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Multiserial and special multiserial algebras and their representations ☆

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In this paper we study multiserial and special multiserial algebras. These algebras are a natural generalization of biserial and special biserial algebras to algebras of wild representation type. We define a module to be multiserial if its radical is the sum of uniserial modules whose pairwise intersection is either 0 or a simple module. We show that all finitely generated modules over a special multiserial algebra are multiserial. In particular, this implies that, in analogy to special biserial algebras being biserial, special multiserial algebras are multiserial. We then show that the class of symmetric special multiserial algebras coincides with the class of Brauer configuration algebras, where the latter are a generalization of Brauer graph algebras. We end by showing that any symmetric algebra with radical cube zero is special multiserial and so, in particular, it is a Brauer configuration algebra.

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1. Introduction

In the study of the representation theory of finite dimensional algebras, the introduction of presentations of algebras by quiver and relations has led to major advances in the field. Such presentations of algebras combined with another combinatorial tool that has proven powerful, the Auslander–Reiten translate and the Auslander–Reiten quiver, have led to many significant advances in the theory, to cite but a selection of these, see for example [14,27,28,41] or for an overview see [7,45,46,8]. The addition of special properties such as semi-simplicity, self-injectivity, Koszulness, finite and tame representation type, and finite global dimension, to name a few, have led to structural results and classification theorems. For example, the Artin–Wedderburn theorems for semi-simple algebras [33], the classification of hereditary algebras of finite representation type [10], Koszul duality [40], classification of Nakayama algebras [8], covering theory of algebras [15], the study of tilted algebras [17,31] and more recently, the study of cluster-tilted algebras beginning with [18,20,6].

Biserial and special biserial algebras have been the object of intense study at the end of the last century. Many aspects of the representation theory of these algebras are well-understood, for example, to cite but a few of the earlier results, the structure of the indecomposable representations [42,29,50], almost split sequences [19], maps between indecomposable representations [22,37], and the structure of the Auslander–Reiten quiver [24]. Recently there has been renewed interested in this class of algebras. On the one hand this interest stems from its connecting with cluster theory. In [5] the authors show that the Jacobian algebras of surface cluster algebras are gentle algebras, and hence a subclass of special biserial algebras. This class has been extensively studied since, see [21,35] for examples of the most recent results. On the other hand with the introduction of τ-tilting and silting theory [2,3], there has been a renewed interested in special biserial algebras and symmetric special biserial algebras, in particular, see [1,38,51,52].

For self-injective algebras, Brauer tree and Brauer graph algebras have been useful in the classification of group algebras and blocks of group algebras of finite and tame representation type [13,16,32,23] and the derived equivalence classification of self-injective algebras of tame representation type, see for example in [4,47] and the references within. In these classifications biserial and special biserial algebras have played an important role.

In this paper, we study two classes of algebras, *multiserial* and *special multiserial* algebras introduced in [49], that are mostly of wild representation type. These algebras generalize biserial and special biserial algebras. In fact, they contain the classes of biserial and special biserial algebras and we will see that they also contain the class of symmetric algebras with radical cube zero. One common feature of these classes is that their representation theory is largely controlled by the uniserial modules. The same is true for multiserial and special multiserial algebras.

We say that a module M over some algebra is *multiserial* if the radical of M is a sum of uniserial modules $U_1, \ldots U_l$ such that, if $i \neq j$, then $U_i \cap U_j$ is either (0) or a simple module. An algebra A is *multiserial* if, as a right and left module, A is multiserial. For the definition of a *special multiserial* algebra see Definition 2.2. We note that the definition of multiserial algebra as well as that of special multiserial algebra first appears in [49]. Subsequently multiserial algebras and rings have been studied in [36] with a focus on hereditary multiserial rings, and with a slightly more general definition of multiserial algebra, they appear in [34,39,12].

One of the main results of this paper is that any module M over a special multiserial algebra is multiserial. As a consequence, a special multiserial algebra is a multiserial algebra, generalizing the work of [48] on special biserial algebras. Since special multiserial algebras are, in general, wild, such a general result on the structure of modules is surprising.

Theorem A. *Let K be a field and let A be a special multiserial K-algebra. Then every finitely generated A-module is a multiserial module.*

Corollary. *Any special multiserial algebra is a multiserial algebra.*

In Section 3, we introduce the concept of a ring having an arrow-free socle. We show that every self-injective finite dimensional algebra has an arrow-free socle. For an algebra with an arrow-free socle, we show that a number of conditions are equivalent to the algebra being special multiserial.

In [30], Brauer configuration algebras were introduced. Their construction is based on combinatorial data, called a Brauer configuration, which generalizes Brauer graphs which in turn generalize Brauer trees. A Brauer configuration algebra is a finite dimensional symmetric algebra. Our next result shows that, over an algebraically closed field, Brauer configuration algebras and symmetric special multiserial algebras coincide.

Theorem B. *Let K be an algebraically closed field and let A be a K-algebra. Then A is a symmetric special multiserial algebra if and only if A is a Brauer configuration algebra.*

Another well-studied class of finite dimensional symmetric algebras is that of symmetric algebras with radical cube zero [9,11,25,26]. We prove that every symmetric algebra, over an algebraically closed field, with radical cube zero is a Brauer configuration algebra.

Theorem C. *Let K be an algebraically closed field. Then every symmetric K-algebra with Jacobson radical cube zero is a special multiserial algebra and, in particular, it is a Brauer configuration algebra.*

In proving Theorems A, B and C, we obtain many structural results on multiserial and special multiserial algebras.

The paper is outline as follows. In Section 2, we define multiserial modules, multiserial algebras and special multiserial algebras. We show that a module over a special multiserial algebra is multiserial. In Section 3 we define algebras with arrow-free socle and show properties of such algebras. Section 4 is on symmetric special multiserial algebras and we prove that an algebra is symmetric special multiserial if and only if it is a Brauer configuration algebra. Finally, in Section 5, we show that symmetric algebras with radical cube zero are special multiserial and hence, they are Brauer configuration algebras.

2. Modules over special multiserial algebras

Let K be a field and let A be a K-algebra. Unless explicitly stated otherwise, all modules considered are finitely generated right modules. Furthermore, let KQ/I be a finite dimensional algebra, for a quiver Q and an admissible ideal I. Denote by Q_0 the set of vertices of Q and by Q_1 the set of arrows in Q. By abuse of notation we sometimes view an element in KQ as an element in KQ/I if no confusion can arise.

Recall that a K-algebra A is *biserial* if for every indecomposable projective left or right module P, there are uniserial left or right modules U and V, such that $\mathrm{rad}(P) = U + V$ and $U \cap V$ is either zero or simple.

The algebra A is *special biserial* if it is Morita equivalent to an algebra of the form KQ/I where KQ is a path algebra and I is an admissible ideal such that the following properties hold

(S1) For every arrow a in Q there is at most one arrow b in Q such that $ab \notin I$ and at most one arrow c in Q such that $ca \notin I$.
(S2) At every vertex v in Q there are at most two arrows in Q starting at v and at most two arrows ending at v.

In particular, property (S2) implies that at every vertex there are at most two incoming and two outgoing arrows. A special multiserial algebra, as defined below, does not satisfy this property instead it only satisfies property (S1).

We now give the definitions of the two main concepts studied in this paper, namely multiserial algebras and special multiserial algebras (Definitions 2.1 and 2.2). These algebras were first defined and studied in [49].

Definition 2.1. Let A be a K-algebra.

1) We say that a left or right A-module is *multiserial* if $\mathrm{rad}(M)$ can be written as a sum of uniserial modules $U_1 \dots U_l$ such that, if $i \neq j$, then $U_i \cap U_j$ either (0) or a simple module.
2) The algebra A is *multiserial* if A, as a left or right A-module is multiserial.

Furthermore, we remark that a multiserial algebra, that satisfies the additional property that the radical is a sum of at most two uniserial module whose intersection is zero or simple (on the left and on the right) is a biserial algebra.

Recall from [49] the definition of special multiserial algebras.

Definition 2.2. Let A be a finite dimensional algebra. We say that A is a *special multiserial algebra* if A is Morita equivalent to a quotient KQ/I of a path algebra KQ by an admissible ideal I such that the following property holds

(M) For every arrow a in Q there is at most one arrow b in Q such that $ab \notin I$ and at most one arrow c in Q such that $ca \notin I$.

We note that the definition of an algebra being special multiserial is left-right symmetric. The following is the main result of this section.

Theorem 2.3. *Let K be a field, A a special multiserial K-algebra, and M a finitely generated A-module. Then M is multiserial.*

Theorem 2.3 has as an immediate consequence the following corollary.

