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Nonlinear Scattering Theory at Low Energy*

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We study the scattering theory of a conservative nonlinear one-parameter group of operators on a Hilbert space X relative to a group of linear unitary operators. Under certain hypotheses, the scattering operator carries a neighborhood of 0 in X into X. The theory is designed to apply to the semilinear Schrödinger and Klein–Gordon equations.

1. INTRODUCTION

In scattering theory one has a Hilbert space X, a "free" group of unitary operators $U_0(t) = \exp(itH_0)$ on X, and a "perturbed" evolution equation. We write the perturbed equation formally as

$$\frac{du}{dt} = iH_0 u + Pu, \tag{1}$$

where P is the perturbation operator. For instance, in quantum mechanics, $H_0 = -\Delta$ and P is the multiplication by a "potential" function $iV(x)$. One looks for conditions under which solutions u of (1) are related to free solutions u_+ and u_-, where $u_\pm(t) = \exp(itH_0) f_\pm$, by the asymptotic condition

$$\|u(t) - u_\pm(t)\|_X \to 0 \qquad \text{as} \quad t \to \pm\infty. \tag{2}$$

The scattering operator is defined as $S(f_-) = f_+$ and the two wave operators as $W_\pm(f_\pm) = u(0)$, if time $t = 0$ is used as the reference time.

In this paper we are concerned with the case where P is nonlinear. We find conditions on $U_0(t)$ and P so that S maps a whole neighborhood of 0 in X into X (Theorem 1). We show in Theorem 2 that S is one–one and continuous in this neighborhood. In Theorem 3 we show that W_+ and W_- map all of X into X. In Theorem 4 we solve the Cauchy problem with $u(0)$

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given small in X and show that f_+ and f_- exist; that is, we construct $W_\pm^{-1}$ locally.

Theorem 1 is applied in Section 7 to the nonlinear Klein–Gordon equation.

$$\text{(NLKG)} \qquad u_{tt} - \Delta u + m^2 u + h(u) = 0,$$

$m > 0$, $x \in R^n$, where

$$|h'(s)| \leqslant c\,|s|^{p-1}, \tag{3}$$

$$1 + 4/n \leqslant p \leqslant 1 + 4/(n-1), \tag{4}$$

and $X = H^1(R^n) \oplus L^2(R^n)$ is the usual Hilbert space of Cauchy data of finite energy. It is applied in Section 6 to the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad iu_t - \Delta u + h(|u|) \arg u = 0,$$

where h satisfies (3), $p = 1 + 4/n$, and $X = H^1(R^n)$. The critical power $1 + 4/n$ is related to the L^∞ decay rate for the free equation, which is the same for the Klein–Gordon and the Schrödinger equations. It appears in these theorems because of the fact that NLKG is conformally invariant and NLS is pseudo-conformally invariant only if $h(s) = cs^{1+4/n}$ [3, 18].

The theory presented in this paper was initiated by Segal in [13, 14]. A second version was formulated by me in [16] with later expositions given by me in [18] and by Reed in [10, 11]. In these versions $W_\pm$ and S are only defined on dense subsets. The present, third version depends (except in Section 5) on the discovery by Segal [15] and Strichartz [21] that arbitrary solutions of finite energy of the linear equations decay as $t \to \infty$ in a certain sense. It is also inspired by the ingenious way that Ginibre and Velo [3] make use of the conservation laws for NLS. I began this third version with the paper [19] where Theorems 3 and 7 were proved under assumptions slightly less than optimal. A special case of Theorem 1 was announced in [20].

In this paper we do not consider the more difficult question of whether S acts on large data, that is, away from a neighborhood of 0 in X.

In Section 5 we dispense with the requirement that the data (f_- or $u(0)$) be arbitrary within a neighborhood in X, with the advantage that a more general perturbation operator is allowed. In Theorem 5 we show under this more general assumption on P that the domain $D(S)$ of S includes a certain large set. In the applications $D(S)$ is dense in a neighborhood of 0 in X. Theorems 6 and 7 are similar analogues of Theorems 3 and 4, respectively. Theorems 5–7 are subsequently applied to NLKG and NLS where h satisfies (3) and

$$\gamma(n) < p < 1 + 4/(n-2) \qquad (< \infty \text{ if } n = 1 \text{ or } 2), \tag{5}$$

where $\gamma(n)$ is the positive root of the quadratic

$$\frac{n}{2}\frac{\gamma-1}{\gamma+1}=\frac{1}{\gamma}. \tag{6}$$

For example, if $n=3$ the interval (4) is $[2\frac{1}{3}, 3]$ while the interval (5) is $(2, 5)$. The condition (3) on h can be weakened to the condition

$$|h(s)|=O(|s|^p) \qquad \text{as} \quad s\to 0 \tag{7}$$

in case we have an a priori bound in $L^\infty(R^n)$ (Theorems 10 and 13). In Section 8 we apply Theorems 5–7 to the nonlinear wave equation (NLKG with $m=0$) and we apply similar ideas to the generalized Korteweg–de Vries equation (GKdV).

The critical power $\gamma(n)$ first appeared explicitly in [19]. John [6] proved that solutions of a nonlinear wave equation in three space dimensions blow up if $1<p<1+\sqrt{2}$ but exist for all time if the data are small and $p>1+\sqrt{2}$. Now $1+\sqrt{2}=\gamma(2)$. We conjecture that the critical power in John's theorem in n space dimensions is $\gamma(n-1)$. The shift of one in the dimension is due to the different rates of decay of the Klein–Gordon equation when $m>0$ and when $m=0$. Glassey [5] proved the blow up theorem in two space dimensions if $1<p<\gamma(1)$. The power $\gamma(n)$ has recently been rediscovered by Dong and Li [1] and by Kato [7] who each prove versions of Theorem 10. Klainerman [8] obtains results like Theorems 10 and 13–15 which permit much more general nonlinear terms which are not necessarily quasilinear but are required to have higher degree, namely, $n(\gamma-1)/(2\gamma)>1/(\gamma-1)$.

It is convenient to rewrite (1) in its integral form

$$(*_s, f) \qquad u(t)=U_0(t)f+\int_s^t U_0(t-\tau)\,Pu(\tau)\,d\tau.$$

Here u is the solution of the differential equation (1) with the initial values $u(s)=U_0(s)f$ at $t=s$. Formally letting $s\to+\infty$ or $s\to-\infty$, we get the integral equations which relate u to f_+ and f_-:

$$(*_{\pm\infty}, f_\pm) \qquad u(t)=U_0(t)f_\pm+\int_{\pm\infty}^t U_0(t-\tau)\,Pu(\tau)\,d\tau.$$

If Y is a Banach space and I is an interval of real numbers, we denote by $L^p(I, Y)$ the usual Lebesgue space of functions: $I\to Y$, by $C(I, Y)$ the space of strongly continuous functions, and by $B(I, Y)$ the space of bounded functions. If X and Y are Banach spaces continuously embedded in some other Banach space and if $X\cap Y$ is dense in X and in Y, then $X\cap Y$ and $X+Y$ are Banach spaces.

2. The Scattering Operator at Low Energy

We make the following hypotheses.

(I) Let X be a Hilbert space with norm $| \ |_2$. Let $U_0(t)$ be a one-parameter group of unitary operators on X.

(II) Let X_1 and X_3 be Banach spaces with norms denoted by $| \ |_1$ and $| \ |_3$, respectively. Let P be an operator which takes a neighborhood of 0 in X_3 into X_1. Assume that $P0 = 0$ and

$$|Pf - Pg|_1 \leqslant c(|f|_3 + |g|_3)^{p-1} |f - g|_3,$$

where p and c are constants, $p > 1$.

