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Quantification of uncertainties in soil–water characteristic curve associated with fitting parameters

Qian Zhai [a], Harianto Rahardjo [b,*]

[a] Division of Infrastructure Systems and Maritime Studies, School of Civil and Environmental Engineering, Nanyang Technological University, Block N1, B4B-07, Nanyang Avenue, Singapore 639798, Singapore

[b] Division of Infrastructure Systems and Maritime Studies, School of Civil and Environmental Engineering, Nanyang Technological University, Block N1, #1B-36, 50 Nanyang Avenue, Singapore 639798, Singapore

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ABSTRACT

Soil–water characteristic curve (SWCC) is commonly expressed using best fit equations with several fitting parameters. These fitting parameters are determined by best fitting experimental data with the best fit equations. Residual errors always exist after the regression procedure for the determination of these fitting parameters. Statistical theory suggests that uncertainties of the determined SWCC can be estimated from the variance of these fitting parameters and the residual errors. In this paper, equations for the confidence limits of the best fitted SWCC are developed to quantify the uncertainties in the determined SWCC associated with the fitting parameters. Applications of the confidence limits in evaluating the performance of best fit equations and suggestion for experimental measurements are presented in this paper.

1. Introduction

Soil–water characteristic curve (SWCC) is a graphical relationship that shows the relationship between the amount of water in a soil (i.e. gravimetric water content w, volumetric water content θ_w or degree of saturation S (Fredlund and Rahardjo, 1993)) and matric suction ψ. As introduced by Fredlund (2006), the entire suction range of the SWCC can be divided into three zones such as boundary effect zone, transition zone and residual zone and they are separated by air-entry value and residual suction as illustrated in Fig. 1.

SWCC is commonly expressed using best fit equations with several fitting parameters. The fitting parameters are determined from limited experimental data by applying a curve fitting technique by minimizing the sum of squared-errors (i.e. $\Sigma w_i * (\theta_i - \theta'_i)^2$, where: θ_i is the measured volumetric water content, θ'_i is the modeled volumetric water content, and w_i is the weighting factor as suggested by Leong and Rahardjo, 1997). Equations for correlation of these fitting parameters and SWCC variables (i.e. air-entry value, slope at the inflection point, residual suction and residual volumetric water content) were developed by Zhai and Rahardjo (2012) as an alternative to the traditional graphical method. In this paper, equations to quantify the uncertainties in SWCC associated with these fitting parameters are developed.

Residual error (i.e. Sum of squared errors) always exists after the regression procedure. Statistical theory (Benjamin and Cornell, 1970) suggests that the uncertainties of SWCC can be estimated from the coefficient of correlation equation and residual error. In this paper, equations for the determination of the variance of these fitting parameters and subsequently confidence limits of the best fitted SWCC and SWCC variables are derived for Fredlund and Xing's (1994) equation.

2. Literature review

Different best fit equations, such as proposed by Brooks and Corey (1964), Van Genuchten (1980), Fredlund and Xing (1994), Kosugi (1996) and Pedroso et al. (2009), have been developed to describe SWCC that relates the amount of water in a soil to the matric suction. Leong and Rahardjo (1997) concluded that Fredlund and Xing's (1994) equation was the best fit equation which could be used for a wide range of soil over the entire range of matric suction. Therefore, in this paper Fredlund and Xing's (1994) equation is selected for best fitting the experimental data for the determination of the SWCC:

$$\theta = C(\psi)\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a}\right)^n\right]\right\}^m} = \left[1-\frac{\ln\left(1+\frac{\psi}{C_r}\right)}{\ln\left(1+\frac{10^6}{C_r}\right)}\right]\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a}\right)^n\right]\right\}^m} \quad (1)$$

where,

a, n, m	fitting parameters;
C_r	an input value related to the residual suction which can be rough estimated as C_r = 1500 kPa for most cases.
θ_s	saturated volumetric water content.

* Corresponding author. Tel.: +65 6790 5246; fax: +65 6791 0676.
E-mail addresses: qzhai1@e.ntu.edu.sg (Q. Zhai), chrahardjo@ntu.edu.sg (H. Rahardjo).
URL: http://research.ntu.edu.sg/expertise/academicprofile/pages/StaffProfile.aspx?ST_EMAILID=CHRAHARDJO (H. Rahardjo).

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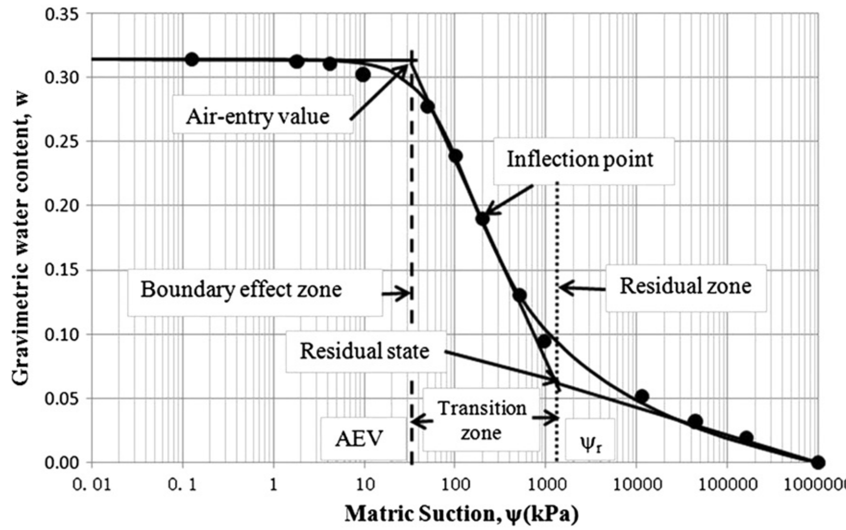

Fig. 1. Illustration of different zones of de-saturation as defined by the soil–water characteristic curve (after Fredlund, 2006).