Corollary 2.4. *Let A be a special multiserial K-algebra. Then A is a multiserial algebra.*

In the case that A is a biserial algebra, we obtain as a Corollary to Theorem 2.3 the following result due to Skowroński and Waschbüsch.

Corollary 2.5. *[48] Let A be a special biserial K-algebra. Then A is a biserial algebra.*

Before proving Theorem 2.3, we present a series of Lemmas that we will use in its proof. We start with a very general Lemma.

Lemma 2.6. *Let W, T, U, V be Λ-modules for a ring Λ such that there is a commutative exact diagram*

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "incl"] \arrow[d] & T \arrow[r, "f"] \arrow[d, "g"] & U \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{soc}(V) \arrow[r, "incl"] \arrow[d] & V \arrow[r] \arrow[d] & U \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

Then $W \supseteq \mathrm{soc}(T)$. In particular, if W is semisimple then $W = \mathrm{soc}(T)$.

Proof. Let S be a simple submodule of T. Suppose that $f(S) \neq 0$. Then $g(S) \nsubseteq \mathrm{soc}(V)$. But $g(S)$ is a simple submodule of V, and hence in $\mathrm{soc}(V)$, a contradiction. $\square$

From now on let $A = KQ/I$ be a special multiserial algebra. For $a, b \in Q_1$ such that the vertex at which a ends equals the vertex at which b starts, we use the convention that ab is the path starting with a followed by b. If M is a right A-module, we say that an element $m \in M$ is *right uniform*, if there is some vertex $v \in Q_0$ such that $me_v = m$ where e_v is the trivial path at vertex v.

By the left-right symmetry of the definition of special multiserial it is sufficient to prove Theorem 2.3 for right modules. The following result was proved, in the special case of modules of the form aA where a is an arrow, in [49].

Lemma 2.7. *Let M be an A-module and let $m \in M$ be right uniform. For $a \in Q_1$, the module generated by ma is uniserial.*

Proof. If $ma = 0$ the result follows. If $ma \neq 0$ there exists at most one arrow b such that $mab \neq 0$. If $mab \neq 0$, there exists at most one arrow c such that $mabc \neq 0$. Continuing in this fashion we see that the submodule generated by ma is uniserial. □

We note that for a general algebra Λ if M is a Λ-module such that $\mathrm{rad}^2(M) = 0$ then $\mathrm{rad}(M)$ is semisimple and thus M is a multiserial module.

Lemma 2.8. *Let M be an A-module with $\mathrm{rad}^2(M) \neq 0$. Then there exist right uniform elements $u_1, \dots, u_t$ in $\mathrm{rad}(M) \setminus \mathrm{rad}^2(M)$ such that*

1) $u_i = m_i a_i$ for some right uniform elements $m_i \in M \setminus \mathrm{rad}(M)$ and $a_i \in Q_1$,
2) $u_i A$ is a uniserial module,
3) $\sum_{i=1}^{t} u_i A = \mathrm{rad}(M)$,
4) $\mathrm{rad}(M)/\mathrm{rad}^2(M) \simeq \bigoplus \pi(u_i)(A/\mathrm{rad}(A))$ where $\pi : \mathrm{rad}(M) \to \mathrm{rad}(M)/\mathrm{rad}^2(M)$ is the canonical surjection.

Proof. Choose a set of right uniform generators of M. Right multiplying these elements by arrows in Q yields a generating set of $\mathrm{rad}(M)$ consisting of right uniform elements.

Applying π to this set we get a generating set of $\mathrm{rad}(M)/\mathrm{rad}^2(M)$, so we may select $u_1, \dots, u_t$ so that $\pi(u_1), \dots, \pi(u_t)$ is a minimal generating set of $\mathrm{rad}(M)/\mathrm{rad}^2(M)$ which is a semi-simple module.

Each $u_i = m_i a_i$ for some right uniform $m_i \in M \setminus \mathrm{rad}(M)$ and $a_i \in Q_1$. By Lemma 2.7 $u_i A$ is a uniserial module. Parts (3) and (4) are clear from the construction. □

Define the following partial order on the set of paths in Q. For p, p' paths in Q, we say $p \geq p'$, if $p = qp'$ for some path q in Q. The following Lemma follows directly from condition (M).

Lemma 2.9. *Let p and p' be paths in Q and $a \in Q_1$. If $pa \neq 0$ and $p'a \neq 0$ then either $p \geq p'$ or $p' > p$. Hence p and p' are comparable.*

For M an A-module, let $\ell(M)$ be the number of non-zero terms in a composition series of M. We introduce the set of short exact sequences satisfying the properties P1)–P3) which are defined below. Let

$$\mathcal{S}_M = \{\varepsilon : 0 \to L \to \bigoplus_i U_i \to \operatorname{rad}(M) \to 0 \mid \varepsilon \text{ is a short exact sequence satisfying P1)–P3) defined below}\}.$$

P1) Each U_i is a uniserial submodule of $\operatorname{rad}(M)$.
P2) $\sum_i U_i = \operatorname{rad}(M)$.
P3) The map $\bigoplus U_i/\operatorname{rad}(U_i) \to \operatorname{rad}(M)/\operatorname{rad}^2(M)$ induced from the map $\bigoplus U_i \to \operatorname{rad}(M)$ is an isomorphism.

Note that by Lemma 2.8, the set $\mathcal{S}_M$ is not empty. Let

$$\alpha = \min\{\ell(L) \mid 0 \to L \to \bigoplus_i U_i \to \operatorname{rad}(M) \to 0 \in \mathcal{S}\}$$

and let

$$\mathcal{S}_M^* = \{\varepsilon \in \mathcal{S}_M \mid \ell(L) = \alpha\}.$$

For a general ring Λ and Λ-modules $N_1, \dots, N_t$, the *support of an element* n, where $n = \sum n_i \in \bigoplus_i N_i$, with $n_i \in N_i$, is the set of all i such that $n_i \neq 0$ and is denoted by $\operatorname{supp}(n)$. The cardinality $|\operatorname{supp}(n)|$ of $\operatorname{supp}(n)$ is the number of components for which $n_i \neq 0$.

Lemma 2.10. *Let $0 \to L \to \bigoplus_i U_i \to \operatorname{rad}(M) \to 0 \in \mathcal{S}_M^*$, $x \in L$ with $x = \sum \lambda_i u_i p_i$, $\lambda_i \neq 0 \in K$, $u_i \in U_i \setminus \operatorname{rad}(U_i)$ and p_i a non-zero path in A, for all i. If for some i and j with $i \neq j$, $p_i \geq p_j$ then there exists, $0 \to L' \to \bigoplus_i U_i' \to \operatorname{rad}(M) \to 0 \in \mathcal{S}_M^*$ such that the following diagram commutes*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "g"] & \bigoplus_i U_i \arrow[r] \arrow[d, "f"] & \operatorname{rad}(M) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & L' \arrow[r] & \bigoplus_i U_i' \arrow[r] & \operatorname{rad}(M) \arrow[r] & 0 \end{tikzcd}$$

where g and f are isomorphisms and where $|\operatorname{supp}(g(x))| < |\operatorname{supp}(x)|$. Note that if $x \notin \operatorname{soc}(L)$ then $g(x) \notin \operatorname{soc}(L')$.

Proof. Let $p_i \geq p_j$ for some i and j. Then $p_i = qp_j$ for some path q. Set $u_l' = u_l$ for all $l \neq j$. Then set $u_j' = u_j + \frac{\lambda_i}{\lambda_j} u_i q$. Let $U_l' = u_l' A$ for all l. It is immediate that $\oplus_l U_l = \oplus_l U_l'$. In fact, $\sum \gamma_l u_l x_l = \sum \gamma_l u_l' x_l - \gamma_j \frac{\lambda_i}{\lambda_j} u_i' q x_j$ where $\gamma_l \in K$ and x_l are paths. Define a map $f : \bigoplus U_l \to \bigoplus U_l'$ given by $f(\sum \gamma_l u_l x_l) = \sum \gamma_l u_l' x_l - \gamma_j \frac{\lambda_i}{\lambda_j} u_i' q x_j$ where $\gamma_l \in K$ and

x_l are paths. Then f just changes generating sets and hence is an isomorphism. Now $f(x) = f(\sum \lambda_l u_l p_l) = \sum \lambda_l u'_l p_l - \lambda_i u'_i p_i = \sum_{l \neq i} \lambda_l u'_l p_l$. Hence $|\mathrm{supp}(f(x))| < |\mathrm{supp}(x)|$. Since g is a restriction of f, $x \in L$ and $g(x) \in L'$, we have $|\mathrm{supp}(g(x))| < |\mathrm{supp}(x)|$. The statement on the socles follows from the fact that g is an isomorphism. $\square$

Our final lemma is the following:

Lemma 2.11. *Suppose there exists a short exact sequence*

$$0 \to L \to \bigoplus_i U_i \to \mathrm{rad}(M) \to 0$$

in $\mathcal{S}_M$, in particular, this implies that the U_i are uniserial submodules of M satisfying P1)–P3) *above. Suppose further that L is semisimple. Then M is multiserial.*

Proof. If M is not a multiserial module, then for every choice of uniserial modules $U_1, \ldots, U_t$ such that $\sum_i U_i = \mathrm{rad}(M)$, for some $i \neq j$, $U_i \cap U_j$ is neither 0 nor a simple module.