(III) Let X_1, X and X_3 be continuously embedded in some Banach space X_4. Let $X \cap X_3$ be dense in both X and X_3. We define $Z = L^{p+1}(R, X_3) \cap B(R, X_3)$.

(IV) For each $f \in X$ assume that the function $t \to U_0(t)f$ belongs to $L^{p+1}(R, X_3)$. Let X be continuously embedded in X_3.

(V) Let $U_0(t)$, restricted to $X \cap X_1$, have a continuous linar extension (still denoted $U_0(t)$) which maps X_1 into X_3 with norm $\leqslant c\,|t|^{-d}$ for $t \neq 0$. We assume c is a constant and

$$d = 2/(p+1). \tag{8}$$

Let $U_0(t)$ (restricted to $X \cap X_3$) also have a continuous extension from X_3 to X_4 such that $U_0(t)\, U_0(s) f = U_0(t+s) f$ for all $f \in X_1$.

(VI) Let G be a functional which maps a neighborhood of 0 in X_3 into R. Assume G is lower semicontinuous and is continuous at 0.

(VII) Whenever I is a time interval, $s \in I, f \in X$, and $u \in Z$, with $\|u\|_Z$ sufficiently small, satisfies the equation $(*_s, f)$ in I, then $u \in C(I, X)$ and

$$\tfrac{1}{2}|u(t)|_2^2 + G(u(t)) \text{ is independent of } t. \tag{9}$$

Theorem 1. *There exists $\delta > 0$ with the following property. If $f_- \in X$, $|f_-|_2 < \delta$, then there exists a unique solution of the integral equation*

$$(*_{-\infty}, f_-) \qquad u(t) = U_0(t) f_- + \int_{-\infty}^{t} U_0(t-\tau)\, Pu(\tau)\, d\tau$$

for $t \in R$ such that

$$u \in C(R, X) \cap L^{p+1}(R, X_3) \tag{10}$$

and

$$|u(t) - U_0(t)f_-|_2 \to 0 \quad \text{as} \quad t \to -\infty. \tag{11}$$

Furthermore, there exists a unique $f_+ \in X$ *such that*

$$|u(t) - U_0(t)f_+|_2 \to 0 \quad \text{as} \quad t \to +\infty. \tag{12}$$

In addition,

$$\tfrac{1}{2}|u(t)|_2^2 + G(u(t)) = \tfrac{1}{2}|f_-|_2^2 = \tfrac{1}{2}|f_+|_2^2 \tag{13}$$

and

$$(*_{+\infty}, f_+) \qquad u(t) = U_0(t)f_+ - \int_t^{+\infty} U_0(t-\tau)\, Pu(\tau)\, d\tau.$$

Remark 1. We do not assume that P maps X into X. Instead we assume it maps another space X_3 into X_1 while $U_0(t)$ maps X_1 back into X_3. In fact, (II) and (V) will always be used together in the form

$$|U_0(t)(Pf - Pg)|_3 \leqslant c\,|t|^{-d}(|f|_3 + |g|_3)^{p-1}|f-g|_3$$

for f, $g \in X_3$ and $t \neq 0$. The intermediate space X_1 is irrelevant in principle. The original Hilbert space X is brought back via hypothesis (VII), which is a kind of regularity statement together with a modified conservation of norm condition.

Remark 2. If P were a locally Lipschitz map from X to X, then the regularity statement in (VII) would be automatic. Indeed, we could then solve $(*_s, f)$ locally in t by the standard iteration method [12]. If we assumed (9) were true for this local solution, then $|u(t)|_2$ would be bounded so that u would extend to a global solution continuous with values in X.

3. Proof of Theorem 1

Lemma 1. *Assume* (II), (III), *and* (V). *There exists* $\delta_1 > 0$ *with the following property. Let* $-\infty \leqslant s \leqslant +\infty$. *Let* $U_0(\cdot)f \in Z$ *with* $\|U_0(\cdot)f\|_Z < \delta_1/2$. *Then there exists a unique solution* v *of equation* $(*_s, f)$ *which satisfies* $v \in Z$ *and* $\|v\|_Z \leqslant 2\,\|U_0(\cdot)f\|_Z$. *The equation* $(*_r, U_0(-r)\,v(r))$ *is satisfied for all* $r \in R$.

Proof. This is simply a consequence of the contraction principle in the space Z. We denote

$$\mathscr{P}v(t) = \int_s^t U_0(t-\tau)Pv(\tau)\, d\tau.$$

By (V) and (II),

$$|\mathscr{P}v(t)|_3 \leqslant c \int_s^t |t-\tau|^{-d} |v(\tau)|_3^p \, d\tau \tag{14}$$

provided $v(\tau)$ belongs to the neighborhood of 0 in X_3 which is referred to in (II). By the singular integral inequality,

$$\int_{-\infty}^{\infty} |\mathscr{P}v(t)|_3^{p+1} \, d\tau \leqslant c \left(\int_{-\infty}^{\infty} |v(\tau)|_3^{p+1} \, d\tau \right)^p$$

since

$$1 + \frac{1}{p+1} = d + \frac{p}{p+1}.$$

We next let $I_1 = [s, t] \cap [t-1, t+1]$ and $I_2 = [s, t] \backslash [t-1, t+1]$. We break the integral defining $\mathscr{P}v$ up into two parts. In the part over the interval I_2 we again use $(p+1)d = 2$ and Hölder's inequality:

$$\begin{aligned} |\mathscr{P}v(t)|_3 &\leqslant c \int_{I_1} |t-\tau|^{-d} \, d\tau \sup_{I_1} |v(\tau)|_3^p \\ &\quad + c \left(\int_{I_2} (t-\tau)^{-2} \, d\tau \right)^{1/(p+1)} \left(\int_{I_2} |v(\tau)|_3^{p+1} \, d\tau \right)^{p/(p+1)} \\ &\leqslant c \, \|v\|_Z^p. \end{aligned} \tag{15}$$

Let $Z(\delta_1) = \{v \in Z : \|v\|_Z \leqslant \delta_1\}$. Thus $\|\mathscr{P}v\|_Z \leqslant c \, \|v\|_Z^p$ for all $v \in Z(\delta_1)$ if δ_1 is sufficiently small. In exactly the same manner we show that

$$\|\mathscr{P}u - \mathscr{P}v\|_Z \leqslant c(\|u\|_Z + \|v\|_Z)^{p-1} \|u - v\|_Z \tag{16}$$

for such u and v. We choose δ_1 so small that $c(2\delta_1)^{p-1} < 1/2$. Then $\mathscr{P}$ is a contraction mapping on $Z(\delta_1)$. Now let f be given and let $y(t) = U_0(t)f$ and $\|y\|_Z \leqslant \delta_1/2$. Then the mapping $v \to y + \mathscr{P}v$ carries $Z(\delta_1)$ into itself and so it has a unique fixed point in $Z(\delta_1)$:

$$v = y + \mathscr{P}v, \qquad v \in Z(\delta_1).$$

It satisfies $\|v\|_Z \leqslant \|y\|_Z + c\delta^{p-1} \|v\|_Z \leqslant \|y\|_Z + \frac{1}{2}\|v\|_Z$, hence $\|v\|_Z \leqslant 2 \, \|y\|_Z$.

Now each term in the equation $(*_s, f)$ belongs to X_3. We let $r \in R$ and we write $(*_s, f)$ with $t = r$. We apply $U_0(t-r)$ to both sides to obtain

$$U_0(t-r) \, v(r) = U_0(t-r) \left[U_0(r)f + \int_s^r U_0(r-\tau) \, Pv(\tau) \, d\tau \right].$$

The integral converges in the space X_3. Since the linear operator $U_0(t-r)$ is continuous from X_3 to X_4, we may write

$$U_0(t)\, U_0(-r)\, v(r) = U_0(t) f + \int_s^r U_0(t-\tau)\, Pv(\tau)\, d\tau.$$

Each term in this equation belongs to X_4. If this equation is subtracted from $(*_s, f)$, we obtain

$$v(t) = U_0(t)[U_0(-r)\, v(r)] + \int_r^t U_0(t-\tau) Pv(\tau)\, d\tau$$

which is equation $(*_r, U_0(-r)\, v(r))$.