There are only three unknown fitting parameters (i.e. a, n and m) in Fredlund and Xing's (1994) equation, C_r is an input value and not a fitting parameter. Zhai and Rahardjo (2012) defined Fredlund and Xing's (1994) equation with correction factor $C(\psi)$ as Method A and Fredlund and Xing's (1994) equation with correction factor $C(\psi) = 1$, which was suggested by Leong and Rahardjo (1997), as Method B.

Mishra et al. (1989) and Phoon et al. (2010) indicated that a first-order error analysis was a reasonable approximation for estimation of uncertainty in a predictive model in view of the lack of complete measurements for calibration data set which would enable more direct assessment. The first-order error analysis was based on Taylor expansion around the mean values of parameters by assuming small parameter perturbations and negligible higher-order terms. On the other hand, laboratory measurement of SWCC is very time consuming and costly because the equilibrium time for each data point can be very long especially for fine-grained soils. Therefore, it is very difficult to obtain sufficient experimental data for direct assessment of uncertainty in the determined SWCC while the first-order error analysis provides an indirect assessment of uncertainty. In this paper the first-order error analysis is adopted to evaluate the uncertainty in SWCC associated with the fitting parameters which are determined from limited experimental data.

Beck and Arnold (1977), Kool et al. (1987), Mishra et al. (1989), and Mishra and Parker (1989) indicated that the covariance matrix C could be used to represent the variances of estimated parameters and also introduced the procedure for estimation of the error covariance matrix C using the first-order error analysis approach as illustrated below.

$$C = E\left[\left(\tilde{b}-b\right)\left(\tilde{b}-b\right)^T\right] \approx \frac{s^2\left(J^TJ\right)^{-1}}{M-P} \tag{2}$$

where:

$\tilde{b}$	is the vector of estimated parameters
b	is the vector of true parameters
E	denotes statistical expectation
s^2	is the sum of squared-error;
M	is the number of experimental data points;

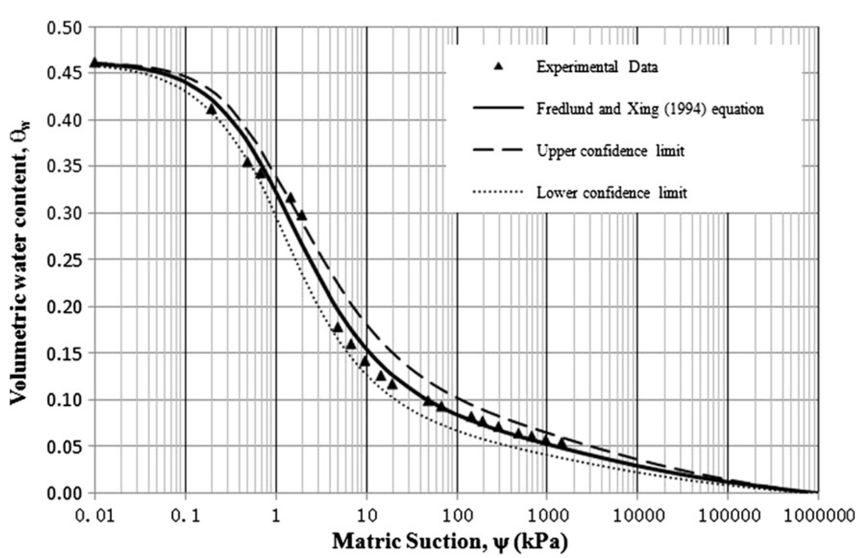

Fig. 2. Illustration of confidence limits of the best fitted SWCC.

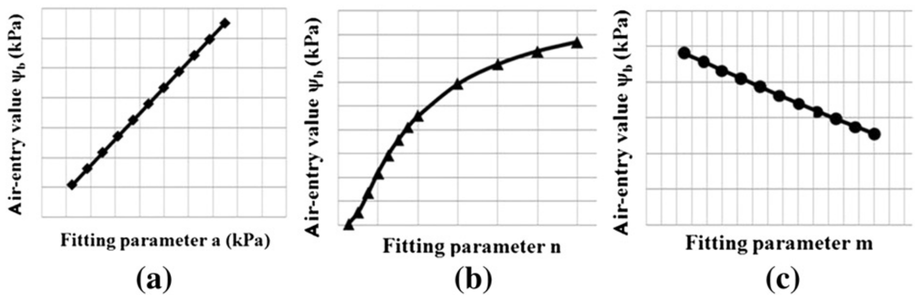

Fig. 3. Relationships between the air-entry value (AEV) and the fitting parameters.

P is the number of fitting parameters;
J is the parameter sensitivity matrix or Jacobian matrix.

$$J_{ij} = -\frac{\partial q_i}{\partial b_j} \approx \frac{\tilde{q}\left(b_j + \delta b_j\right) - \tilde{q}\left(b_j\right)}{\delta b_j} \tag{3}$$

With the forward difference increment taken to be $\delta b_j = 0.01 b_j$.

Kool and Parker (1988) indicated that confidence limits of the parameters could be determined from individual parameter variance by approximately using t-statistics. In this paper, two-sided confidence limits with α% significance level and t-statistic tool are adopted for the determination of the confidence limits of the fitting parameters. Theory of statistics suggests that confidence intervals of the fitting parameters can be determined as follows:

$$\begin{aligned} a &\sim \left[a - t_{\alpha/2}\sqrt{\mathrm{Var}(a)}, a + t_{\alpha/2}\sqrt{\mathrm{Var}(a)}\right], \\ n &\sim \left[n - t_{\alpha/2}\sqrt{\mathrm{Var}(n)}, n + t_{\alpha/2}\sqrt{\mathrm{Var}(n)}\right], \\ m &\sim \left[m - t_{\alpha/2}\sqrt{\mathrm{Var}(m)}, m + t_{\alpha/2}\sqrt{\mathrm{Var}(m)}\right]. \end{aligned}$$

3. Formulation

Equations for estimation of the variance of the fitting parameters for Fredlund and Xing's (1994) equation are derived in this section. Subsequently, the determination of confidence limits of the best fitted SWCCs and SWCC variables obtained from Fredlund and Xing's (1994) equation is also presented.

