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Alpha labelings of full hexagonal caterpillars

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Abstract

Barrientos and Minion (2015) introduced the notion of generalized snake polyomino graphs and proved that when the cells are either squares or hexagons, then they admit an alpha labeling. Froncek et al. (2014) generalized the notion by introducing straight simple polyominal caterpillars with square cells and proved that they also admit an alpha labeling.

We introduce a similar family of graphs called full hexagonal caterpillars and prove that they also admit an alpha labeling. This implies that every full hexagonal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.
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Keywords: Alpha labeling; Graceful labeling; Graph decomposition

1. Introduction

At the Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton in March 2014, Minion presented her joint results with Barrientos on alpha labelings of snake polyominoes and other related graphs (they later published it in [1]). Froncek, Kingston, and Vezina [2] generalized the notion by introducing straight simple polyominal caterpillars with square cells and proved that they also admit an alpha labeling. In this paper, we introduce a similar class of graphs with hexagonal cells, called full hexagonal caterpillars and prove that they also admit an alpha labeling.

Barrientos and Minion [1] define a *snake polyomino* as a chain of m edge-amalgamated cycles $C^1, C^2, \ldots, C^m$ of the same length with the property that C^1 shares one edge with C^2, C^m shares one edge with C^{m-1}, and for $i = 2, 3, \ldots, m-1$, each C^i shares one edge with C^{i-1} and another edge with C^{i+1}. Note that no edge appears in more than two of those cycles. They proved that such a snake polyomino has an alpha labeling whenever the cycles are of length four or six.

Froncek, Kingston, and Vezina [2] generalized this notion for square polyominoes and defined a *straight simple polyominal caterpillar* as follows. The *spine* of the caterpillar is a straight snake polyomino in which the edges of C^i shared with C^{i-1} and C^{i+1} are non-adjacent, which means that every vertex is of degree at most three. The spine can be also viewed as the Cartesian product $P_{m+1} \square P_2$. We denote the vertices of the two paths as $x_0, x_1, \ldots, x_m$

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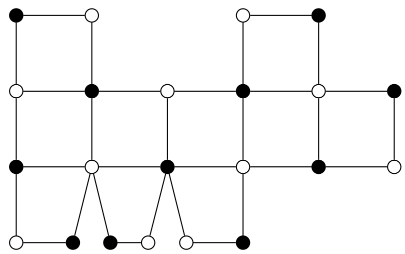

Fig. 1. Straight simple polyominal caterpillar.

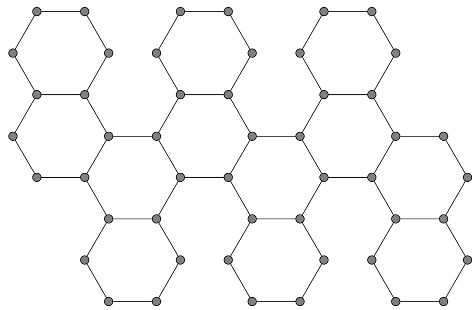

Fig. 2. Full hexagonal caterpillar H_6.

and $y_0, y_1, \dots, y_m$, respectively. A straight simple polyominal caterpillar then can be constructed by amalgamating at most one four-cycle to each of the edges $x_j x_{j+1}$ and $y_l y_{l+1}$ for $j, l \in \{0, 1, \dots, m-1\}$. Notice that we can amalgamate the four-cycles to none, one, or both of the two edges $x_j x_{j+1}$ and $y_j y_{j+1}$ for any admissible value of j. The number of four-cycles in the spine is the *length* of the caterpillar. An example is shown in Fig. 1.

We generalize the notion for hexagonal polyominoes in a rather restricted form and define *full hexagonal caterpillars* of length m as follows. The *spine* is a hexagonal chain (see [1]) consisting of m six-cycles, $C^1, C^2, \dots, C^m$, where C^i consists of edges $e_1^i, e_2^i, \dots, e_6^i$ numbered consecutively clockwise. Cycle C^1 shares edge e_5^1 with C^2, C^2 shares edge e_1^2 with C^1 and e_3^2 with C^3, C^3 shares edge e_1^3 with C^2 and e_5^3 with C^4, and so on. More precisely, we have

$$e_5^1 = e_1^2, e_3^2 = e_1^3, e_5^3 = e_1^4, \dots, e_5^{2i-1} = e_1^{2i}, e_3^{2i} = e_1^{2i+1}, \dots, e_5^{m-1} = e_1^m$$

when m is even and

$$e_5^1 = e_1^2, e_3^2 = e_1^3, e_5^3 = e_1^4, \dots, e_5^{2i-1} = e_1^{2i}, e_3^{2i} = e_1^{2i+1}, \dots, e_3^{m-1} = e_1^m$$

when m is odd.

