k;1) \) counts elements of \( B \cap (n;k) \) having one rooted block, the thesis follows from (71).

We illustrate the preceding theorem with an example. Suppose that \( n = 2 \): then there are eight partial signed partitions of \( [2] \) that have one rooted non-special block; namely, \(-1 - 1 - 2, 1 - 2, -1 - 2, 1 - 2, -1 - 2 \) and \(-1 - 2 \) (where we have distinguished the rooted block with a bar). Therefore

\[
P(D_2; x) = (x - 1)^2 + 4(x - 1) + 4.
\]

5. CONJECTURES AND OPEN PROBLEMS

There are several open problems which are suggested by the present work.

As noted in Section 1, the polynomials \( P(W; x) \) are actually Poincaré polynomials of certain (toric) projective varieties. Therefore, algebra-geometric proofs of all the results presented in this paper would be interesting.

From the enumerative point of view the main one is probably whether there is any \( q > 0 \) such that \( D_n(x; q) \) has only real zeros. In this respect, we feel that this is true at least for \( q = 1 \).

**Conjecture 5.1.** For \( n \in \mathbb{P} \), the polynomial \( P(D_n; x) \) has only real zeros. In particular, \( P(D_n; x) \) is log-concave.

The above conjecture has been verified for \( n \leq 23 \). It is possible that Conjecture 5.1 may be solved using Theorem 4.7 and a more precise knowledge of the location of the zeros of \( P(A_n; x) \) and \( P(B_n; x) \). More generally, we feel that the following statement is true.

**Conjecture 5.2.** For every finite Coxeter group \( W \), the polynomial \( P(W; x) \) has only real zeros. In particular, \( P(W; x) \) is log-concave.

By Corollary 3.7, Conjecture 5.2 holds if \( W \) is of type \( A_n \) and \( B_n \). Also, one can check, with the aid of a computer, that \( P(H_3; x), P(H_4; x), P(F_4; x), P(E_6; x), P(E_7; x) \) and \( P(E_8; x) \) all have only real zeros. Furthermore, it is easy to see that if \( (W_1, S_1) \) and \( (W_2, S_2) \) are (finite) Coxeter systems, then \( P(W_1 \times W_2, x) = P(W_1; x)P(W_2; x) \) (where
$W_1 \times W_2$ denotes the direct product of $W_1$ and $W_2$. Therefore, $P(W_1 \times W_2; x)$ has only real zeros (respectively, is log-concave) if both $P(W_1; x)$ and $P(W_2; x)$ have only real zeros (respectively, are log-concave). This shows that it is enough to study Conjecture 5.2 for irreducible (finite) Coxeter systems. Therefore, by the classification of such Coxeter systems (see, e.g., [11]), to settle either part of Conjecture 5.2 it is enough to decide if $P(D_n; x)$ has only real zeros or is log-concave for all $n \in \mathbb{N}$.

From a combinatorial point of view, we feel that the main open problems are the following.

**Problem 5.3.** Is there an analogue of part (i) of Theorem 3.4 for the polynomials $P(D_n; x)$? (Or, more generally, for the $D_n(x; q)$'s?)

An affirmative answer to the above question would probably yield a simple recurrence relation which could, in turn, be used to attack Conjecture 5.1 with ‘interlacing roots’ arguments such as those used, e.g., in [3]. It is conceivable that if a geometric proof of (18) can be found, for $q = 1$, then this would give some insight on what a corresponding recursion for the $D_{n,i}(1)'$s might look like, and thus on Problem 5.3.

**Problem 5.4.** Is there a ‘natural’ notion of exceedance, for $D_n$, which is equidistributed with $D$-descents in $D_n$?

By ‘natural’ we mean here that this notion of exceedance should coincide, when $\sigma \in S_n$, with the usual one and that the definition of whether $\sigma \in D_n$ has an exceedance at $i \in [n]$ should be independent of whether $\sigma$ has an exceedance at $j \in [n]$, for $j \neq i$.

Computer evidence suggests the following conjecture. For $n \in \mathcal{P}$ and $0 \leq i, k \leq n$, let

$$A_{n,k,i} \overset{\text{def}}{=} |\{\sigma \in B_i : N(\sigma) = k, d_\partial(\sigma) = i\}|,$$

so that $A_{n,k,i}(x) = \sum_{i=0}^{n} A_{n,k,i}x^i$, where $A_{n,k}(x)$ is defined by (24).

**Conjecture 5.5.** For $n \in \mathcal{P}$ and $1 \leq k \leq n - 1$, there exists a simplicial convex $n$-dimensional polytope $\mathcal{P}_{n,k}$ such that $\sum_{i=0}^{n} h_i(\mathcal{P}_{n,k})x^i = 1 + A_{n,k}(x) + A_{n,n-k}(x) + x^n$.

Note that, if the preceding conjecture is correct, then Theorem 3.14 yields an explicit combinatorial description of the $f$-vector of $\mathcal{P}_{n,k}$. It is well known that $h$-vectors of simplicial convex polytopes have been completely characterized. Using this characterization (see, e.g., [20]) and Theorem 3.11 we obtain that Conjecture 5.5 is equivalent to the existence of a standard graded $\mathcal{Q}$-algebra $R$ such that $H(R; 1) = A_{n,k,1} + A_{n,n-k,1}$, and $H(R; i) = A_{n,k,i} - A_{n,k,i-1} + A_{n,n-k,i} - A_{n,n-k,i-1}$ for $2 \leq i \leq \lfloor n/2 \rfloor$, (where $H(R; i)$ denotes the Hilbert function of $R$—see [18] for definitions). There is a numerical characterization of Hilbert functions of standard graded $\mathcal{Q}$-algebras, (see, e.g., [18, Thm 2.2]), which is essentially due to Macaulay. Using this characterization (and a computer), we have verified Conjecture 5.5 for $1 \leq k \leq n - 1 \leq 10$.

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