3.1. First-order error analysis

Consider fitting parameters a, n and m for Fredlund and Xing's (1994) equation as unknown variables and the function of θ can be expressed based on Taylor expansion by neglecting higher-order terms as follows:

$$\theta \approx \theta(x) = \theta(\hat{x}) + \frac{\partial \theta}{\partial x}(x - \hat{x}) \tag{4}$$

where:

x is parameter vector [**a**, **n**, **m**]
$\hat{x}$ is the vector of best fitted parameter [a, n, m]

Applying the expected value operator on both sides of the equation as follows:

$$E[\theta(x)] \approx \theta(\hat{x}) + \frac{\partial \theta}{\partial x} E[(x - \hat{x})]. \tag{5}$$

Since small parameter perturbations are assumed around the mean values, Eq. (5) can be simplified as

$$E[\theta(x)] \approx \theta(\hat{x}). \tag{6}$$

The variance of θ is defined as:

$$\mathrm{Var}[\theta] \approx E\left[(\theta - E[\theta])^2\right]. \tag{7}$$

Substituting Eqs. (4) and (6) into Eq. (7), the following equation can be obtained:

$$\mathrm{Var}[\theta] \approx E\left[\left(\theta(\hat{x}) + \frac{\partial \theta}{\partial x}(x - \hat{x}) - \theta(\hat{x})\right)^2\right] = \sum \left(\frac{\partial \theta}{\partial x}\right)^T \frac{\partial \theta}{\partial x} E\left[(x - \hat{x})^2\right]. \tag{8}$$

The expected value of $(x - \hat{x})$ can be expressed as follows:

$$E\left[(x - \hat{x})^2\right] = \mathrm{Var}[x - \hat{x}] + (E[x - \hat{x}])^2 = \mathrm{Var}[x] + 0. \tag{9}$$

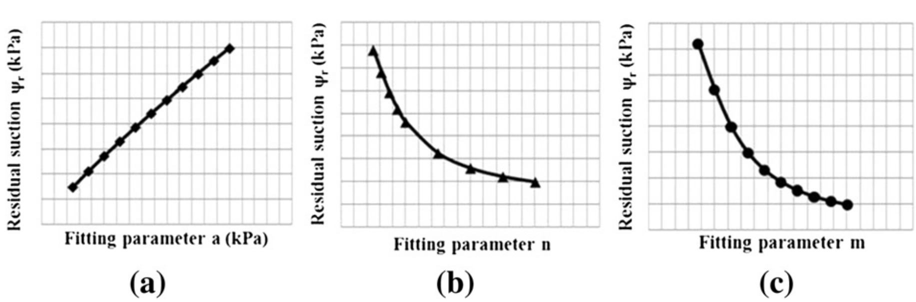

Fig. 4. Relationships between the residual suction ψ_r and the fitting parameters.

Table 1
Soil properties of silty sand.

Soil properties	Silty sand from Becher, 1970
USCS soil classification	SM
Sand%	88%
Fines%	12%
D_{60} (mm)	0.36
D_{30} (mm)	0.22
D_{10} (mm)	0.07
Specific gravity, Gs	2.56
Dry density, ρ_d (Mg/m^3)	1.37
Void ratio, e	0.86
Water content, w(%)	–

Substituting Eq. (9) into Eq. (8) results in the following Eq. (10):

$$\mathrm{Var}[\theta]\approx\sum\left(\frac{\partial\theta}{\partial x}\right)^T\frac{\partial\theta}{\partial x}\mathrm{Var}[x]. \tag{10}$$

Replacing parameter vector x with [a, n, m] in Eq. (10) results in the following Eq. (11):

$$\mathrm{Var}[x]=C=\begin{bmatrix}\mathrm{Var}(a) & \mathrm{Cov}(a,n) & \mathrm{Cov}(a,m)\\ \mathrm{Cov}(n,a) & \mathrm{Var}(n) & \mathrm{Cov}(n,m)\\ \mathrm{Cov}(m,a) & \mathrm{Var}(m,n) & \mathrm{Var}(m)\end{bmatrix}=\frac{\mathrm{Var}[\theta]}{\left(\sum\left(\frac{\partial\theta}{\partial x}\right)^T\frac{\partial\theta}{\partial x}\right)}. \tag{11}$$

There are M experimental data points and three unknown variables, resulting in M-3 degrees of freedom and $\mathrm{Var}[\theta]=\frac{SSE}{M-3}$. Eq. (11) can be rearranged as follows:

$$C=\frac{SSE}{(M-3)\left(\sum\left(\frac{\partial\theta}{\partial x}\right)^T\frac{\partial\theta}{\partial x}\right)}. \tag{12}$$

Eq. (12) has a similar form as Eq. (2). The variances of these fitting parameters can be determined by Eq. (12) using Microsoft Excel.

Eq. (11) also indicates that these three fitting parameters [a, n, m] of Fredlund and Xing (1994) equation are interdependent because the covariance of these fitting parameters (i.e. Cov(a,n), Cov(a,m) and Cov(n,m)) is not equal to zero. This point is in agreement with the conclusion from probabilistic analyses of experimental data from database by Phoon et al. (2010).