The *legs* are six-cycles $D^1, D^2, \dots, D^m$, where D^i consists of edges $f_1^i, f_2^i, \dots, f_6^i$ numbered consecutively clockwise. For $i = 1, 2, \dots, m$, cycle D^i shares edge f_1^i with C^i, where $f_1^i = e_3^i$ when i is odd, and $f_1^i = e_5^i$ when i is even. An example of a full hexagonal caterpillar H_6 of length $m = 6$ is shown in Fig. 2.

2. Supporting results and tools

Rosa [3] introduced in 1967 certain types of vertex labelings as important tools for decompositions of complete graphs K_{2n+1} into graphs with n edges.

A *labeling* ρ of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \dots, 2n\}$ of elements of the additive group Z_{2n+1}. Let ρ be the injection. The *length* of an edge xy is defined as $\ell(x, y) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. The subtraction is performed in Z_{2n+1} and hence $0 < \ell(x, y) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \dots, n\}$ and $S \subseteq \{0, 1, \dots, 2n\}$, then ρ is a *rosy labeling* (called originally ρ-valuation by Rosa); if $S \subseteq \{0, 1, \dots, n\}$ instead, then ρ is a *graceful labeling* (called β-valuation by

Rosa). A graph admitting a graceful labeling is called a *graceful graph*. A graceful labeling ρ is said to be an *alpha labeling* if there exists a number λ (called the *boundary value*) with the property that for every edge $xy \in G$ with $\rho(x) < \rho(y)$ it holds that $\rho(x) \leq \lambda < \rho(y)$. Obviously, G must be bipartite to allow an alpha labeling. A graph admitting an alpha labeling is called an α*-graph*. For an exhaustive survey of graph labelings, see [4] by Gallian.

Let G be a graph with at most n vertices. We say that the complete graph K_n has a G*-decomposition* if there are subgraphs $G_0, G_1, G_2, \dots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. Such a decomposition is called *cyclic* if there exists a graph isomorphism φ such that $\varphi(G_i) = G_{i+1}$ for $i = 0, 1, \dots, s-1$ and $\varphi(G_s) = G_0$.

Each graceful labeling is of course also a rosy labeling. The following theorem was proved by Rosa in [3].

Theorem 2.1. *A cyclic G-decomposition of K_{2n+1} for a graph G with n edges exists if and only if G has a rosy labeling.*

The main idea of the proof is the following. K_{2n+1} has exactly $2n + 1$ edges of length i for every $i = 1, 2, \dots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding an element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \dots, 2n$.

For graphs with an alpha labeling, even stronger result was proved by Rosa.

Theorem 2.2. *If a graph G with n edges has an alpha labeling, then there exists a G-decomposition of K_{2kn+1} for any positive integer k.*

The proof is based on the observation that if we increase all labels in the partite set with labels exceeding λ by t, then all edge lengths will also stretch by t. Hence, we can take k copies of G and increase the labels in the upper partite set in the jth copy by jn, where $j = 0, 1, \dots, k-1$. This way we obtain edge lengths $1, 2, \dots, nk$, each exactly once.

Barrientos and Minion [1] made the following observation.

Theorem 2.3. *If G_1 of order v_1 with n_1 edges and G_2 of order v_2 with n_2 edges are two α-graphs with boundary values λ_1 and λ_2, respectively, then there exists their edge-amalgamation Γ of order $v_1 + v_2 - 2$ with $n_1 + n_2 - 1$ edges that is also an α-graph with boundary value $\lambda = \lambda_1 + \lambda_2$.*

For $i = 1, 2$ let X_i be the partite sets with the lower labels, that is, at most λ_i, and Y_i the sets with the upper labels. Call the respective labelings f_1 and f_2. Further, let $e_1 = x_1 y_1$ be the longest edge of G_1 of length n_1 and $e_2 = x_2 y_2$ the shortest edge of G_2 of length 1. Then indeed $f_1(x_1) = 0$, $f_1(y_1) = n_1$, $f_2(x_2) = \lambda_2$, and $f_2(y_2) = \lambda_2 + 1$.

Barrientos and Minion observed that one can amalgamate x_1 with x_2 and y_1 with y_2 and increase the labels in X_1 and Y_1 by λ_2 and labels in Y_2 by $n_1 - 1$ to obtain the desired graph Γ. The amalgamated edge arising from e_1 and e_2 is called the *link*. Notice that the shortest edge of Γ is in the subgraph arising from G_1 while the longest one is in the subgraph arising from G_2. The edge-amalgamation of G_1 and G_2 as described above will be denoted as $G_1 \parallel G_2$.