But $U_i \cap U_j$ is isomorphic to a submodule of the kernel of the canonical surjection $\bigoplus_i U_i \to \mathrm{rad}(M)$. Hence, since $U_i \cap U_j$ is a submodule of a uniserial module, it follows that $\mathrm{rad}(U_i \cap U_j)$ is nonzero. Thus, the kernel of any map $\bigoplus_i U_i \to \mathrm{rad}(M)$ is never semisimple. The result follows. $\square$

Proof of Theorem 2.3: Without loss of generality, we may assume that M is indecomposable. If $\mathrm{rad}^2(M) = 0$, we have seen that the result is true. Assume that $\mathrm{rad}^2(M) \neq 0$. If there exists $0 \to L \to \bigoplus_i U_i \to \mathrm{rad}(M) \to 0 \in \mathcal{S}^*_M$ such that L is semi-simple then the result follows from Lemma 2.11.

Suppose no such L exists, that is, L is not semi-simple for every short exact sequence in $\mathcal{S}^*_M$. We will now show that this leads to a contradiction. Consider the set

$$\mathcal{X} = \{x \mid \text{ there is a s.e.s } 0 \to L \to \bigoplus_i U_i \to \mathrm{rad}(M) \to 0 \text{ in } \mathcal{S}^*_M \text{ and } x \in L \setminus \mathrm{soc}(L)\}.$$

Let $\mathcal{X}_{min} = \{x \in \mathcal{X} \mid |\mathrm{supp}(x)| \text{ is minimal in } \{|\mathrm{supp}(y)|, y \in \mathcal{X}\}\}$.

Let $x \in \mathcal{X}_{min}$ where $0 \to L \to \bigoplus_i U_i \to \mathrm{rad}(M) \to 0$ and $x \in L \setminus \mathrm{soc}(L)$. Then $x = \sum \lambda_i u_i p_i$ with $\lambda_i \in K$ and with $u_i \in U_i \setminus \mathrm{rad}(U_i)$. Note that by choice of x the number of non-zero $\lambda_i u_i p_i$ is as small as possible.

By Lemma 2.10 and minimality of $|\mathrm{supp}(x)|$, all the p_i are incomparable in the partial order on the paths in KQ defined earlier. Since $x \notin \mathrm{soc}(L)$, there exists an arrow $a \in Q_1$ such that $xa \neq 0$.

Since $xa \neq 0$ there exists $u_i p_i a \neq 0$ for some i. If for some $j \neq i$, $u_j p_j a \neq 0$ then $p_i a \neq 0$ and $p_j a \neq 0$ and hence by Lemma 2.9, p_i is comparable to p_j. Then by Lemma 2.10 we obtain a contradiction to the minimality of $|\mathrm{supp}(x)|$. Hence $xa = \lambda_i u_i p_i a$. So we have $xa \in L \cap U_i$ for exactly one i.

Now consider

$$\begin{tikzcd}
 & 0 & 0 & & \\
(1) \quad 0 \arrow[r] & L/(L\cap U_i) \arrow[r] \arrow[u] & U_i/(L\cap U_i)\oplus(\bigoplus_{j\neq i}U_j) \arrow[r] \arrow[u] & \mathrm{rad}(M) \arrow[r] & 0 \\
(2) \quad 0 \arrow[r] & L \arrow[r] \arrow[u] & \bigoplus_i U_i \arrow[r] \arrow[u] & \mathrm{rad}(M) \arrow[r] \arrow[u] & 0 \\
 & L\cap U_i \arrow[r, "="] \arrow[u] & L\cap U_i \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Since the short exact sequence in line (2) is in $\mathcal{S}^*_M$, clearly the one in line (1) is in $\mathcal{S}_M$. But $\ell(L/(L\cap U_i) < \ell(L)$ thus contradicting the minimality of L and so the assumption that L is not semi-simple is false and the result follows. $\square$

More structural results on the uniserial modules over a special multiserial algebra. Due to the importance of uniserial modules, we end this section with a few structural results about such modules over a special multiserial algebra. In Section 4 we show that a symmetric special multiserial algebra is a Brauer configuration algebra. In this case, further structural results on the uniserial modules can be found in [30].

Let $A = KQ/I$ be a finite dimensional special multiserial algebra, let J be the ideal in KQ generated by the arrows in Q, and let $N \geq 2$ be an integer such that $J^N \subseteq I \subseteq J^2$. If $x \in KQ$, we denote its image in A by $\bar{x}$. If p is a path in Q, the *length* $\ell(p)$ *of* p is the number of arrows in p.

For every arrow in Q, we now define a set of paths starting or ending with that arrow. Let a be an arrow in Q and let i be a non-negative integer. We set $p_0(a) = a$ and we define $p_i(a) = aq$ where q is the unique path in Q of length i such that $aq \notin I$ if such a path q exists. If no such path exists, then $p_i(a)$ is not defined. Thus $p_1(a) = ab$ for a unique arrow b such that $ab \notin I$ if such an arrow b exists and $p_2(a) = abc$ for a unique arrows b and c such that $abc \notin I$ if such b and c exist. Now define $p_{-j}(a) = qa$ where q is the unique path of length j in Q such that $qa \notin I$ if such a q exists. Again the uniqueness of q, if it exists, follows from (M). Let $t(a)$ denote the largest non-negative integer i such that $p_i(a) \notin I$. Let $s(a)$ be the largest non-negative integer j such that $p_{-j}(a) \notin I$. Note that $0 \leq s(a), t(a) \leq N-1$. Furthermore, $\overline{p_{t(a)}(a)}$ is in the right socle of A and $\overline{p_{-s(a)}(a)}$ is in the left socle of A.

The next lemma provides a number of results about these paths.

Lemma 2.12. *Let $A = KQ/I$ be a special multiserial K-algebra.*

(1) Suppose that q, q' are paths in KQ and a is an arrow in Q such that $qaq' \notin I$. Then $qa = p_{-\ell(q)}(a)$ and $aq' = p_{\ell(q')}(a)$.
(2) Suppose that $q = a_1a_2 \cdots a_{i-1}a_ia_{i+1} \cdots a_r$ and $q' = b_1 \cdots b_{i-1}a_ib_{i+1} \cdots b_r$ are paths in KQ such that $q \notin I$ and $q \neq q'$. Then $q' \in I$.
(3) For $0 \leq i \leq j \leq t(a)$, $p_i(a)q = p_j(a)$, for some path q.
(4) For $0 \leq i \leq j \leq s(a)$ $qp_{-i}(a) = p_{-j}(a)$, for some path q.

Proof. The proof is just repeated applications of condition (M). □

We now examine the structure of uniserial modules defined by an arrow in Q.

Lemma 2.13. *Let $A = KQ/I$ be a special multiserial algebra and let a be an arrow in Q. Set $U_a = aA$. Then*

(1) The A-module U_a is uniserial.
(2) The A-module $U_a/\operatorname{soc}(U_a)$ is uniserial.
(3) We have $\operatorname{soc}(U_a) = U_a \cap \operatorname{soc}(A)$.
(4) We have $\operatorname{rad}(A) = \sum_{b \in Q_1} U_b$.
(5) The set $\{\overline{p_0(a)}, \overline{p_1(a)}, \ldots, \overline{p_{t(a)}(a)}\}$ is a K-basis of U_a.

Proof. Let $a \in Q_1$. If U_a is a simple A-module, part (1) follows. Assume that U_a is not a simple module. By condition (M), there is at most one arrow b such that $ab \notin I$. It follows that $U_a/U_a \operatorname{rad}(A)$ and $U_a \operatorname{rad}(A)/U_a \operatorname{rad}^2(A)$ are both simple modules. Continuing in this fashion proves part (1).

Part (2) follows from part(1).

Part (3) follows from the observation that $p_{t(a)}(a)$ is in the right socle of A and that it is also in the socle of U_a.