3.2. Determination of confidence limits of the best fitted SWCC

Define the confidence limits of the fitting parameters as follows:

$$\begin{aligned}a_{max}&=a+t_{a/2}\sqrt{\mathrm{Var}(a)}, & n_{max}&=n+t_{a/2}\sqrt{\mathrm{Var}(n)}, & m_{max}&=m+t_{a/2}\sqrt{\mathrm{Var}(m)}\\ a_{min}&=a-t_{a/2}\sqrt{\mathrm{Var}(a)}, & n_{min}&=n-t_{a/2}\sqrt{\mathrm{Var}(n)}, & m_{min}&=m-t_{a/2}\sqrt{\mathrm{Var}(m)}\end{aligned}.$$

Different combinations of the fitting parameters represent different SWCCs. The upper and lower confidence limits can be obtained from the combinations of these fitting parameters. The correction factor $C(\psi)$ in Method A does not contain any fitting parameter, which means that the variances of the fitting parameters do not result in any changes in the correction factor $C(\psi)$. Therefore, the correction factor $C(\psi)$ can be considered as a constant in the determination of the confidence limits of the best fitted SWCC.

Table 2
Evaluation results of Fredlund and Xing (1994) equation for a silty sand.

Fredlund and Xing (1994) equation	R^2	SSE	SSE_{norm}	ARE
Method A	98.98%	3.44%	10.52%	6.49%
Method B	99.03%	3.27%	8.50%	5.93%

In order to observe the combinations of the fitting parameters that correspond to the upper and lower confidence limits of the best fitted SWCC, mathematical deduction is carried out as illustrated in Appendix A. It indicates that when $\psi < a_{min}$, the (a_{max}, n_{max}, m_{min}) combination gives the upper confidence limit while the (a_{min}, n_{min}, m_{max}) combination gives the lower confidence limit. When $a_{min} < \psi < a_{max}$, the (a_{max}, n_{max}, m_{min}) combination gives the upper confidence limit while the (a_{min}, n_{max}, m_{max}) combination gives the lower confidence limit. When $a_{max} < \psi$, the (a_{max}, n_{min}, m_{min}) combination gives the upper confidence limit while the (a_{min}, n_{max}, m_{max}) combination gives the lower confidence limit. Equations for the determination of confidence limits of the best fitted SWCC are presented in Eqs. (13) to (14), and illustration of confidence limits of the best fitted SWCC is shown in Fig. 2.

When $0<\psi<a_{max}$ $\quad$ When $\psi > a_{max}$

$$\theta_{upper}=C(\psi)\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right]\right\}^{m_{min}}}\qquad \theta_{upper}=C(\psi)\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right]\right\}^{m_{min}}} \tag{13}$$

When $0<\psi<a_{min}$ $\quad$ When $\psi > a_{min}$

$$\theta_{lower}=C(\psi)\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right]\right\}^{m_{max}}}\qquad \theta_{lower}=C(\psi)\frac{\theta_s}{\left\{\ln\left[e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right]\right\}^{m_{max}}} \tag{14}$$

3.3. Determination of confidence limits of the SWCC variables

Correlation equations between SWCC variables and fitting parameters have been proposed by Zhai and Rahardjo (2012). Air-entry value (AEV or ψ_b) and residual suction ψ_r are most commonly used variables for estimation of other unsaturated soil properties such as shear strength, volume change and permeability. The relationships between air-entry value and residual suction are plotted using equations developed by Zhai and Rahardjo (2012) as illustrated in Figs. 3 and 4.

Fig. 3 indicates that the air-entry value increases with the increase in the fitting parameters "a" and "n" and decreases with the increase in the fitting parameter "m". Fig. 4 indicates that the residual suction ψ_r increases with the increase in the fitting parameter "a" and decreases with the increase in the fitting parameters "n" and "m". Therefore, the combination of (a_{max}, n_{max}, and m_{min}) which defines the upper confidence limit gives the maximum value of air-entry value while the combination of (a_{min}, n_{min} and m_{max}) which defines the lower confidence limit gives the minimum value of air-entry value. The combination of (a_{max}, n_{min}, and m_{min}) which defines the upper confidence limit gives the maximum value of residual suction ψ_r while the combination of (a_{min}, n_{max} and m_{max}) which defines the lower confidence limit gives the minimum value of residual suction ψ_r. Therefore, the confidence limits of air-entry value and residual suction can be determined by the following equations:

$$AEV_{max}=\psi_b(a_{max},\ n_{max},\ m_{min});\ AEV_{min}=\psi_b(a_{min},\ n_{min},\ m_{max}); \tag{15}$$

$$\psi_{rmax}=\psi_r(a_{max},\ n_{min},\ m_{min});\psi_{rmin}=\psi_r(a_{min},\ n_{max},\ m_{max}); \tag{16}$$

Where:

ψ_b(a, n, m) and ψ_r(a, n, m) are the function for the determination of air-entry value and residual suction (Zhai and Rahardjo, 2012), respectively.