LEMMA 2. *Assume* (II), (III), *and* (V). *Let* $U_0(\cdot) f_- \in Z$ *be given such that* $|U_0(t) f_-|_3 \to 0$ *as* $t \to -\infty$ *and* $\| U_0(\cdot) f_- \|_Z \leqslant \delta_1/2$. *Let* u *be a solution in* Z *of* $(*_{-\infty}, f_-)$ *and let* v_s *be a solution in* Z *of* $(*_s, f_-)$, *where* $-\infty < s$. *Then*

$$|u(t)|_3 \to 0 \qquad as \quad t \to -\infty \tag{17}$$

$$\int_{-\infty}^{\infty} |v_s(t) - u(t)|_3^{p+1}\, dt \to 0 \qquad as \quad s \to -\infty. \tag{18}$$

Proof. We have $u = u_- + \mathscr{P} u$. As in (15) we have

$$|\mathscr{P} u(t)|_3 \leqslant c \left(\int_{-\infty}^{t-1} |u(\tau)|_3^{p+1}\, d\tau \right)^{p/(p+1)}$$
$$+ c \left(\int_{t-1}^{t} |u(\tau)|_3^{p(1+\epsilon)/\epsilon}\, d\tau \right)^{\epsilon/(1+\epsilon)},$$

where c is independent of t and ε is fixed so that $d(1+\varepsilon) < 1$. Since $u \in L^{p+1}(R, X_3) \cap L^{p(1+\epsilon)/\epsilon}(R, X_3)$, we have $|\mathscr{P} u(t)|_3 \to 0$ as $t \to -\infty$. This proves (17).

Now we subtract the equations satisfied by v_s and u:

$$v_s(t) - u(t) = \int_s^t U_0(t-\tau)[Pv_s(\tau) - Pu(\tau)]\, d\tau$$
$$- \int_{-\infty}^{s} U_0(t-\tau)\, Pu(\tau)\, d\tau.$$

Hence

$$|v_s(t)-u(t)|_3$$
$$\leqslant c\int_s^t |t-\tau|^{-d}(|v_s(\tau)|_3+|u(\tau)|_3)^{p-1}|v_s(\tau)-u(\tau)|_3\,d\tau$$
$$+\ c\int_{-\infty}^s |t-\tau|^{-d}|u(\tau)|_3^p\,d\tau$$

whence

$$\|v_s-u\|_B\leqslant c\,(\|v_s\|_B+\|u\|_B)^{p-1}\|v_s-u\|_B$$
$$+c\left(\int_{-\infty}^s |u(\tau)|_3^{p+1}\,d\tau\right)^{p/(p+1)},$$

where $B=L^{p+1}(R,X_3)$. Since

$$c(\|v_s\|_B+\|u\|_B)^{p-1}\leqslant c(2\delta_1)^{p-1}<\tfrac{1}{2},$$

it follows that

$$\|v_s-u\|_B\leqslant 2c\left(\int_{-\infty}^s |u(\tau)|_3^{p+1}\,d\tau\right)^{p/(p+1)}\to 0 \qquad \text{as} \quad s\to-\infty.$$

LEMMA 3. *Assume* (I)–(III) *and* (V)–(VII). *Given* f_- *and* u *as in Lemma* 2. *If* $f_-\in X$, *then* $u\in C(R,X)$ *and*

$$\tfrac{1}{2}|u(t)|_2^2+G(u(t))\leqslant\tfrac{1}{2}|f_-|_2^2. \tag{19}$$

Proof. By (VII), $v_s\in C(R,X)$ and

$$|v_s(t)|_2^2+2G(v_s(t))=|U_0(s)f_-|_2^2+2G(U_0(s)f_-). \tag{20}$$

By (I) the first term on the right equals $|f_-|_2^2$. By (VI), $G(U_0(s)f_-)\to 0$ as $s\to-\infty$. We assume of course that $\{h\in X_3:|h|_3\leqslant\delta_1\}$ is contained in the domain of G and that G is bounded there. From (18) it follows that there exists a sequence $s_j\to-\infty$ suh that $|v_{s_j}(t)-u(t)|_3\to 0$ for almost every t. We fix any such t. By (VI),

$$G(u(t))\leqslant\liminf G(v_{s_j}(t)).$$

From (20) it follows that $|v_{s_j}(t)|_2$ is bounded. Each weak accumulation point in X of $v_{s_j}(t)$ must be equal to $u(t)$. Hence $u(t)\in X$ and $v_s(t)\to u(t)$ weakly in X. Hence

$$|u(t)|_2^2\leqslant\liminf|v_{s_j}(t)|_2^2.$$

Therefore (20) implies (19). By (VII) and the last part of Lemma 1, $u \in C(R, X)$ and (9) is valid.

LEMMA 4. *Under the same assumptions as Lemma* 3,

$$|u(t) - U_0(t)f_-|_2 \to 0 \qquad as \quad t \to -\infty \tag{11}$$

and

$$\tfrac{1}{2}|u(t)|_2^2 + G(u(t)) = \tfrac{1}{2}|f_-|_2^2. \tag{21}$$

Proof. As in the proof of Lemma 1, we may apply $U_0(-t)$ to both sides of equation $(*_{-\infty}, f_-)$ to obtain

$$U_0(-t)u(t) - f_- = \int_{-\infty}^{t} U_0(-\tau)\, Pu(\tau)\, d\tau. \tag{22}$$

Hence

$$\begin{aligned}
&|U_0(-t)\, u(t) - f_-|_3 \\
&\leqslant \int_{-\infty}^{t} c\,|\tau|^{-d}\, |u(\tau)|_3^p\, d\tau \\
&\leqslant \left(\int_{-\infty}^{t} c\tau^{-2}\, d\tau\right)^{1/(p+1)} \left(\int_{-\infty}^{t} |u(\tau)|_3^{p+1}\, d\tau\right)^{p/(p+1)}
\end{aligned}$$

tends to zero as $t \to -\infty$. On the other hand we know from (19) that $|U_0(-t)\, u(t)|_2 = |u(t)|_2$ is bounded. Hence $U_0(-t)\, u(t)$ converges weakly in X to f_- as $t \to -\infty$. By (17) and (VI), $G(u(t)) \to 0$ as $t \to -\infty$. Hence

$$\begin{aligned}
|f_-|_2^2 &\leqslant \liminf |U_0(-t)\, u(t)|_2^2 = \liminf |u(t)|_2^2 \\
&= \liminf |u(t)|_2^2 + 2G(u(t)) \leqslant |f_-|_2^2
\end{aligned}$$

by Lemma 3. It follows that (21) holds and that the weak limit is strong. Therefore

$$|u(t) - U_0(t)f_-|_2 = |U_0(-t)\, u(t) - f_-|_2 \to 0.$$

LEMMA 5. *Under the same assumptions, define*

$$f_+ = f_- + \int_{-\infty}^{\infty} U_0(-\tau) Pu(\tau)\, d\tau. \tag{23}$$

Then $f_+ \in X$. *If* $|U_0(t)f_+|_3 \to 0$ *as* $t \to +\infty$, *then* $|f_+|_2 \leqslant |f_-|_2$ *and* $|u(t)|_3 \to 0$ *as* $t \to +\infty$.