Part (4) holds since $\sum_{b \in Q_1} U_b$ is the right submodule of A generated by all arrows in Q. Hence $\sum_{b \in Q_1} U_b = J/I = \operatorname{rad}(A)$, where J is the ideal in KQ generated by the arrows of Q.

We now prove part (5). It is clear that $\{\overline{p_0(a)}, \overline{p_1(a)}, \ldots, \overline{p_{t(a)}(a)}\}$ generates U_a. So we are left to show that if $\sum_{i=0}^{t(a)} \lambda_i p_i(a) \in I$, with $\lambda_i \in K$, then $\lambda_i = 0$ for all i. Suppose for contradiction that there is an integer i, $0 \leq i \leq t(a)$ such that $\lambda_i \neq 0$. Let i_0 be the smallest such i.

By Lemma 2.12 (4), $p_j(a) = p_{i_0}(a)q_j$, for $j \geq i_0$ and some path q_j of length $j - i_0$ starting at the vertex, w, at which $p_{i_0}(a)$ ends. Thus

$$\sum_{i=0}^{t(a)} \lambda_i p_i(a) = p_{i_0}(a) \sum_{j=i_0}^{t(a)} \lambda_j q_j.$$

Let e_w be the idempotent in KQ associated to w. Then, noting that if $j = i_0$, q_j is of length 0 and hence $q_j = e_w$, we have

$$\sum_{j=i_0}^{t(a)} \lambda_j q_j = \lambda_{i_0} e_w + \sum_{j=i_0+1}^{t(a)} \lambda_j q_j.$$

Let $x = \sum_{j=i_0+1}^{t(a)} \lambda_j q_j$. We have $x \in J$ and since $J^N \subset I$, there is some $y \in KQ$ such that

$$(\lambda_{i_0} e_w + x)y + I = e_w + I$$

where y is obtained as follows: $(\lambda_{i_0} e_w + x)(\lambda_{i_0}^{-1} e_w - x) = e_w - x^2$. Then $(e_w - x^2)(e_w + x^2) = e_w - x^4$ and continuing in this fashion we finally obtain $(\lambda_{i_0} e_w + x)(\lambda_{i_0}^{-1} e_w - x)(e_w + x^2)(e_w + x^4) \cdots = e_w - x^{2n}$ and $x^{2n} \in I$. Hence

$$\sum_{i=0}^{t(a)} \lambda_i p_i(a) y + I = p_{i_0}(a) + I.$$

But by assumption $\sum_{i=0}^{t(a)} \lambda_i p_i(a) \in I$ and hence $p_{i_0}(a) \in I$, a contradiction. This completes the proof. $\square$

3. Algebras with arrow-free socles

In this section we introduce the concept of an algebra with arrow-free socle. We show that the socle of a self-injective algebra of radical series with length at least 3 is arrow-free. We also show that for an algebra with arrow-free socle, condition (M) is equivalent to a stronger condition (M′) defined below.

We fix the following notation. We let $A = KQ/I$ be an indecomposable finite dimensional algebra with I an admissible ideal in the path algebra KQ. Denote by $\pi : KQ \to A$ the canonical surjection and let $\bar{a} = \pi(a)$, for $a \in Q_1$.

Definition 3.1. We say that the socle of A is *arrow-free* if, for each $a \in Q_1$, we have $\bar{a} \notin \operatorname{soc}({}_AA)$ and $\bar{a} \notin \operatorname{soc}({}_AA)$ where ${}_AA$ denotes the left A-module A and A_A the right A-module A.

We first show that the socle of a self-injective algebra is arrow-free.

Proposition 3.2. *Let A be self-injective and* $\operatorname{rad}^2(A) \neq 0$. *Then the socle of A is arrow-free.*

Proof. Suppose $\bar{a} \in \operatorname{soc}(A_A)$ and suppose that a is an arrow from a vertex v to a vertex w in Q. If $v = w$ then A is isomorphic to $K[x]/(x^2)$ since A is self-injective.

Now suppose that $v \neq w$. If there is another arrow starting at v, by multiplying by arrows we would obtain a path $p \neq a$ such that $\bar{p}$ is a non-zero element in $\mathrm{soc}(A_A)$. Since A is self-injective we get $p - \lambda a \in I$, for some non-zero $\lambda \in K$, contradicting that I is admissible. Thus a is the only arrow starting at v. Suppose that b is an arrow ending at v. If $ba \notin I$ then $\overline{ba} \in \mathrm{soc}(A_A)$ and hence, for some $\lambda \neq 0$, we have $ba - \lambda a \in I$, which contradicts that I is admissible. Thus $ba \in I$ and $\bar{b} \in \mathrm{soc}(A_A)$. Continuing in this fashion, since A is indecomposable, we see that every arrow is in $\mathrm{soc}(A_A)$ and thus $\mathrm{rad}^2(A) = 0$. □

We note that the converse does not hold in general.

The following Lemma follows immediately from the definition of arrow-free.

Lemma 3.3. *If the socle of A is arrow-free then for all arrows a in Q there are arrows b and c such that $ab \notin I$ and $ca \notin I$.*

From condition (M) it follows that understanding the paths of length 2 is crucial. Set

$$\Pi = \{ab | a, b \in Q_1, ab \notin I\}.$$

We say that a cycle C is *basic* if there are no repeated arrows in C. We say that a set $\{C_1, \ldots, C_r\}$ of basic cycles is *special* if the following conditions hold

(1) for each arrow a in Q, a occurs in exactly one C_i, $i = 1, \ldots r$,
(2) the path ab is in Π if and only if ab is a subpath of some cycle $C_i = c_1 \ldots c_n$ where we consider $c_n c_1$ a subpath of C_i.

We show that for an algebra with arrow-free socle the following condition is equivalent to condition (M). Set condition

(M′) For every arrow a in Q there exists exactly one arrow b in Q such that $ab \notin I$ and exactly one arrow c in Q such that $ca \notin I$.

Proposition 3.4. *Let $A = KQ/I$ be a finite dimensional indecomposable algebra with I an admissible ideal of KQ. Suppose that the socle of A is arrow-free. Then the following are equivalent*

(1) Condition (M) holds, that is, A is special multiserial.
(2) Condition (M′) holds.
(3) The map $\varphi : \Pi \to Q_1$ given by $\varphi(ab) = a$ is bijective.
(4) The map $\psi : \Pi \to Q_1$ given by $\psi(ab) = b$ is bijective.
(5) There exists a special set of cycles.

Proof. The implication (1) implies (2) follows from Lemma 3.3.

To see that (2) implies (3), let $ab \in \Pi$. Then by (M′), b is unique and hence φ is one-to-one and well-defined. Again by (M′), given $a \in Q_1$ there exists $b \in Q_1$ such that $ab \in \Pi$ and hence φ is onto.

For the implication (3) implies (2), let $a \in Q_1$. By (3) there exists a unique arrow b such that $ab \notin I$. Now by Lemma 3.3 there exists an arrow c_1 such that $c_1 a \notin I$. Again by Lemma 3.3 there exists an arrow c_2 such that $c_2 c_1 \notin I$. Continue in this way until the first repeat of an arrow, that is, we have some path $c_n \dots c_s \dots c_1 c_0$ where $c_0 = a$. If the first repeat is $c_n = c_s$, we show that $s = 0$. If not then since φ is bijective, we have $c_{n-1} = c_{s-1}$, a contradiction. It now follows that c_1 is unique and (2) follows.

That (2) is equivalent to (4) is similar to the equivalence between (2) and (3).

Next we show that (2) implies (5). Let $a = a_0 \in Q_1$. Then there exists a unique $a_1 \in Q_1$ such that $a_0 a_1 \notin I$ and there exists a unique $a_2 \in Q_1$ such that $a_1 a_2 \notin I$. Continue in this way until the first repeat of an arrow to obtain a sequence of arrows $a_0 \dots a_n$. As above we have $a_n = a_0$. So $a_0 \dots a_{n-1}$ is a basic cycle C_1. If there is some arrow b_0 such $b_0 \neq a_i$, for $i = 1, \dots, n$, then continue in the same fashion to obtain a cycle $C_2 = b_0 \dots b_m$. By uniqueness, no $b_i = a_j$. Either all the arrows occur in C_1 and C_2 or we can continue this process and construct a C_3. Eventually one obtains a special set $\{C_1, \dots, C_r\}$ of special cycles.