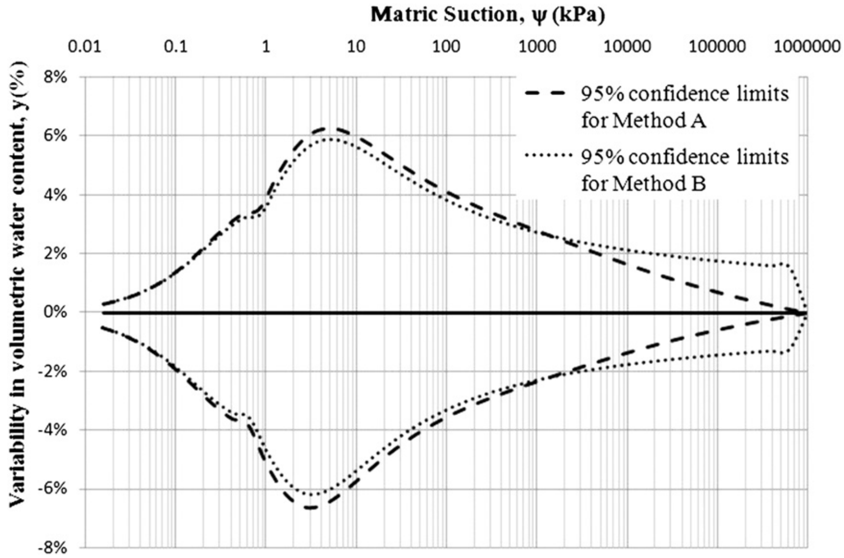

Fig. 5. Variability y in volumetric water content θ_w for sand with silt with respect to Method A and Method B.

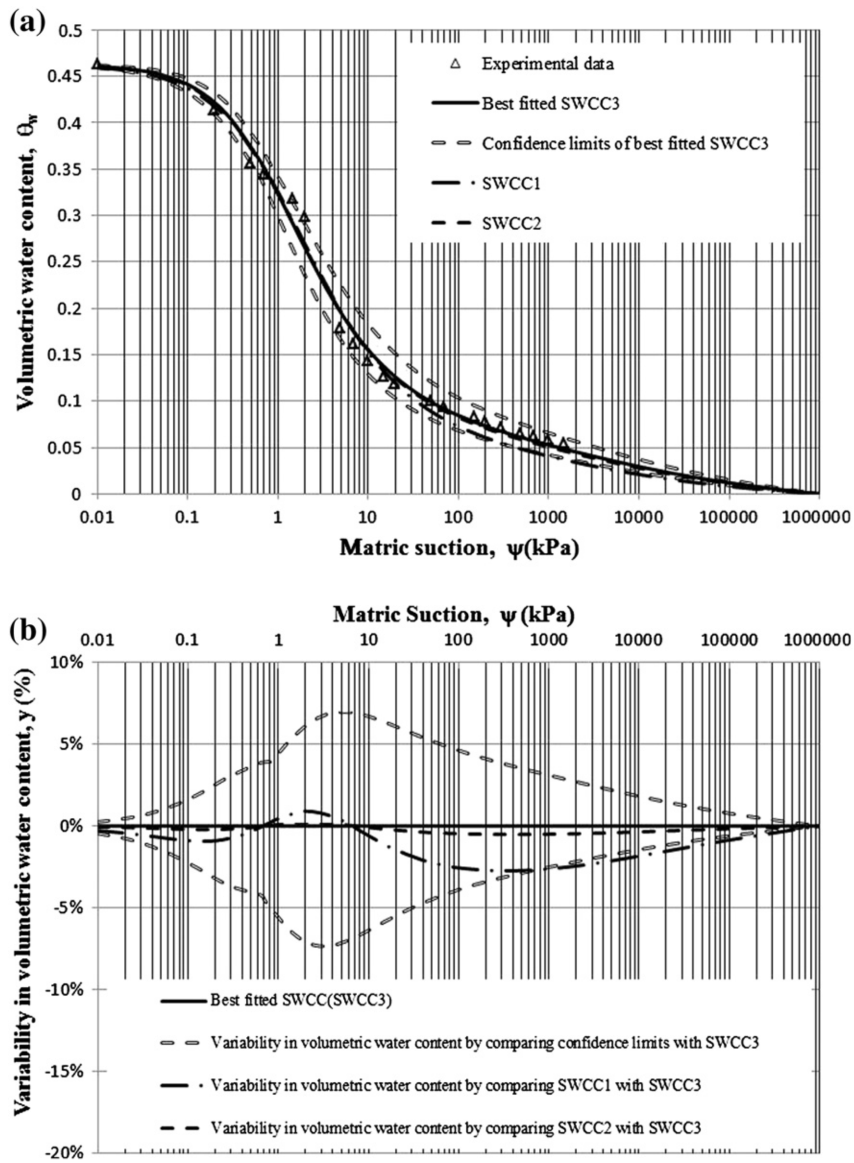

Fig. 6. Best fitted SWCCs and the variability y in volumetric water content compared with SWCC3.

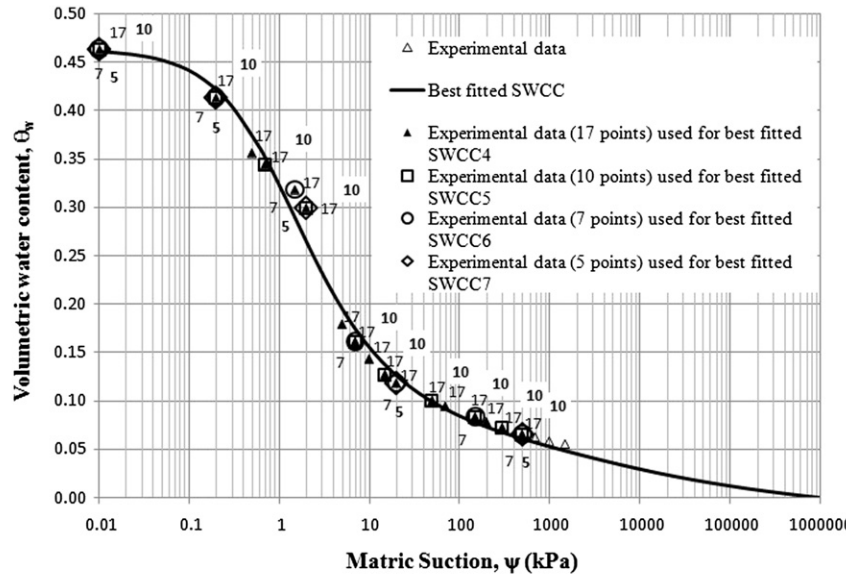

Fig. 7. Selections of data points for the best fitting of SWCC.