Proof. We first show that the integral converges in X_3. For any t let $I_1 = [t-1, t+1]$ and $I_2 = (-\infty, t-1) \cup (t+1, \infty)$. Exactly as in the proof of Lemma 1,

$$\int_{-\infty}^{\infty} |U_0(t-\tau)Pu(\tau)|_3 \, d\tau \leqslant c \, \|u\|_Z^p.$$

When $t=0$ this shows that the integral in (23) converges in X_3. Let us write $u_\pm(t) = U_0(t)f_\pm$. We apply the operator $U_0(t)$ to (23) so that

$$u_+(t) = u_-(t) + \int_{-\infty}^{\infty} U_0(t-\tau)\, Pu(\tau)\, d\tau. \tag{24}$$

As in Lemma 1, this implies that

$$\|u_+\|_Z \leqslant \|u_-\|_Z + c\, \|u\|_Z^p. \tag{25}$$

From $(*_{-\infty}, f_-)$ and (24) we obtain

$$u(t) = u_+(t) - \int_t^{\infty} U_0(t-\tau)\, Pu(\tau)\, d\tau. \tag{26}$$

From this equation or from (22) and (23), we have

$$f_+ - U_0(-t)\, u(t) = \int_t^{\infty} U_0(-\tau)\, Pu(\tau)\, d\tau.$$

Hence

$$|f_+ - U_0(-t)u(t)|_3 \leqslant c \left(\int_t^{\infty} \tau^{-2}\, d\tau \right)^{1/(p+1)} \|u\|_Z^p$$

tends to zero as $t \to +\infty$. On the other hand, $|U_0(-t)\, u(t)|_2 = |u(t)|_2$ is bounded. So $f_+ \in X$ and

$$U_0(-t)\, u(t) \to f_+ \text{ weakly in } X \text{ as } t \to +\infty. \tag{27}$$

If $|U_0(t)f_+|_3 \to 0$, then it follows from (26) that $|u(t)|_3 \to 0$ as $t \to +\infty$, exactly as in the proof of (17). By (VI), $G(u(t)) \to 0$. From (27),

$$\begin{aligned} |f_+|_2 &\leqslant \liminf |U_0(-t)u(t)|_2^2 = \liminf |u(t)|_2^2 \\ &= |u(t)|_2^2 + 2G(u(t)) = |f_-|_2^2. \end{aligned}$$

LEMMA 6. *$|f_+|_2 = |f_-|_2$ and* (12) *holds.*

Proof. We assume $\|u_-\|_Z \leqslant \delta_1/4$. By (25) and Lemma 1, $\|u_+\|_Z \leqslant \delta_1/2$. If $s < +\infty$, we let w_s be the solution of equation $(*_s, f_+)$ given by Lemma 1.

Exactly as in Lemma 2 (but with $+\infty, f_+, w_s$ playing the roles of $-\infty, f_-, v_s$) we have

$$\int_{-\infty}^{\infty} |w_s(t) - u(t)|_3^{p+1}\, dt \to 0 \qquad \text{as} \quad s \to +\infty.$$

We choose a sequence $s_j \to +\infty$ so that $|w_{s_j}(t) - u(t)|_3 \to 0$ almost everywhere. We fix any such t. By (VII), $w_s \in C(R, X)$ and

$$|w_s(t)|_2^2 + 2G(w_s(t)) = |U_0(s)f_+|_2^2 + 2G(U_0(s)f_+). \tag{28}$$

By (I) and (VI), the right side of this identity tends to $|f_+|_2^2$ as $s \to +\infty$. So $w_s(t)$ is bounded in X as $s \to +\infty$. Hence $w_{s_j}(t) \to u(t)$ weakly in X, and

$$|u(t)|_2^2 \leqslant \liminf |w_{s_j}(t)|_2^2.$$

By (VI)

$$G(u(t)) \leqslant \liminf G(w_{s_j}(t).$$

Therefore (28) implies that

$$|u(t)|_2^2 + 2G(u(t)) \leqslant |f_+|_2^2.$$

By Lemmas 4 and 5, $|f_-|_2 = |f_+|_2$. So the weak convergence in (27) is strong and

$$|u(t) - U_0(t)f_+|_2 = |U_0(-t)u(t) - f_+|_2 \to 0.$$

LEMMA 7. *Assume* (I), (III), *and* (IV). *Let f be an arbitrary element of X. Then $U_0(\cdot)f \in Z$,*

$$\|U_0(\cdot)f\|_Z \leqslant k\,|f|_2 \tag{29}$$

and

$$|U_0(t)f|_3 \to 0 \qquad \text{as} \quad t \to \pm\infty. \tag{30}$$

Proof. We have $c\,|U_0(t)f|_3 \leqslant |U_0(t)f|_2 = |f|_2$. Hence $U_0(\cdot)f \in Z$. By the closed graph theorem, (29) is true. Now

$$|U_0(t)f - U_0(r)f|_2 = |U_0(t-r)f - f|_2 \to 0$$

as $t - r \to 0$. So $U_0(\cdot)f$ is uniformly continuous with values in X_3. Since it belongs to $L^{p+1}(R, X_3)$, it follows that (30) is true.

Now assume (I)–(VII). Let f_- be an arbitrary element of X with $|f_-|_2 < \delta_1/(4k)$. By Lemma 7, $\|U_0(\cdot)f_-\| < \delta_1/4$ and the hypotheses of Lemmas 1–6 are satisfied. This completes the proof of Theorem 1.

4. The Operators S and $W_{\pm}$

THEOREM 2. *Let* $X(\delta)=\{f\in X:|f|_2<\delta\}$. *The (nonlinear) operator* $S:f_-\to f_+$ *which is defined in Theorem* 1 *carries* $X(\delta)$ *onto* $X(\delta)$. *It is a homeomorphism.*

Proof of Theorem 2. Let $f_+^{(1)}=f_+^{(2)}$. For $j=1,2$, the equation $(*_{+\infty}, f_+^{(j)})$ has a unique solution among the elements of Z of small norm. Thus $u^{(1)}=u^{(2)}$. By (23), $f_-^{(1)}=f_-^{(2)}$. This proves that S is one–one.

Now let $f_-^{(n)}\in X(\delta)$ such that $|f_-^{(n)}-f_-|_2\to 0$ as $n\to\infty$. By (IV), $\|u_-^{(n)}-u_-\|_Z\to 0$, where $u_-^{(n)}(t)=U_0(t)f_-^{(n)}$. The solution $u^{(n)}$ of the equation $(*_{-\infty}, f_-^{(n)})$ also tends to u strongly in Z. Indeed $u^{(n)}=u_-^{(n)}+\mathscr{P}u^{(n)}$ so that

$$\|u^{(n)}-u\|_Z\leqslant\|u_-^{(n)}-u_-\|_Z+\tfrac{1}{2}\|u^{(n)}-u\|_Z$$

since $\mathscr{P}$ is a contraction; this tends to zero. By (23) $|f_+^{(n)}-f_+|_3\to 0$. Now

$$|f_+^{(n)}|_2=|f_-^{(n)}|_2\to|f_-|_2.$$

Hence

$$f_+^{(n)}\to f_+ \text{ weakly in } X \text{ and}$$
$$|f_+|_2\leqslant\liminf_{n\to\infty}|f_+^{(n)}|_2=|f_-|_2=|f_+|_2.$$

So $f_+^{(n)}\to f_+$ strongly in X. This proves the continuity of S.

The next result says that the wave operators $W_-(T): u_-(T)\to u(T)$ map *all* of X into X. It was proved in [19] in slightly less generality and with a small error in the proof.