Finally we prove that (5) implies (1). Let $a \in Q_1$. By the definition of a special set of cycles $\{C_1, \dots, C_r\}$, there exists an i such that $a \in C_i$. The second part of the definition of a special set of cycles implies that there exists unique arrows b and c such that $ab \notin I$ and $ca \notin I$. $\square$

Remark 3.5. (1) The above Proposition does neither assume that the algebra is self-injective nor that it is special multiserial.

(2) Suppose that A is special multiserial and arrow-free. If there are paths p, q in KQ with $\ell(p) \geq \ell(q)$ and $a \in Q_1$ such that $pa \notin I$ and $qa \notin I$ then there exists a unique path r such that $rq = p$.

4. Symmetric special multiserial algebras and Brauer configuration algebras

In this section we study special multiserial algebras that have the additional property of being symmetric algebras. In the case of symmetric special biserial algebras, it is proved in [43,44] that the class of symmetric special biserial algebras coincides with the class of Brauer graph algebras. We will show in this section that an analogous results holds for symmetric special multiserial algebras. Namely, the main result of this section is to show that the class of symmetric special multiserial algebras coincides with the class of Brauer configuration algebras. Brauer configuration algebras have been defined in [30] and they can be seen as generalizations of Brauer graph algebras. We will recall their definition below. Note that in the present paper, we assume all Brauer configurations to be reduced.

Before recalling definitions and further analysing the structure of symmetric special multiserial algebras, we first state the main result of this section.

Theorem 4.1. *Let $A = KQ/I$ be an indecomposable finite dimensional algebra over an algebraically closed field K such that I is an admissible ideal and* $\mathrm{rad}(A)^2 \neq 0$*. Then A is a symmetric special multiserial algebra if and only if A is a Brauer configuration algebra.*

4.1. Definition of Brauer configuration algebras

We recall from [30] the definition of a (reduced) Brauer configuration algebra. We start with the definition of a Brauer configuration, which generalizes a Brauer graph. A *Brauer configuration* Γ is a tuple $(\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$, where

(1) Γ_0 is a finite set of elements called *vertices*.
(2) Γ_1 is a finite collection of finite multisets of vertices which are called *polygons*. Recall that a multiset is a set where elements can occur multiple times.
(3) $\mu \colon \Gamma_0 \to \{1, 2, 3, \dots\}$ is a set function called the *multiplicity function*.
(4) A vertex α is called *truncated* if it occurs once in exactly one polygon and $\mu(\alpha) = 1$. The sum over the polygons $V \in \Gamma_1$ of the number of times a vertex α occurs in V is denoted $\mathrm{val}(\alpha)$. We say $\mathfrak{o}$ is an *orientation* which means that, for each nontruncated vertex α, there is a chosen cyclic ordering of the polygons that contain α, counting repetitions. See the example and the discussion below.

We require that $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ satisfies

C1. Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1.
C2. Every polygon in Γ_1 has at least two vertices.
C3. Every polygon in Γ_1 has at least one vertex α such that $\mathrm{val}(\alpha)\mu(\alpha) > 1$.
C4. If α is a vertex in polygon V and $\mathrm{val}(\alpha)\mu(\alpha) = 1$, that is, α is truncated then V is a 2-gon.

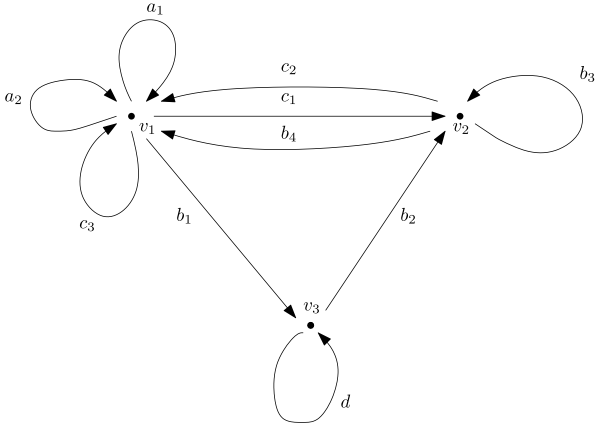
We note that C4 does not occur in the definition of a Brauer configuration in [30]. In that paper a Brauer configuration was called *reduced* if it satisfied C4. In this paper, all Brauer configurations are "reduced".

Example 4.2. We give an example of a Brauer configuration. Let $\Gamma_0 = \{1, 2, 3, 4\}$, $\Gamma_1 = \{V_1, V_2, V_3\}$ where $V_1 = \{1, 1, 2, 3, 3\}$, $V_2 = \{2, 2, 3\}$ and $V_3 = \{2, 4\}$, and $\mu(i) = 1$, except that $\mu(1) = 3$ and $\mu(4) = 2$. To give an orientation, for each nontruncated vertex, we need to be given a cyclic order of the polygons that contain the vertex. If a vertex occurs in a polygon more than once, we will use superscripts to denote these occurrences. Thus for vertex 1, we need to order $V_1^{(1)}, V_1^{(2)}$, for vertex 2, we must order $V_1, V_2^{(1)}, V_2^{(2)}, V_3$, for vertex 3 we must order $V_1^{(1)}, V_1^{(2)}, V_2$, etc. So for vertex 1 we must have $V_1^{(1)} < V_1^{(2)}$, and to make it cyclic, we implicitly have $V_1^{(2)} < V_1^{(1)}$. For vertex 2, there are many choices of cyclic orderings, and for example, we will use $V_1 < V_3 < V_2^{(1)} < V_2^{(2)}$, and to make it cyclic, we implicitly have $V_2^{(2)} < V_1$. Note that equivalently we could have taken any cyclic permutation of $V_1 < V_3 < V_2^{(1)} < V_2^{(2)}$. For vertex 3, take $V_1^{(1)} < V_1^{(2)} < V_2$ or a cyclic permutation of this; vertex 4, since $\mu(4) = 2$ is not truncated and we take the cyclic ordering to be just V_3 (with implicitly $V_3 < V_3$).

We call $V_{i_1} < V_{i_2} < \cdots V_{i_m}$ a *successor sequence of* V_{i_1} *at* α if α is a vertex in Γ_0 and $V_{i_1} < V_{i_2} < \cdots V_{i_m}$ is a cyclic ordering, obtained from the orientation $\mathfrak{o}$, of the polygons containing α as an element. In this case, we say the $V_{i_{j+1}}$ is the successor of V_{i_j} at α, for $j = 1, \ldots, m$ where $V_{i_{m+1}} = V_{i_1}$.

Fix a field K. We now define the Brauer configuration algebra A, associated to a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ via a quiver with relations. That is, we will define a quiver Q and a set of relations ρ in the path algebra KQ such that A is isomorphic to KQ/I, where I is the ideal generated by ρ. The vertex set of Q is in one-to-one correspondence with Γ_1, the set of polygons of Γ. If V is a polygon in Γ_1, we will denote the associated vertex in Q by v. If the polygon V is a successor to the polygon V' at α, there is an arrow from v to v', where v is the vertex in Q associated to V and v' is the vertex in Q associated to V' in Q. This gives a one-to-one correspondence between the set of successors in Γ and the arrow set in Q.

Example 4.3. The quiver Q of Example 4.2 is



Here a_1 corresponds to $V_1^{(2)}$ being a successor of $V_1^{(2)}$ and a_2 corresponds to $V_1^{(1)}$ being a successor of $V_1^{(2)}$. The arrows labelled b_1, b_2, b_3, b_4 correspond to the successor sequence at vertex 2 of Γ. The arrows labelled c_1, c_2, c_3 correspond to the successor sequence at vertex 3 of Γ. Finally, the arrow labelled d corresponds to the successor sequence at vertex 4 of Γ.

Before we define a generating set for the ideal of relations of A, we introduce the definition of a special α-cycle at v, for a nontruncated vertex α in Γ_0 and vertex $v \in (Q_\Gamma)_0$. Let $V_1 < V_2 < \ldots < V_{\mathrm{val}(\alpha)}$ be the successor sequence of V_1 at α. For each j, let $C_j = a_j a_{j+1} \cdots a_{\mathrm{val}(\alpha)} a_1 \cdots a_{j-1}$ be the cycle in $\mathcal{Q}_\Gamma$, where the arrow a_r corresponds to the polygon V_{r+1} being the successor of the polygon V_r at the vertex α. Let V be the polygon in Γ_1 associated to the vertex $v \in (Q_\Gamma)_0$ and suppose that α occurs t times in V, with $t \geq 1$. Then there are t indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. We define the *special α-cycles at v* to be the cycles $C_{i_1}, \ldots, C_{i_t}$. We remark that each C_{i_j} is a cycle in Q_Γ, beginning and ending at the vertex v. Note that if α occurs only once in V, then there is only one special α-cycle at v. Furthermore, if V' is a polygon consisting of n vertices, counting repetitions, then there are a total of n different special α'-cycles at v', for $\alpha' \in (Q_\Gamma)_0$. We will sometimes write *special α-cycle* (omitting the vertex) or simply *special cycle* omitting both α and v, when no confusion can arise.