It is easy to observe that this concept can be used for consecutive amalgamation of any number of α-graphs into a larger α-graph. We will use that observation in the next section.

3. Construction

Using the above result, we now prove that every full hexagonal caterpillar is an α-graph. The proof will be performed by strong induction. We first prove the result for full hexagonal caterpillars of even length. We start with the base case for $m = 2$.

The assertion of the lemma below follows directly from the labelings in Fig. 3.

Lemma 3.1. *The full hexagonal caterpillar H_2 of length 2 has an alpha labeling such that the edge e_1^1 has length 21 and the edge e_3^2 has length 1.*

Now we are ready to prove our result for m even.

Theorem 3.2. *The full hexagonal caterpillar H_m of an even length m has an alpha labeling such that the edge e_1^1 has length $10m + 1$ and the edge e_3^m has length 1.*

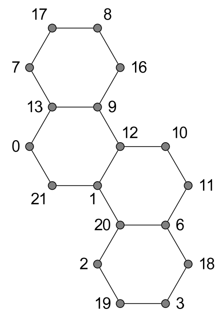

Fig. 3. Full hexagonal caterpillar H_2.

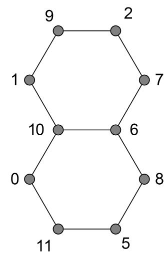

Fig. 4. Full hexagonal caterpillar H_1.

Proof. The base case has been established in Lemma 3.1. Now suppose we have a full hexagonal caterpillar H_m of an even length m. Obviously, it has $10m + 1$ edges. We remove the cycles C^{m-1}, C^m, D^{m-1}, D^m except edge e_1^{m-1}, which is equal to e_3^{m-2} and belongs to C^{m-2} as well. We obtain a full hexagonal caterpillar H_{m-2} of even length $m-2$. By our induction hypothesis, H_{m-2} has an alpha labeling with e_1^1 of length $10(m-2)+1$ and e_3^{m-2} of length 1.

Now we apply Theorem 2.3, setting $G_1 = H_2$ and $G_2 = H_{m-2}$. It should be clear that H_m is the amalgamation $H_{m-2} \parallel H_2$ of the two full hexagonal caterpillars H_{m-2} and H_2 with the required alpha labeling. □

An analogous result for m odd is slightly weaker, as we do not require the shortest edge to be in any specific position.

Lemma 3.3. *The full hexagonal caterpillar H_1 of length 1 has an alpha labeling such that the edge e_1^1 has length 11.*

The labeling is shown in Fig. 4.

Theorem 3.4. *The full hexagonal caterpillar H_m of an odd length m has an alpha labeling such that the edge e_1^1 has length $10m + 1$.*

Proof. The base case follows from Lemma 3.3. We start with a full hexagonal caterpillar H_m of an odd length m. Obviously, it has $10m + 1$ edges. We remove the cycles C^m and D^m except edge e_1^m, which is equal to e_3^{m-1} and belongs to C^{m-1} as well. Now we have a full hexagonal caterpillar H_{m-1} of even length $m - 1$. By Theorem 3.2, H_{m-1} has an alpha labeling with e_1^1 of length $10(m-1)+1$ and e_3^{m-1} of length 1.

We again apply Theorem 2.3 with $G_1 = H_1$ and $G_2 = H_{m-1}$. It should be clear that the amalgamation $H_{m-1} \parallel H_1$ of the two full hexagonal caterpillars H_{m-1} and H_1 is H_m with the required alpha labeling. □

Combining Theorems 3.2 and 3.4, we get our main result immediately.

Theorem 3.5. *Every full hexagonal caterpillar admits an alpha labeling.*

The result on decompositions of complete graphs into full hexagonal polyominal caterpillars follows directly from Theorems 3.5 and 2.2.

Corollary 3.6. *Every full hexagonal polyominal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.*

4. Conclusion

There are two obvious directions for further research. One is looking at hexagonal caterpillars that are not full. That in our notation means that only some of the leg hexagons D_i would be present. Although it does not seem so hard at the first glance, one has to realize that because a hexagon itself does not admit an alpha labeling, and also an alpha labeling of H_1 with the proper placement of the edge of length one is not known (and although we are not going to present a proof here, we are reasonably sure that it does not exist), we would have to consider eight starting cases for caterpillars of length three. Studying only hexagonal caterpillars of even length is not much easier either, since we were not able to find a proper alpha labeling for a spine segment of length two with no legs attached (again, we are reasonably sure that it does not exist).

Another possibility would be to look at the case where more than one leg hexagon would be attached to the same spine edge. This seems to be even more complex problem than the above one. Finally, while Barrientos and Minion proved that all hexagonal snakes (they called them chains) admit alpha labelings, one could ask under what conditions there would exist alpha labelings of hexagonal rings.

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