THEOREM 3. *Assume* (I)–(VII). *If* $f_-\in X$, *there exists a time* $T>-\infty$ *and a unique solution* u *of the integral equation* $(*_{-\infty}, f_-)$ *in the time interval* $I=(-\infty, T]$ *such that*

$$u\in C(I,X)\cap L^{p+1}(I,X_3)$$

and (11) *and* (21) *hold.*

Proof. We iterate in the space $Z=L^{p+1}(I,X_3)\cap B(I,X_3)$. We are given $u_-(t)=U_0(t)f_-$. By Lemma 7, u_- belongs to this space. We choose T so large negative that

$$\|u\|_Z=\left(\int_{-\infty}^{T}|u_-(t)|_3^{p+1}\,dt\right)^{1/p+1}+\sup_{(-\infty,T]}|u_-(t)|_3 \tag{31}$$

is less than $\delta_1/2$. Theorem3 follows immediately from Lemmas 1–4.

Now we consider the ordinary initial-value problem with initial data small in X but otherwise arbitrary.

THEOREM 4. *Assume* (I)–(VII). *If* $f_0 \in X$, $|f_0|_2 < \delta$, *then there exists a unique solution u of equation* $(*_0, f_0)$ *for* $t \in R$ *satisfying* (9). *Furthermore there exist unique* f_+ *and* $f_- \in X$ *with the same properties as in Theorem* 1. *The mappings* $f_0 \to f_\pm$ *are one–one and continuous from* $X(\delta)$ *into* $X(2\delta)$.

Proof. By Lemma 1, u exists. Define

$$f_\pm = f_0 + \int_0^{\pm\infty} U_0(-\tau)\, Pu(\tau)\, d\tau. \tag{32}$$

The proof continues exactly as in Lemmas 5 and 6 and Theorem 2.

5. A DENSE CLASS OF SOLUTIONS

We introduce the norm

$$\|v\|_V = \sup_t\, (1 + |t|)^d\, |v(t)|_3 \tag{33}$$

and the space $V = \{v \in C(R, X_3) : \|v\|_V < \infty\}$.

THEOREM 5. *Assume* (I)–(III) *and* (V)–(VII) *but allow*

$$p^{-1} < d < 1 \tag{34}$$

instead of (8). *In* (VII) *replace* Z *by* V. *Assume that* $f_- \in X$ *and* $U_0(\cdot)f_- \in V$. *There exists* $\delta < 0$ *such that if* $\|U_0(\cdot)f_-\|_V < \delta$, *then all the conclusions of Theorem* 1 *are valid and* $u \in V$.

Remark. Note that $p^{-1} < 2(p+1)^{-1} < 1$ for $p > 1$. Hence the assumption on d is weaker than in Theorem 1. However the assumption on f_- is stronger. Note also that $V \subset Z$ since $d(p+1) > dp > 1$.

Proof. We first prove the analogue of Lemma 1 in the space V. We have

$$|\mathscr{P}v(t)|_3 \leqslant c \int_s^t |t-\tau|^{-d}\, |v(\tau)|_3^p\, d\tau$$

$$\leqslant c\, \|v\|_V^p \int_{-\infty}^{\infty} |t-\tau|^{-d} (1 + |\tau|)^{-dp}\, d\tau.$$

Since $d < 1$, the integral is bounded. We break up the integral into the part with $|t-\tau| < |t|/2$ and its complement. Since $dp > 1$, the integral is $O(|t|^{-d})$

as $|t| \to \infty$. Therefore $\|\mathscr{P}v\|_V \leqslant c\,\|v\|_V^p$. Similarly we prove the analogue of (16) for the space V. The rest of Lemma 1 is identical, except for using the space V.

As an analogue of (18) we claim that

$$\|v_s - u\|_V \to 0 \qquad \text{as} \quad s \to -\infty$$

(with v_s and u defined as in Lemma 2). We now have

$$\begin{aligned} &|v_s(t) - u(t)|_3 \\ &\quad \leqslant c \int_s^t |t-\tau|^{-d}(1+|\tau|)^{-dp}\, d\tau (\|v_s\|_V + \|u\|_V)^{p-1}\, \|v_s - u\|_V \\ &\qquad + c \int_{-\infty}^s |t-\tau|^{-d}(1+|\tau|)^{-dp}\, d\tau\, \|u\|_V^p. \end{aligned}$$

These integrals are $\leqslant c(1+|t|)^{-d}$. The second integral is less than $\omega(s)(1+|t|)^{-d}$ where $\omega(s) \to 0$ as $s \to -\infty$. This can be seen by breaking up the integral as above. Therefore $\|v_s - u\|_V \leqslant \omega(s)$.

Lemma 3 is unchanged. In Lemmas 4 and 5 we estimate

$$|U_0(-t)\, u(t) - f_\pm|_3 \leqslant c\, \|u\|_V^p \int_{\pm\infty}^t |\tau|^{-d}(1+|\tau|)^{-dp}\, |d\tau|,$$

which tends to zero as $t \to \pm\infty$. We assume the space V in place of Z. In Lemma 6 we prove $\|w_s - u\|_V \to 0$ as $s \to +\infty$. Otherwise there are no changes from the proof of Theorem 1.

THEOREM 6 (The Wave Operator). *Under the same assumptions as in Theorem* 5, *but without assuming the norm of* $U_0(\,\cdot\,)f_-$ *is small, there exists a time* $T > -\infty$ *and a unique solution* u *of* $(*_{-\infty}, f_-)$ *in* $I = (-\infty, T]$ *such that* $u \in C(I, X \cap X_3)$,

$$\sup_{-\infty < t \leqslant T} (1+|t|)^d |u(t)|_3 < \infty$$

and (11) *and* (21) *hold.*

Proof. As in Theorem 5 we prove

$$\|\mathscr{P}u - \mathscr{P}v\|_V \leqslant \varepsilon(T)(\|u\|_V + \|v\|_V)^{p-1}\|u - v\|_V$$

where we take the norms only over I and

$$\varepsilon(T) = \sup_{t \leqslant T} (1+|t|)^{-d} \int_{-\infty}^T |t-\tau|^{-d}(1+|\tau|)^{-dp}\, d\tau$$

tends to zero as $T \to -\infty$. We let $k = 2 \| U_0(\cdot) f_- \|_V$ and choose T so that $\varepsilon(T)(2k)^{p-1} < 1/2$. Then $\mathscr{S}$ is a contraction map on $\{v \in V \mid \|v\|_V \leqslant k\}$. The proof continues exactly as in Theorem 5 (up to Lemma 4).

THEOREM 7 (The Initial-Value Problem). *Assume* (I)–(III) *and* (V)–(VII) *but allow* $p^{-1} < d < 1$. *In* (VII) *replace* Z *by* V. *Let* $f_0 \in X$ *and* $U_0(\cdot) f_0 \in V$. *There exists* $\delta > 0$ *such that if* $\| U_0(\cdot) f_0 \|_V < \delta$, *then there exists a unique solution* u *of equation* $(*_0, f_0)$, $u \in C(R, X) \cap V$. *There exist unique* f_+ *and* f_- *in* X *with the same properties as in Theorem* 1.

Proof. By the analogue of Lemma 1 in the space V, u exists. Define f_+ and f_- by (32). The proof continues just as in Lemmas 5 and 6.

6. The Nonlinear Schrödinger Equation

By $W^{k,p}(R^n)$ we denote the usual Sobolev space where k is the number of derivatives. The norm is denoted $\| \ \|_{k,p}$. Consider the ordinary Schrödinger equation

$$\text{(LS)} \qquad i \frac{\partial u}{\partial t} - \Delta u = 0$$

and the nonlinear Schrödinger equation NLS with the nonlinear term $h(|u|) \arg u$.