Example 4.4. Continuing the example, the special 1-cycles at v_1 are a_1a_2 and a_2a_1, the special 2-cycle at v_1 is $b_1b_2b_3b_4$, the special 3-cycles at v_1 are $c_1c_2c_3$, and $c_3c_1c_2$, and there are no special 4-cycles at v_1. There are no special 1-cycles at v_2, the special 2-cycles at v_2 are $b_3b_4b_1b_2$ and $b_4b_1b_2b_3$, the special 3-cycle at v_2 is $c_2c_3c_1$, and there are no special 4-cycles at v_2. Finally there are no special i-cycles at v_3 for $i = 1, 3$. The special 2-cycles at v_3 is $b_2b_3b_4b_1$ and the special 4-cycle at v_3 is d.

There are three types of relations forming the generating set of relations ρ_Γ.

Relations of type one. For each polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of nontruncated vertices α_i and α_j in V, ρ_Γ contains all relations of the form $C^{\mu(\alpha_i)} - (C')^{\mu(\alpha_j)}$ or $(C')^{\mu(\alpha_j)} - C^{\mu(\alpha_i)}$ where C is a special α_i-cycle at v and C' is a special α_j-cycle at v.

Relations of type two. The type two relations are all paths of the form $C^{\mu(\alpha)}a_1$ where $C = a_1 \cdots a_m$ is a special α-cycle for some vertex α.

Relations of type three. These relations are quadratic monomial relations of the form ab in KQ_Γ where ab is not a subpath of any cycle C where C is a special cycle.

Definition 4.5. Let K be a field and Γ a Brauer configuration. The *Brauer configuration algebra Λ_Γ associated to* Γ is defined to be KQ_Γ/I_Γ, where Q_Γ is the quiver associated to Γ and I_Γ is the ideal in KQ_Γ generated by the set of relations ρ_Γ of types one, two and three.

Example 4.6. Continuing our example, some type one relations are $(a_1a_2)^3 - (a_2a_1)^3$, $(a_1a_2)^3 - b_1b_2b_3b_4$, $(a_2a_1)^3 - b_1b_2b_3b_4$, $(a_1a_2)^3 - c_1c_2c_3$, $c_1c_2c_3 - c_3c_1c_2$, $b_2b_3b_4b_1 - d^2$. Some type two relations are $(a_1a_2)^3a_1$, $(a_2a_1)^3a_2$, $b_1 \cdots b_4b_1$, $c_1c_2c_3c_1$, and d^3. Some type three relations are any a_ib_j, b_ja_i, a_ic_k, a_id, db_j, for $1 \leq i \leq 2, 1 \leq j \leq 4, 1 \leq k \leq 3$. The set of relations of types one, two, and three generate the ideal of relations, but this set contains a large number of redundant relations and the set is usually not a minimal generating set for the ideal of relations.

4.2. Properties of symmetric special multiserial algebras

Before proving Theorem 4.1, we analyse the structure of symmetric special multiserial algebras in more detail. For this we do not necessarily need to assume that K is algebraically closed.

From now on and for the remainder of Section 4, in addition to our previous assumptions, we always assume that all K-algebras considered are indecomposable. In fact we assume for the rest of this subsection, unless otherwise stated, that $A = KQ/I$ is an indecomposable symmetric special multiserial algebra where I is an admissible ideal and where $\mathrm{rad}(A)^2 \neq 0$.

Remark 4.7. Since A is symmetric, it is self-injective, and therefore by Proposition 3.2 its socle is arrow-free. So in particular, the condition (M′) holds for A.

For the special multiserial algebra A, the additional property of being symmetric implies the existence of a permutation on the set of arrows of Q. Namely, let $\pi\colon KQ \to A$ be the canonical surjection. Suppose that a is an arrow in Q. Since $\mathrm{rad}(A)^2 \neq 0$ and A is symmetric, $\pi(a) \notin \mathrm{soc}(A)$ and hence there is some arrow b such that $ab \notin I$. By condition (M), b is unique. This leads to the following definition.

Definition 4.8. Let $A = KQ/I$ be an indecomposable symmetric special multiserial algebra where I is an admissible ideal and such that $\mathrm{rad}(A)^2 \neq 0$. Letting $\sigma(a)$ denote the unique arrow such that $a\sigma(a) \notin I$, the assignment $\sigma : Q_1 \to Q_1$ given by $a \mapsto \sigma(a)$ defines a permutation on the set of arrows Q_1 of Q. We call it the *permutation induced by* I.

We remark that a similar construction has been observed by S. Ladkani in the context of Brauer configuration algebras.

Let σ be the permutation induced by I. Note that if $b \neq a$, then $\sigma(a) \neq \sigma(b)$ since there is at most one arrow c such that $c\sigma(a) \notin I$. Thus σ is bijective and $\sigma^{-1}(a)$ is the unique arrow such that $\sigma^{-1}(a)a \notin I$. Since σ is an isomorphism, Q_1 is partitioned into the orbits of σ, which we denote by $\{O_1, \ldots, O_m\}$. These orbits will play an important role in what follows. Note that if O is an orbit and a is an arrow in O, then $O = \{a, \sigma(a), \sigma^2(a), \ldots, \sigma^s(a)\}$ where the cardinality $|O|$ of O is $s + 1$.

Since A is symmetric, by definition, there is a non-degenerate K-linear form $f\colon A \to K$ such that, for all $x, y \in A$, $f(xy) = f(yx)$ (that is, the form f is symmetric) and $\ker(f)$ contains no nonzero two-sided ideals of A. Furthermore, the fact that A is symmetric implies that A is self-injective, and (since A is basic) the left socle of A is equal to the right socle of A and they are both equal to the two-sided socle of A. Note that $\ker(f)$ contains no nonzero two-sided ideals if and only if $\ker(f) \cap \mathrm{soc}(A) = \{0\}$.

We now prove a technical lemma about the orbits of σ.

Lemma 4.9. *Let $A = KQ/I$ be an indecomposable symmetric special multiserial K-algebra with $\operatorname{rad}(A)^2 \neq 0$ and let σ be the permutation on Q induced by I. Given a σ-orbit O and an arrow a in O, we set*

$$c_a = a\sigma(a)\sigma^2(a)\dots\sigma^{|O|-1}(a).$$

Then the following hold:

(1) *The path c_a is a cycle in Q.*
(2) *The paths $c_{\sigma^i(a)}$ are cycles in Q, for $0 \leq i \leq |O| - 1$.*
(3) *There is an integer $m_a > 0$ such that $\overline{c_a^{m_a}}$ is a nonzero element in $\operatorname{soc}(A)$.*
(4) *We have $m_a = m_{\sigma^i(a)}$ and $\overline{(c_{\sigma^i(a)})^{m_a}}$ is a nonzero element in $\operatorname{soc}(A)$, for $0 \leq i \leq |O| - 1$.*
(5) *We have $f(\overline{c_a^{m_a}}) = f(\overline{(c_{\sigma^i(a)})^{m_a}})$, for $0 \leq i \leq |O| - 1$, where f is the symmetric linear form defined above.*

Proof. By definition, $\overline{\sigma^i(a)\sigma^{i+1}(a)} \neq 0$, for $1 \leq i \leq |O| - 1$. Therefore c_a is a path in Q. Since $\sigma^{|O|}(a) = a$, the arrow $\sigma^{|O|-1}(a)$ ends at the same vertex at which a starts. Thus c_a is a cycle and (1) is proved.

Statement (2) follows from (1) by replacing a by $\sigma^i(a)$.

By the definition of σ, there must be an integer s such that $\overline{a\sigma(a)\cdots\sigma^{s-1}(a)}$ is a non-zero element in $\operatorname{soc}(A)$. Since $\sigma^{|O|}(a) = a$, there exists an integer m_a such that

$$a\sigma(a)\cdots\sigma^{s-1}(a) = (c_a^{m_a-1})a\cdots\sigma^i(a),$$

for $1 \leq i \leq |O|-1$. Now there are two cases. Firstly, if $i = |O|-1$, then (3) directly follows. Secondly, suppose $i < |O| - 1$. We will show that this is not possible since it leads to a contradiction, thus proving (3). So if $i < |O| - 1$ then $f(\overline{(c_a^{m_a-1})a\cdots\sigma^i(a)}) \neq 0$. But $\overline{\sigma^i(a)a} = 0$ since $a \neq \sigma^{i+1}(a)$. Thus $\overline{\sigma^i(a)(c_a^{m_a-1})a\cdots\sigma^{i-1}(a)} = 0$. But $f(\overline{(c_a)^{m_a-1}a\cdots\sigma^i(a)}) = f(\overline{\sigma^i(a)(c_a^{m_a-1})a\cdots\sigma^{i-1}(a)})$, a contradiction.