4. Application and discussion

Rahardjo et al. (2012) presented the variability of residual soil properties by analyzing the experimental data such as grain size distribution data (GSD), soil–water characteristic curve (SWCC), liquid limit (LL), plastic limit (PL), natural water content (w), void ratio (e), effective cohesion (c′), effective friction angle (ϕ′), and angle indicating the change in shear strength due to a change in suction (ϕ^b) with depth for the residual soils from Bukit Timah Granite, Jurong Formation and Old Alluvium. Rahardjo et al. (2012)'s work suggested that the variability of SWCC existed in the results obtained from undisturbed residual soils from the field. However, additional variability might also exist from different interpretations of experimental data as presented in this paper.

A silty sand (Becher, 1970) is selected for the illustration of the application of confidence limits for the best fitted SWCC and illustration of confidence limits of air-entry value and residual suction. The soil properties of the silty sand are summarized in Table 1.

4.1. Application one: evaluation of the performance of the best fit equation

The commonly adopted criteria for evaluation of the performance of the best fit equation (Leong and Rahardjo, 1997; Goh et al., 2010) are coefficient of determination R^2, sum of squared-error SSE, normalized sum of squared-error SSE_{norm} and average relative error ARE. The determinations of these statistical parameters are illustrated in Eqs. (17) to (19).

$$R^2 = 1 - \frac{SSE}{SST}, SST = \sum_i \left(\theta_i - \overline{\theta}\right)^2, SSE = \sum_i \left(\theta_i - \theta_i'\right)^2 \tag{17}$$

Where: θ_i is the observed value, $\overline{\theta}$ is the average of observed value, and θ'_i is the modeled value.

$$SSE_{Norm} = \sum_i \left(\frac{\theta_i - \theta_i'}{\theta_i}\right)^2 \tag{18}$$

$$ARE = \frac{1}{N} \sum_i \left|\frac{\theta_i - \theta_i'}{\theta_i}\right| \tag{19}$$

A large value of R^2 means better performance of the best fit equation. On the other hand, a large value of SSE, SSE_{norm} and ARE means worse performance of the best fit equation. The evaluation results of the performance of Fredlund and Xing's (1994) equation in accordance with Method A and Method B for the silty sand are summarized in Table 2.

The variability in volumetric water content θ_w can be defined in the following equation, which is similar to the one presented by Zapata (1999):

$$\text{variability } y = \frac{\theta_i - \hat{\theta}_i}{\theta_s} 100\% \tag{24}$$

where:

- θ_i predicted volumetric water content at ith suction level, from the confidence limit of the best fitted SWCC or from the experimental data.
- $\hat{\theta}_i$ best estimated volumetric water content at ith suction level, from the best fitted SWCC.
- θ_s saturated volumetric water content.

The variability y in volumetric water content θ_w can be plotted as shown in Fig. 5 for Method A and Method B.

Table 2 indicates that Method B has R^2 that is slightly higher than Method A. On the other hand, Method B has SSE, SSE_{norm} and ARE that are smaller than those associated with Method A. Fig. 5 also indicates that Method B produces less variability in volumetric water content than Method A when suction ψ is less than 1000 kPa and Method B produces high variability in volumetric water content than Method A when suction ψ is greater than 1000 kPa. This illustrates that the criteria of R^2, SSE, SSE_{norm} and ARE can be used to evaluate the overall performance of best fit equation while the confidence limit can be used to evaluate the performance of best fit equation with respect to different suction ranges. Therefore, in addition to the evaluation criteria (i.e. R^2, SSE, SSE_{norm} and ARE) that are commonly adopted by researchers, confidence limits can also be used as a tool to evaluate the performance of the best fit equations with respect to different suction ranges.