We assume that h is a real function and that $h(0) = 0$. Let H be the primitive of h which vanishes at 0. The invariants for NLS are

$$\int |u(x,t)|^2 \, dx \qquad \text{and} \qquad \int \left[\tfrac{1}{2} |\nabla u(x,t)|^2 + H(|u(x,t)|)\right] dx.$$

We call a solution of NLS for which both invariants are finite a finite-energy solution.

THEOREM 8. *Assume that* h *is of class* C^1 *and* $|h'(s)| \leqslant c\,|s|^{4/n}$ *for all* s.

(a) *If* u_- *is* any *solution of LS of finite energy, then there exists a unique solution* u *of NLS in some time interval such that*

u is continuous with values in $W^{1,2}(R^n)$,

$$\iint_{-\infty}^{T} |u(x,t)|^{2+4/n} \, dx \, dt < \infty,$$

$$\|u(t) - u_-(t)\|_{1,2} \to 0 \qquad \textit{as} \quad t \to -\infty.$$

(b) *If the energy of u_- is sufficiently small, then u exists globally, $|u|^{2+4/n}$ is integrable over all space-time, and there exists a unique solution u_+ of LS such that*

$$\|u(t) - u_+(t)\|_{1,2} \to 0 \qquad \textit{as} \quad t \to +\infty.$$

For all t,

$$\int u^2\,dx = \int u_-^2\,dx = \int u_+^2\,dx,$$

$$\int [|\nabla u|^2 + 2H(|u|)]\,dx = \int |\nabla u_-|^2\,dx = \int |\nabla u_+|^2\,dx.$$

(c) *If u_0 is a solution of LS of sufficiently small energy, then there exist unique u, u_+, and u_- with the same properties such that $u(0) = u_0(0)$.*

Proof. We choose $X = W^{1,2}$, $X_3 = L^{p+1}$, $X_1 = L^{1+1/p}$, $p = 1 + 4/n$, and $U_0(t)$ is the evolution operator for LS. We verify hypotheses (I)–(VII). (I) and (III) are obvious. The assumption on h implies the Lipschitz condition

$$|h(s) - h(r)| \leqslant c(|s| + |r|)^{p-1}\,|s - r|. \tag{35}$$

We take $(Pf)(x) = h(|f(x)|) \arg f(x)$. Let $s = f(x)$ and $r = g(x)$ where $f, g \in X_1$. Taking the X_1 norm of the inequality and using Hölder's inequality, we obtain exactly (II). A simple dilation argument shows that there is only one value of p for which (IV) could be true, namely, $p = 1 + 4/n$. For that p, (IV) is proved by Strichartz [21]. Now $U_0(t)$ is unitary from L^2 onto L^2 and, from the explicit formula for its kernel, it carries L^1 into L^∞ with norm $\leqslant ct^{-n/2}$ for $t > 0$. By interpolation, it carries X_1 into X_3 with norm $\leqslant ct^{-d}$ where $d = (n/2)(p-1)/(p+1)$. For hypothesis (V) we require

$$\frac{n}{2}\,\frac{p-1}{p+1} = \frac{2}{p+1}.$$

This is true only if $p = 1 + 4/n$ (again!). We may choose $X_4 = W^{-m,p+1}$ for an appropriate m since $(1 + |\xi|^2)^{-m} \exp it|\xi|^2$ is a multiplier on L^{p+1}.

Next we let $G(u) = \int H(|u(x)|)\,dx$. Because of (3) we have

$$|H(s) - H(r)| \leqslant c(|s| + |r|)^p\,|s - r|$$

so that G is a continuous functional on L^{p+1}. Finally we sketch the proof of (VII). Let a solution u of NLS be given with small norm in Z, say less than δ, and $u(s) = f$. Let h_ν be a sequence of smooth functions which are globally Lipschitz such that $h_\nu(s) \to h(s)$ and $|h_\nu(s)| \leqslant |h(s)|$ for all s. By Remark 2 of

Section 2 there exists $u_\nu \in C(I, X)$ which satisfies NLS with the nonlinearity h_ν and $u_\nu(s) = f$. Furthermore u_ν is smooth [10, 12] and

$$E_\nu = \tfrac{1}{2}\| u_\nu(t)\|_{1,2}^2 + \int H_\nu(|u_\nu(t)|)\, dx$$

is independent of t. Since u_ν satisfies equation $(*_s, f)$ modified by the approximate nonlinearity h_ν,

$$\sup_I |u_\nu(t)|_3 \leqslant \| u_\nu\|_Z \leqslant 2\delta + c\, \| u_\nu\|_Z^p \leqslant 4\delta.$$

Hence the second term in E_ν is bounded and so is the first term, independently of t and ν. Pick a subsequence $u_\nu \to w$ weakly in $L^\infty(I, X)$ and in Z and by compactness $u_\nu \to w$ a.e. Then $h_\nu(u_\nu) \to h(w)$ a.e. Since $h_\nu(u_\nu)$ is bounded in $L^{1+1/p}$, $h_\nu(u_\nu) \to h(w)$ strongly in L^1_{loc}. Hence w satisfies NLS and $\| w\|_Z \leqslant 4\delta$. By uniqueness (in Lemma 1) $w = u$. It follows by passage to the limit that the inequality

$$\tfrac{1}{2}\| u(t)\|_{1,2}^2 + \int H(|u(t)|)\, dx \leqslant \tfrac{1}{2}\| f\|_{1,2}^2 + \int H(|f|)\, dx$$

holds, just as in the proof of Lemma 3, where $f = u(s)$. If the roles of s and t are reversed, the same argument shows the opposite inequality, hence equality. Equality together with weak convergence implies the strong continuity $u \in C(I, X)$.

Part (a) follows from Theorem 3, part (b) from Theorem 1, and part (c) from Theorem 4.

Remark. In a sequel to this paper, we show that the condition on p in Theorem 8 can be weakened to

$$1 + 4/n \leqslant p < 1 + 4/(n-2). \tag{36}$$

Remark. The solution in part (a) exists globally if we assume

$$|H(s)| = o(s^{2+4/n}) \qquad \text{as} \quad s \to +\infty.$$

Indeed, this implies the inequality $|H(s)| \leqslant c_\varepsilon s^2 + \varepsilon s^{2+4/n}$ for all s, $\varepsilon > 0$. Hence we have the energy bound

$$\begin{aligned}\int |\nabla u|^2\, dx &\leqslant c + c_\varepsilon \int |u|^2\, dx + k\varepsilon \left(\int u^2\, dx\right)^{2/n} \left(\int |\nabla u|^2\, dx\right)\\ &\leqslant c'_\varepsilon + k\varepsilon \int |\nabla u|^2\, dx\end{aligned}$$

by Sobolev's inequality. Picking $\varepsilon = 1/2k$ we have u bounded in $W^{1,2}(R^n)$. This implies the global existence of solutions [2, 4].

We now consider various classes of data which are dense in X. We will have the condition (5) where $\gamma(n)$ is given by (6).

$$\gamma(n) = [n + 2 + (n^2 + 12n + 4)^{1/2}]/2n.$$

Approximately $\gamma(1) = 3.56$, $\gamma(2) = 2.41$, $\gamma(3) = 2.00$, $\gamma(4) = 1.78$, $\gamma(5) = 1.64$, and $\gamma(n) \to 1$ as $n \to \infty$. An equivalent way to state (5) and (6) is (37) below.