We now prove (4). Suppose that $\overline{c_a^{m_a}}$ is a non-zero element in $\operatorname{soc}(A)$. It suffices to show $\overline{c_{\sigma(a)}^{m_a}} \in \operatorname{soc}(A)$. First note that, using $f(xy) = f(yx)$, for any $x, y \in A$, we see that $f(\overline{c_a^{m_a}}) = f(\overline{c_{\sigma(a)}^{m_a}})$. Hence $\overline{c_{\sigma(a)}^{m_a}} \neq 0$. Suppose for contradiction that $\overline{c_{\sigma(a)}^{m_a}} \notin \operatorname{soc}(A)$, then $\overline{ac_{\sigma(a)}^{m_a}} \neq 0$ since a is the only arrow such that $\overline{a\sigma(a)} \neq 0$. But

$$ac_{\sigma(a)}^{m_a} = c_a^{m_a}a$$

and hence $\overline{a}\overline{c_{\sigma(a)}^{m_a}} = 0$ since $\overline{c_{\sigma(a)}^{m_a}}$ is in the (left) socle of A, a contradiction.

Part (5) follows since $f(xy) = f(yx)$,

$$c_a^{m_a} = (a\cdots\sigma^{i-1}(a))(c_{\sigma^i(a)}^{m_a-1}\sigma^i(a)\cdots\sigma^{|O|-1}) \text{ and}$$

$$c_{\sigma^i(a)}^{m_a} = (c_{\sigma^i(a)}^{m_a-1}\sigma^i(a)\cdots\sigma^{|O|-1})(a\cdots\sigma^{i-1}(a)). \quad \square$$

Lemma 4.10. *Let $A = KQ/I$ be an indecomposable symmetric special multiserial K-algebra and let $f: A \to K$ be a non-degenerate symmetric K-linear form such that* $\ker(f)$ *contains no two-sided ideals in A. Let e be a primitive idempotent in A and let p and p' be nonzero elements in $e \operatorname{soc}(A)e$ such that $f(p) = f(p')$. Then $p = p'$.*

Proof. Since $\ker f$ contains no non-zero two-sided ideals and since $\dim_K(e \operatorname{soc}(A)e) = 1$, we have that f restricted to $e \operatorname{soc}(A)e$ is an isomorphism. The result follows. □

For the next result we need to assume that the field K is algebraically closed. Keeping the notations as above, we have the following.

Proposition 4.11. *Let K be an algebraically closed field, let A be a basic indecomposable symmetric special multiserial K-algebra, and let Q be the quiver of A. Then there exists a surjection $\pi: KQ \to A$ such that*

(1) $\ker(\pi)$ *is admissible, and*
(2) if a and b are arrows starting at a vertex v in Q, then $\pi(c_a^{m_a}) = \pi(c_b^{m_b})$.

Proof. Since A is assumed to be finite dimensional and basic, there exists a surjection $\pi': KQ \to A$ such that $\ker(\pi')$ is admissible. Let $f: A \to K$ be a non-degenerate symmetric K-linear form with no two sided ideal in its kernel. We now construct a surjection $\pi: KQ \to A$ by defining, for each arrow a in Q, a non-zero constant $\lambda_a \in K$ such that by setting $\pi(a) = \lambda_a \pi'(a)$ the desired properties hold. Since $\ker(\pi')$ is admissible, clearly $\ker(\pi)$ is admissible.

We show that (2) holds one σ-orbit at a time. Let O be a σ-orbit and $a \in O$. Fix a nonzero element $k \in K$ and consider the cycle c_a. Then by Lemma 4.9 (1) and (3), $\pi'(c_a^{m_a}) \in e_v \operatorname{soc}(A)e_v$ where v is the vertex at which the arrow a starts and e_v is the associated primitive idempotent in A. We know that $f(\pi'(c_a^{m_a})) \neq 0$. Let $\lambda_a = (\frac{k}{f(\pi'(c_a^{m_a}))})^{1/m_a}$. Note that if we set $\pi(a) = \lambda_a \pi'(a)$ and $\pi(\sigma^i(a)) = \pi'(\sigma^i(a))$, for $1 \leq i \leq |O| - 1$, then $f(\pi(c_a^{m_a})) = k$. By Lemma 4.9(5), $f(\pi(c_{\sigma^i(a)}^{m_a})) = k$ for $1 \leq i \leq |O| - 1$. That is, we define $\lambda_b = 1$ if $b \in O$ and $b \neq a$.

Let $\pi: Q \to A$ be the resulting surjection from the construction above carried out for every σ-orbit. Then we have that for each arrow a in Q, $f(\pi(c_a^{m_a})) = k$. Applying Lemma 4.10, we get the desired result. □

4.3. Proof of Theorem 4.1

We are now able to prove Theorem 4.1, which states that a Brauer configuration algebra is a special multiserial algebra and conversely, that every symmetric special multiserial algebra is a Brauer configuration algebra.

Proof of Theorem 4.1. First assume that $A = KQ/I$ is symmetric special multiserial. By Proposition 4.11 we can assume that there is a surjection $\pi: KQ \to A$ with $I = \ker \pi$

such that if a and b are arrows in Q starting at the same vertex then $\pi(c_a^{m_a}) = \pi(c_b^{m_b})$ where c_a, m_a, c_b, m_b are as defined in Lemma 4.9.

Let $\sigma\colon Q_1 \to Q_1$ be the permutation induced by I. For each σ-orbit O, choose an arrow $a \in O$, and let L_O denote the multiset consisting of the vertices occurring in c_a, counting repetitions. More precisely, if $c_a = a\sigma(a)\cdots\sigma^{|O|-1}(a)$ and if $\sigma^i(a)$ is an arrow from v_{j_i} to $v_{j_{i+1}}$ then $L_O = \{v_{j_0}, v_{j_1}, \ldots v_{j_{|O|-1}}\}$. Note, for $i = 0$, $\sigma^0(a) = a$ is an arrow from v_{j_0} to v_{j_1} and $\sigma^{|O|-1}(a)$ is an arrow from $v_{j_{|O|-1}}$ to v_{j_0}; that is, $v_{j_{|O|}} = v_{j_0}$. By construction, the set L_O is independent of the choice of $a \in O$ since if $a' \in O$, then $c_{a'}$ is a cyclic permutation of the arrows of c_a.

We now construct the desired Brauer configuration algebra which we denote by Γ. We begin with a set $\Gamma_0^* \subset \Gamma_0$ which is in one-to-one correspondence with the set of σ-orbits $\mathcal{O} = \{O_1, \ldots, O_m\}$ of σ. We let $\Gamma_0^* = \{\alpha_1, \ldots, \alpha_m\}$ where α_i corresponds to O_i. These will be the nontruncated vertices of Γ. The polygons of Γ are in one-to-one correspondence with the vertices of Q such that if $Q_0 = \{v_1, \ldots, v_n\}$ then we set $\Gamma_1 = \{V_1, \ldots, V_n\}$ where the polygon V_i corresponds to the vertex v_i of Q.

We need to describe the truncated vertices in Γ_0, the elements that occur in each polygon V_i, the multiplicity function μ and the orientation $\mathfrak{o}$. We begin fixing $\alpha \in \Gamma_0^*$, $V \in \Gamma_1$ and determine how many times α occurs as an element in V. Suppose that α corresponds to the σ-orbit O and V corresponds to $v \in Q_0$. Then α occurs in V the number of times v occurs in L_O.

Next we define the set of truncated vertices in Γ. For each polygon V that consists of exactly one nontruncated vertex, say α, we add a new vertex α_V to the vertex set Γ_0 and to V. Thus $V = \{\alpha, \alpha_V\}$. We set $\mu(\alpha_V) = 1$ and hence α_V is a truncated vertex of Γ. In this way, we have defined the truncated vertices and we see that condition C3 is satisfied. We also see that condition C2, namely that $|V| \geq 2$, is satisfied. From this construction it is clear that Γ satisfies condition C4.

For each $\alpha \in \Gamma_0^*$, let $\mu(\alpha) = m_a$, where a is an arrow in the σ-orbit corresponding to α. By Lemma 4.9(4), μ is independent of the choice of a. We have defined μ to be 1 on truncated vertices and hence, we have completed the definition of the multiplicity function μ.