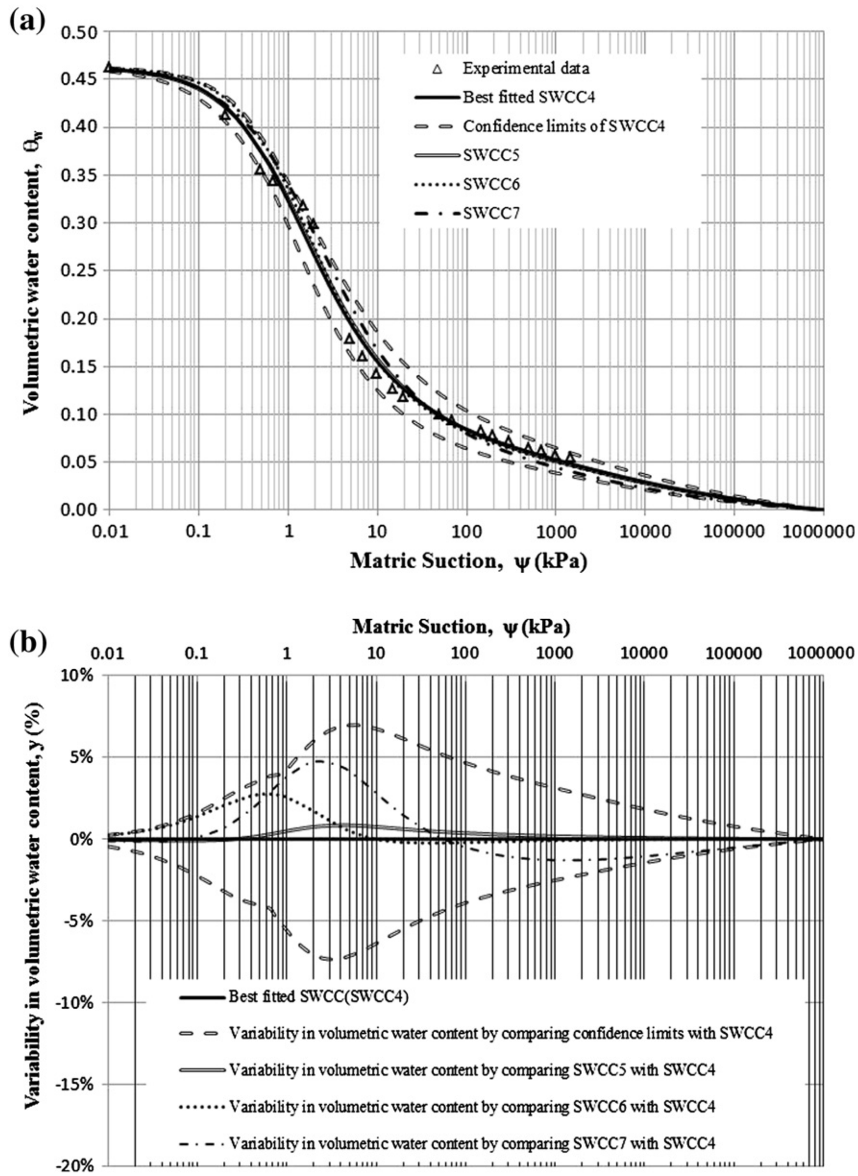

Fig. 8. Best fitted SWCCs and the variability y in volumetric water content compared with SWCC4.

4.2. Application two: suggestion for experimental measurement from confidence limits of SWCC

Fig. 5 indicates that high variability occurs in the transition zone which is defined in Fig. 1. It is suggested that more experimental data in the transition zone are obtained in order to have more accurate SWCC.

Table 3
Air-entry values and residual suctions as determined from the best fitted SWCC together with their confidence limits.

SWCC variables	Combination of fitting parameters	Value (kPa)
AEV	(a, n, m)	0.162
AEV_{max}	(a_{max}, n_{max}, m_{min})	0.209
AEV_{min}	(a_{min}, n_{min}, m_{max})	0.121
ψ_r	(a ,n, m)	21.99
ψ_{rmax}	(a_{max}, n_{min}, m_{min})	34.06
ψ_{rmin}	(a_{min}, n_{max}, m_{max})	14.32

The suction range of measurement data is always limited by the capacity of the measurement apparatus (i.e. Tempe cell, 5 Bar pressure plate, 15 Bar pressure plate, etc.). Due to the expensive cost and long testing time associated with the SWCC measurement, only few points are normally measured for the determination of SWCC. Question can be raised on the maximum suction that has to be measured and the minimum number of data points needed to obtain an acceptable (accurate) SWCC. The confidence limits of SWCC can be used to help investigate the maximum suction and minimum data points needed. If the SWCC determined from data points with less suction or less data points is still within the confidence limits then it is acceptable, otherwise it is rejected (or inaccurate).

In this study, SWCC1 refers to the use of experimental data up to 100 kPa for the best fit procedure while SWCC2 and SWCC3 refer to the use of experimental data up to 500 kPa and 1500 kPa, respectively for the best fit procedure. Method A is selected as the best fit equation for best fitting the experimental data. The best fitted SWCCs and 95% confidence limits of SWCC3 are shown in Fig. 6(a). The variability in

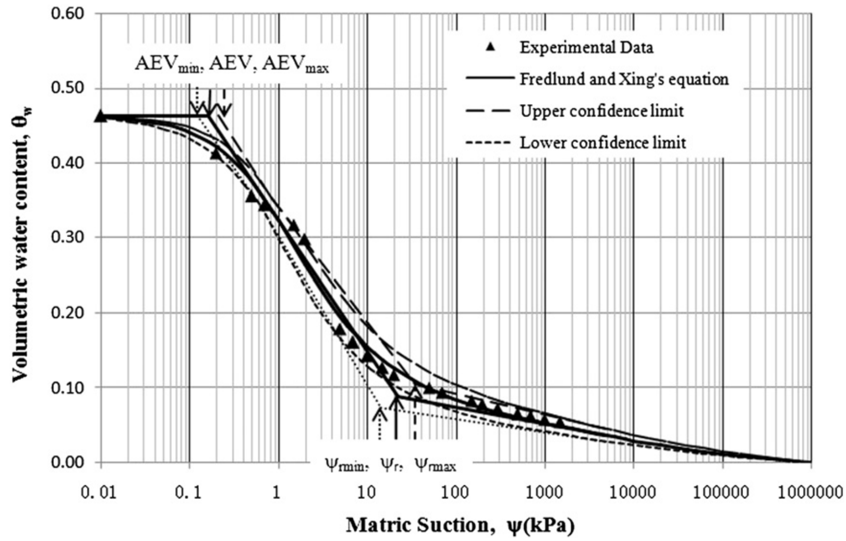

Fig. 9. Air-entry values and residual suctions as determined from the best fitted SWCC together with their confidence limits.

volumetric water content with respect to different best fitted SWCCs (i.e. SWCC1, SWCC2 and SWCC3) is illustrated in Fig. 6(b). Fig. 6 indicates that SWCC1 exceeds the band of confidence limits at high suction range while SWCC2 is within the band of confidence limits throughout the entire suction range. Therefore, SWCC1 is rejected while SWCC2 is accepted. In other words, experimental data up to 100 kPa (or 1 Bar Tempe cell) are insufficient for obtaining an accurate SWCC while experimental data up to 500 kPa (or 5 Bar pressure plate) are sufficient for this type of soil. The maximum suction needed for SWCC measurement can be suggested for different types of soil using the confidence limit analyses of SWCC experimental data from database.