THEOREM 9. *Let h satisfy* (3) *where*

$$\frac{1}{p} < \frac{n}{2}\,\frac{p-1}{p+1} < 1. \tag{37}$$

(a) *If $f_- \in L^2(R^n) \cap L^{1+1/p}(R^n)$, then there exists a unique solution u of NLS in some time interval $(-\infty, T]$ such that*

$$u \text{ is continuous with values in } L^2(R^n) \cap L^{p+1}(R^n), \tag{38}$$

$$\int |u(x,t)|^{p+1}\, dx \leqslant c(1 + |t|)^{-n(p-1)/2}, \tag{39}$$

$$\int |u(x,t) - u_-(x,t)|^2\, dx \to 0 \qquad \text{as} \quad t \to -\infty. \tag{40}$$

(b) *If $f_- \in W^{d,2}(R^n) \cap L^{1+1/p}(R^n)$ where $d = n(p-1)/(2(p+1))$ and f_- has small norm in this space, then u exists and satisfies* (38) *and* (39) *for all time and there exists u_+ asymptotic to u in the space $L^2(R^n)$ as $t \to +\infty$.*

(c) *If $f_- \in W^{1,2}(R^n) \cap L^{1+1/p}(R^n)$ and f_- has small norm in this space, then the asymptotes are valid in the space $W^{1,2}(R^n)$.*

(d) *If $f_0 \in W^{d,2}(R^n) \cap L^{1+1/p}(R^n)$ has small norm, then there exists u such that $u(0) = f_0$ and there exist u_+ and u_- asymptotic to u in the space $L^2(R^n)$. If in addition $f_0 \in W^{1,2}(R^n)$, then the asymptotes are valid in $W^{1,2}(R^n)$.*

Proof. In parts (a) and (b), we take $X = L^2(R^n)$, $X_3 = L^{p+1}(R^n)$, $X_1 = L^{1+1/p}(R^n)$, $G = 0$. Then (I), (II), (III), and (V) are proved as in Theorem 8. Recall that $d = (np - n)/(2p + 2)$. To prove (VII), we let u be the unique solution (given by Lemma 1) in V. We approximate h by h_ν as in the proof of Theorem 8. Then $u_\nu \to u$ strongly in the space V (as in the proof of Theorem 2, because of the smallness assumption). We have $\{u_\nu\}$ bounded in X. Hence $u_\nu \to u$ weakly in X and $|u(t)|_2 \leqslant |u(s)|_2$. Reversing the roles of s and t, we have $|u(t)|_2 = |u(s)|_2$ and so $u \in C(I, X)$. In part (a) we use

Theorem 6. Since $f_- \in X_1$ and since (V) is valid, $u_- = U_0(\cdot)f_-$ satisfies (39) for $t \leqslant T < 0$. In part (b) we apply Theorem 5. But now (39) must hold for all t, including small t. Since $f_- \in H^d(R^n)$, we have

$$u_- \in C(R, H^d(R^n)) \subset C(R, L^{p+1}(R^n)),$$

which is the required condition.

In part (c), we take $X = W^{1,2}(R^n)$ and $G(f) = \int H(|f(x)|)\, dx$. The hypotheses are verified exactly as in Theorem 8. In part (d) we apply Theorem 7 in exactly the same way.

In the next theorem no assumption is made on $h(s)$ for large s.

THEOREM 10. *Assume* (37). *Let k be an integer greater than $np/(p+1)$. If p is not an integer, assume $k \leqslant p$. Let h be a C^k function such that*

$$|h(s)| = O(|s|^p) \qquad \text{as} \quad s \to 0. \tag{7}$$

Let $f_- \in W^{k+1,2}(R^n) \cap W^{k,1+1/p}(R^n)$. Then the same conclusions hold as in Theorem 9(a) *and also in* 9(c) *if f_- has small norm in this space. Also*

$$\|u(t)\|_{k,p+1} \leqslant c(1 + |t|)^{-d}. \tag{41}$$

If f_0 has small norm in this space, then there exists u with $u(0) = f_0$ and u_+ and u_- asymptotic to u in the space $W^{1,2}(R^n)$.

Proof. We choose $X = W^{1,2}(R^n)$, $X_3 = W^{k,p+1}(R^n)$, and $X_1 = W^{k,1+1/p}(R^n)$. Because of the choice of k we have $X_1 \subset L^\infty(R^n)$. Hence it is well known that (7) implies (II) and (VI). Just as in Theorem 9, $u_- \in C(R, W^{k+1,2}) \subset C(R, W^{k,p+1})$. We apply Theorems 5, 6, and 7 as before.

This theorem was proved in [17] in case $n = 1$ under the assumptions $p > 4$ and $f_- \in W^{1,2} \cap L^1$.

7. THE NONLINEAR KLEIN–GORDON EQUATION

Consider the Klein–Gordon equation

$$\text{(KG)} \qquad u_{tt} - \Delta u + m^2 u = 0$$

with $m > 0$ and NLKG with the nonlinear term $h(u)$. We consider real solutions only. (We could allow complex solutions with the nonlinear term $h(|u|) \arg u$.) Let $X = W^{1,2}(R^n) \oplus L^2(R^n)$, the space of Cauchy data of finite energy. It is provided with the energy norm

$$\|[f_1, f_2]\|_2^2 = \int \{|\nabla f_1|^2 + m^2 f_1^2 + f_2^2\}\, dx, \tag{42}$$

where $f = [f_1, f_2]$ denotes an element of X. Because the energy norm is an invariant for KG, the evolution operator $U_0(t)$, which acts on Cauchy data, is a unitary operator on X. The perturbation operator P acts on Cauchy data $[f_1, f_2]$ by $P[f_1, f_2] = [0, h(f_1)]$. The only useful invariant for NLKG is $\frac{1}{2}|f|_2^2 + G(f)$ where $G[f_1, f_2] = \int H(f_1(x))\,dx$.

THEOREM 11. *Let $1 + 4/n \leqslant p \leqslant 1 + 4/(n-1)$. Let h be a C^1 function which satisfies* (3).

(a) *If u_- is* any *solution of KG of finite energy, there exists a unique solution u of NLKG in some time interval $(-\infty, T]$ such that*

u is continuous with values in X,

$$\int_{-\infty}^{T} |u(x,t)|^{p+1}\,dx\,dt < \infty,$$

the energy of $(u - u_-)$ tends to zero as $t \to -\infty$.

(b) *If the energy of u_- is sufficiently small, then u exists for all time, $|u|^{p+1}$ is integrable over all space–time, and there exists a unique solution u_+ of KG such that*

the energy of $(u - u_+)$ tends to zero as $t \to +\infty$,

the energies of u, u_-, and u_+ are equal.

(c) *If u_0 is a solution of KG of sufficiently small energy, then there exists unique u, u_+, and u_- with the same properties and with the Cauchy data of u equal to that of u_0 at $t = 0$.*

Proof. We choose $X_3 = L^{p+1} \oplus W^{-1,p+1}$, $X_1 = \{0\} \oplus L^{1+1/p}$. Hypotheses (I) and (III) are obvious. Hypothesis (II) is equivalent to the inequality

$$\left(\int |h(f_1) - h(g_1)|^{1+1/p}\,dx\right)^p \leqslant \left(\int |f_1|^{p+1} + |g_1|^{p+1}\,dx\right)^{p-1} \left(\int |f_1 - g_1|^{p+1}\,dx\right)$$

which follows from (3) and Hölder's inequality. It is proved in [9] that (V) is true if and only if p satisfies the given inequality. It is shown in [21] that (IV) is also true for such p (for some larger p as well). We may take $X_4 = W^{-1,p+1} \oplus W^{-2,p+1}$, since the operator T_t of [9] takes L^{p+1} into L^{p+1}. Clearly G is a continuous functional on X_3. Now we verify (VII). If $n \leqslant 3$, then $p \leqslant 1 + 2/(n-2)$, hence $W^{1,2}(R^n) \subset L^{2p}(R^n)$, hence P is a locally Lipschitz operator from X to X. Then (VII) follows from Remark 2 of

Section 2. For general n, we approximate $h_\nu \to h$ and use the same argument as in the proof of Theorem 8 to prove (VII). Part (a) follows from Theorem 3, part (b) from Theorem 1, and part (c) from Theorem 4.

Remark. The solution in part (a) exists globally if we assume

$$\liminf_{|s|\to\infty} H(s)/s^2 > -\infty.$$

For then we get from the energy identity (9) an a priori bound for the energy norm. This implies the global existence of solutions.

THEOREM 12. *Let* $\gamma(n) < p \leqslant 1 + 4/(n-1)$ *and let* h *satisfy* (3).

(a) *If* $f_- \in [W^{1,2} \cap W^{1,1+1/p}] \oplus [L^2 \cap L^{1+1/p}]$, *then there exists* u *as in Theorem* 11(a).

(b) *If* f_- *has small norm in this space, then the conclusion of Theorem* 11(b) *holds.*

(c) *If* f_0 *has small norm in this space, then the conclusion of Theorem* 11(c) *holds.*

Proof. We make the same choices of X, X_3, and X_1. We need only check that $u_- = U_0(\cdot)f_- \in V$. From (V) it follows that if $f_- \in W^{1,1+1/p} \oplus L^{1+1/p}$, then $|u_-(t)|_3 \leqslant c\,|t|^{-d}$. Since $f_- \in X$, $u_- \in C(R, X) \subset C(R, X_3)$. Hence $u_- \in V$. Theorem 12 follows from Theorems 6, 5, and 7.

THEOREM 13. *Let* p, k, *and* h *satisfy the same conditions as in Theorem* 10. *Let*

$$f_- \in [W^{k+1,2} \cap W^{k+1,1+1/p}] \oplus [W^{k,2} \cap W^{k,1+1/p}].$$

Then the same conclusions hold as in Theorem 12(a) *and also in Theorem* 12(b) *if* f_- *has small norm. If* f_0 *has small norm in this space, the same conclusions as in Theorem* 12(c) *hold.*

Proof. We choose $X = W^{1,2} \oplus L^2$ but now $X_3 = W^{k,p+1} \oplus W^{k-1,p+1}$ and $X_1 = W^{k+1,1+1/p} \oplus W^{k,1+1/p}$. By choice of k, $W^{k,1+1/p} \subset L^\infty$. Therefore (II) and (VI) follows from (7). Theorem 13 follows from Theorems 6, 5, and 7.

8. TWO OTHER EXAMPLES

In order to further illustrate the abstract theorems, we now consider the nonlinear wave equation

(NLW)
$$u_{tt} - \Delta u + h(u) = 0,$$

where $h(0) = h'(0) = 0$. The only useful conservation law is once again the energy. The space $X = \tilde{W}^{1,2} \oplus L^2$ is provided with the energy norm (42) where $m = 0$ now. We denote by $\tilde{W}^{k,q}$ the closure of the test functions with respect to the norm

$$\sum_{1 \leqslant |\alpha| \leqslant k} \left(\int |D^\alpha \phi|^q \, dx \right)^{1/q}.$$

We also denote $W_*^{k,q} = (-\Delta) \tilde{W}^{k+2,q}$. We choose

$$X_3 = L^{p+1} \oplus W_*^{-1,p+1}, \qquad X_1 = \tilde{W}^{1,1+1/p} \oplus L^{1+1/p}$$

in analogy to the Klein–Gordon case. By dilation we can see that there is only one value of p for which (IV) can be true, namely, $p = 1 + 6/(n-2)$. See [21]. There is also only one value of p for which (V) can be true. According to [9], (V) is valid for $1 \leqslant p \leqslant 1 + 4/(n-1)$, where $d(p+1) = (n-1)p - (n+1)$. In order that $d(p+1) = 2$, we would require $p = 1 + 4/(n-1)$. This conflicts with the preceding requirement. It is interesting to note that (IV) would be true with $p = 1 + 4/(n-1)$ if we took for X the Lorentz–invariant Hilbert space (but then we would lose the conservation law). We can, however, apply Theorem 5, 6, and 7.

THEOREM 14. *Let*

$$\frac{n + 2 + \sqrt{n^2 + 8n}}{2(n-1)} < p \leqslant 1 + \frac{4}{n-1} \qquad (<\infty \text{ if } n = 1).$$

If h satisfies (3) *and*

$$f_- \in [\tilde{W}^{1,2} \oplus \tilde{W}^{1,1+1/p}] \oplus [L^2 \cap L^{1+1/p}],$$

then the analogue of Theorem 12 *holds. If k and h satisfy the conditions of Theorem* 10 *and*

$$f_- \in [\tilde{W}^{k+1,2} \oplus \tilde{W}^{k+1,1+1/p}] \oplus [W^{k,2} \cap W^{k,1+1/p}],$$

then the analogue of Theorem 13 *holds.*

Proof. We require $p^{-1} < d < 1$, where d is given above. This determines the possible values of p. The lower bound for p is larger than the critical power (5) for NLKG. But it is not sharp: Klainerman [8] obtains a number which is smaller if $n \leqslant 3$.

Now consider the generalized Korteweg–deVries equation

$$\text{(GKdV)} \qquad u_t + u_{xxx} + h(u)_x = 0,$$

where $x \in R$. We limit ourselves to an analogue of Theorem 7.

THEOREM 15. *Let* h *satisfy* $|h'(s)| = O(|s|^{\gamma-1})$ *as* $s \to 0$ *where* $\gamma > (5+\sqrt{21})/2 \doteq 4.79$. *Let* $p = 2\gamma - 1$ *and* $d = (\gamma-1)/3\gamma$. *For any function* f_0 *with* $\int |f_0|^{1+1/p}\, dx$ *and* $\int |(f_0)_x|^2\, dx$ *sufficiently small, there is a solution* $u(x,t)$ *of GKdV and two solutions* u_+ *and* u_- *of the Airy equation such that*

$$\|u(t) - u_\pm(t)\|_{1,2} \to 0 \qquad \text{as} \quad t \to \pm\infty.$$

Proof. This result is a very slight improvement over [17] and the proof is a slight variation of that one. It also appears as an example in [8]. The existence of a global solution with values in $X = W^{1,2}(R)$ is proved in [17]. We will prove by iteration as in Theorem 5 that $\|u(t)\|_{p+1} = O(|t|^{-d})$. As usual we take $X_3 = L^{p+1}(R)$ and $X_1 = L^{1+1/p}(R)$. We estimate

$$\begin{aligned} \|h(u)_x\|_{1+1/p} &\leqslant \|h'(u)\|_r \|u_x\|_2, \qquad r = 2(p+1)/(p-1) \\ &\leqslant c\,\|u\|_{p+1}^{\gamma-1} \|u_x\|_2 \end{aligned}$$

by the choice of p and the condition on h. Now the evolution operator $U_0(t)$ for the Airy equation takes $L^2 \to L^2$ and $L^1 \to L^\infty$ with norm $O(|t|^{-1/3})$. Hence $U_0(t)$ takes X_1 into X_3 with norm $O(|t|^{-d})$, where $d = (p-1)/3(p+1) = (\gamma-1)/3\gamma$. Hence

$$\begin{aligned} &\int \|U_0(t-\tau)h(u(\tau))_x\|_{p+1}\, d\tau \\ &\quad \leqslant c \int |t-\tau|^{-d} \|u(\tau)\|_{p+1}^{\gamma-1}\, d\tau \\ &\quad \leqslant c(1+|t|)^{-d} [\sup_\tau (1+|\tau|)^d \|u(\tau)\|_{p+1}]^{\gamma-1} \end{aligned}$$

provided $d(\gamma-1) > 1$. This is the condition on γ in the theorem. An iteration then proves the decay of the L^{p+1} norm. Then u_+ and u_- are defined as in (26) and the proof concludes as in Lemmas 5 and 6.

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