A similar approach can be applied to investigate the minimum number of data points needed to obtain an acceptable SWCC. Different sets of experimental data up to a maximum suction of 500 kPa are selected for best fitting as illustrated in Fig. 7. SWCC4 refers to the use of 17 data points for the best fitting procedure while SWCC5, SWCC6 and SWCC7 refer to the use of 10, 7, and 5 data points for the best fitting procedure, respectively. The selected data points are distributed evenly for every log-cycle within the range from 0.01 kPa to 500 kPa as shown in Fig. 7. The determined SWCCS and 95% confidence limits of SWCC4 are illustrated in Fig. 8(a).

The variability in volumetric water content with respect to different sets of data points used for the determination of SWCC is illustrated in Fig. 8(b). Fig. 8 indicates that SWCC5, SWCC6 and SWCC7 are all within the band of confidence limits. In other words, 5 points are sufficient to obtain an acceptable SWCC for this type of soil.

As illustrated in this paper, confidence limits of SWCC can be used for the investigation of maximum matric suction and minimum number of data points needed to obtain an acceptable SWCC, providing guidelines for cost and time savings in experimental measurements of SWCC.

4.3. Application two: determination of confidence limits of SWCC variables

The air-entry value and residual suction ψ_r as determined from the best fitted SWCC together with their confidence limits for the silty sand are summarized in Table 3 and Fig. 9. Fig. 9 indicates the possible ranges of air-entry values that can be suggested as given in Table 3, instead of the traditionally single air-entry value for the determined set of air-entry value and residual suction.

5. Conclusions

The first-order error analysis is adopted for quantification of uncertainties in SWCC associated with the fitting parameters of Fredlund and Xing's (1994) equation. Equations for the determination of confidence limits of SWCC and SWCC variables from fitting parameters of the best fit equation are developed. High variability in water content occurs in the transition zone, suggesting that more data points need to be measured within the transition zone in order to obtain a more accurate SWCC. Maximum suction and minimum number of data points needed to obtain an acceptable SWCC can be analyzed for different types of soil using the confidence limits of best fitted SWCC from database.

Appendix A. Determination of confidence limits of best fitted SWCC for Fredlund and Xing's (1994) equation

It is known that if $x_1 > x_2 > 0$: when $a > 1$, then $a^{x_1} > a^{x2}$; when $0 < a < 1$, then $a^{x_1} > a^{x2}$; and when $a > 0$, then $x_1^a > x_2^a$. Since all these fitting parameters are positive, the following relationships can be obtained:

$$\text{when } 0<\psi<a_{min}, \text{then} \left(\frac{\psi}{a_{max}}\right)^{n_{max}} < \left(\frac{\psi}{a_{max}}\right)^{n_{min}} \text{or} < \left(\frac{\psi}{a_{min}}\right)^{n_{max}} < \left(\frac{\psi}{a_{min}}\right)^{n_{min}} < 1 \tag{29}$$

$$\text{when } a_{min}<\psi<a_{max}, \text{then} \left(\frac{\psi}{a_{max}}\right)^{n_{max}} < \left(\frac{\psi}{a_{max}}\right)^{n_{min}} < 1 < \left(\frac{\psi}{a_{min}}\right)^{n_{min}} < \left(\frac{\psi}{a_{min}}\right)^{n_{max}} \tag{30}$$

$$\text{when } \psi > a_{max}, \text{then} \left(\frac{\psi}{a_{min}}\right)^{n_{max}} > \left(\frac{\psi}{a_{max}}\right)^{n_{max}} \text{or} > \left(\frac{\psi}{a_{min}}\right)^{n_{min}} > \left(\frac{\psi}{a_{max}}\right)^{n_{min}} > 1 \tag{31}$$

$\ln\left(e + \left(\frac{\psi}{a}\right)^n\right)$ is always greater than 1,

$$\text{therefore, } \left\{\ln\left(e + \left(\frac{\psi}{a}\right)^n\right)\right\}^{m_{min}} < \left\{\ln\left(e + \left(\frac{\psi}{a}\right)^n\right)\right\}^{m_{max}}. \tag{32}$$

It can be concluded from inequalities (29) to (31):when $0 < \psi < a_{min}$; then

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)\right\}^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)^{m_{min}}} \text{ or } > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{min}}} \tag{33}$$

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)\right\}^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)^{m_{max}}} \text{ or } > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{max}}} \tag{34}$$

when $a_{min} < \psi < a_{max}$; then

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)\right\}^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{min}}} \tag{35}$$

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)\right\}^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{max}}} \tag{36}$$

when $a_{max} < \psi$

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)\right\}^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{min}}} \text{ or } \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)^{m_{min}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{min}}} \tag{37}$$

$$\frac{\theta_s}{\left\{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{min}}\right)\right\}^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{min}}\right)^{m_{max}}} \text{ or } \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{max}}\right)^{n_{max}}\right)^{m_{max}}} > \frac{\theta_s}{\ln\left(e+\left(\frac{\psi}{a_{min}}\right)^{n_{max}}\right)^{m_{max}}}. \tag{38}$$

Therefore, it can be concluded that:

when $\psi < a_{min}$, (a_{max}, n_{max}, m_{min}) gives the upper confidence limit while (a_{min}, n_{min}, m_{max}) gives the lower confidence limit;
when $a_{min} < \psi < a_{max}$, (a_{max}, n_{max}, m_{min}) gives the upper confidence limit while (a_{min}, n_{max}, m_{max}) gives the lower confidence limit;
when $a_{max} < \psi$, (a_{max}, n_{min}, m_{min}) gives the upper confidence limit while (a_{min}, n_{max}, m_{max}) gives the lower confidence limit.

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