Finally, we need to describe the orientation $\mathfrak{o}$. For this, we let α be a vertex in Γ_0^* and assume that α corresponds to the σ-orbit $O = \{a, \sigma(a), \sigma^2(a), \ldots, \sigma^{|O|-1}(a)\}$. Then, as above, $\sigma^i(a)$ is an arrow from v_{j_i} to $v_{j_{i+1}}$, for $0 \leq i \leq |O|-1$ and $v_{j_{|O|}} = v_{j_0}$. If V_{j_i} is the polygon corresponding to the vertex v_{j_i}, then we let $V_{j_0} < \cdots < V_{j_{|O|-1}}$ be the successor sequence at α. Varying $\alpha \in \Gamma_0^*$ yields an orientation $\mathfrak{o}$.

We have now constructed a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$. Let Λ be the Brauer configuration algebra associated to Γ. We show that Λ is isomorphic to A. Let Q_Λ be the quiver of Λ. We begin by showing that there is an isomorphism of quivers from Q_Λ to Q. By our construction of the quiver of Q_Λ given in the beginning of this section, we see that the vertices of Q_Λ correspond to the polygons in Γ, which in turn correspond to the vertices of Q. Thus, we get a one-to-one correspondence between the vertices of Q_Λ and the vertices of Q. Again by our construction of the quiver of Q_Λ

given in the beginning of this section, the arrows of Q_Λ correspond to successors in the successor sequences of Γ. But the successor sequences in Γ correspond to the σ-orbits and each arrow in Q occurs once, in exactly one σ-orbit. Thus, the quivers Q_Λ and Q are isomorphic.

An isomorphism from Q_Λ to Q induces an isomorphism of path algebras $KQ_\Lambda \to KQ$. Thus, we obtain a surjection $\varphi\colon KQ_\Lambda \to A$. It is straightforward to see that relations of types one, two and three are all in $\ker(\varphi)$. Hence φ induces a surjection from Λ to A. To complete the proof, we consider the uniserial modules in both algebras, that is in the Brauer configuration algebra Λ and in A. We now apply the results from Section 3 in [30] on uniserial modules in a Brauer configuration algebra and the results from Section 2 of this paper on uniserial modules in a symmetric special multiserial algebra. It follows that the uniserial Λ-modules U such that U is not a projective Λ-module and such that U is maximal with this property, correspond to the uniserial A-modules U' such that U' is not a projective A-module and such that U' is maximal with this property. Thus the dimensions of $\mathrm{rad}(\Lambda)$ and $\mathrm{rad}(A)$ are equal. It follows that the surjection from Λ to A is an isomorphism and we are done.

The converse immediately follows from Proposition 2.8 in [30]. □

5. Symmetric algebras with radical cube zero are special multiserial

In this section we show that the class of special multiserial algebras contains another class of well-studied algebras. Namely that of symmetric algebras with radical cube zero. We show that basic symmetric algebras with radical cube zero are special multiserial and hence that they are Brauer configuration algebras. We remark that in [30], it is proved that the class of symmetric algebras with radical cube zero associated to a symmetric matrix with non-negative integer coefficients is the same as the class of Brauer configuration algebras in which the polygons have no repeated vertices. Our main results of this section show that dropping this restriction on polygons classifies all symmetric algebras with radical cube zero.

More precisely, we prove the following result.

Theorem 5.1. *Let K be an algebraically closed field and let $A \cong KQ/I$ be a finite dimensional indecomposable K-algebra. Assume that $\mathrm{rad}^3(A) = 0$ but $\mathrm{rad}^2(A) \neq 0$. Then the following statements are equivalent.*

(1) A is a symmetric K-algebra.
(2) A is a symmetric multiserial K-algebra.
(3) A is a symmetric special multiserial K-algebra.
(4) A is isomorphic to an indecomposable Brauer configuration algebra.

Proof. Clearly (2) implies (1). We see that (3) implies (2) by Theorem 2.3. By Theorem 4.1, (3) holds if and only if (4) holds. It remains to show that (1) implies (4). For this we start with some preliminary results, culminating in Theorem 5.6, which is the desired result. □

For the remainder of this section, we let $A = KQ/I$ be an indecomposable symmetric K-algebra such that $\mathrm{rad}^3(A) = 0$ but $\mathrm{rad}^2(A) \neq 0$. We assume that I is an admissible ideal and let $f\colon A \to K$ be a non-degenerate symmetric linear form such that $\ker(f)$ does not contain a two-sided ideal of A. If M is a right Λ-module, then the *Loewy length* of M is n if $M\,\mathrm{rad}^{n-1}(A) \neq 0$ and $M\,\mathrm{rad}^n(A) = 0$. We fix a surjection $\pi\colon KQ \to A$ with kernel I and if $x \in KQ$, we will write $\bar{x}$ for $\pi(x)$. More generally, we will write $\bar{a}$ for elements in A.

The next result is well-known but we include a proof for completeness.

Lemma 5.2. *Keeping the notations and assumptions above, every indecomposable projective A-module has Loewy length* 3*.*

Proof. Let P be an indecomposable projective A-module of Loewy length 2. Then P is the projective cover of a simple A-module S. Since A is symmetric, P has simple top and simple socle isomorphic to S and these are the only composition factors of P. Thus P is an extension of S by S and if S is the simple at vertex v in Q then there is a loop at v in Q. Furthermore, there is no other arrow leaving v. Since A is symmetric, P is also the injective hull of S and so there is no arrow entering v. Thus there is a loop at v and no other arrow entering or leaving v. Thus there is a factor $K[x]/(x^2)$ of A, contradicting the indecomposability of A. □

Lemma 5.3. *Keeping the notations and assumptions above, let e_v be the idempotent at a vertex v in KQ and let x be a linear combination of paths of length* 2 *such that $e_v x e_v = x$. Then $x \in I$ if and only if $f(\bar{x}) = 0$.*

Proof. Suppose $x \notin I$. Then $\bar{x}$ is a nonzero element of the socle of A. If $f(\bar{x}) = 0$ then the K-span of $\bar{x}$ is in $\ker(f)$. Since $\mathrm{soc}(A)$ is semisimple and each simple A-module is one dimensional, we obtain a two sided ideal in $\ker(f)$ which is a contradiction. Hence $f(\bar{x}) \neq 0$. Next suppose that $x \in I$. Then $\bar{x} = 0$ and we conclude that $f(\bar{x}) = 0$. □

Our next lemma shows the special nature of symmetric algebras with radical cube zero.

Lemma 5.4. *Keeping the notations and assumptions above, let a and b be arrows in Q. Then the following statements are equivalent:*

(1) $ab \notin I$.
(2) $ba \notin I$.
(3) $\overline{ab}$ is a nonzero element of $\mathrm{soc}(A)$*.*
(4) $\overline{ba}$ is a nonzero element of $\mathrm{soc}(A)$*.*

(5) $f(\overline{ab}) \neq 0$.
(6) $f(\overline{ba}) \neq 0$.

Proof. Note that since A is symmetric, if a and b are arrows with $\overline{ab} \in \operatorname{soc}(A)$ and $\overline{ab} \neq 0$, then there is a vertex v such that $e_v abe_v = ab$ where e_v is the corresponding idempotent in KQ. Using Lemma 5.3 and that $f(\overline{ab}) = f(\overline{ba})$, it is clear that parts (1), (2), (5), and (6) are equivalent. Using that $\ker(f)$ cannot contain any non-zero two-sided ideals and that $\operatorname{rad}^3(A) = 0$, we obtain that part (3) is equivalent to part (1) and part (4) is equivalent part (2). □

The next result shows that in general, for a basic indecomposable symmetric K-algebra A such that $\operatorname{rad}^3(A) = 0$ but $\operatorname{rad}^2(A) \neq 0$ there is a special way of presenting A as KQ/I. For this we define a set *Arr* whose K-span equals the K-span of the image of the arrows in Q and such that *Arr* satisfies a tight set of multiplicative properties.

Proposition 5.5. *Let K be an algebraically closed field and let A be a basic indecomposable symmetric K-algebra such that* $\operatorname{rad}^3(A) = 0$ *but* $\operatorname{rad}^2(A) \neq 0$.

Then there is a K-linearly independent set Arr $\subset \operatorname{rad}(A)$ *with the following properties.*

(1) Arr generates $\operatorname{rad}(A)$ *as a two-sided ideal.*
(2) If $\bar{x}$ is a nonzero linear combination of elements in Arr then $\bar{x} \notin \operatorname{rad}^2(A)$.
(3) If $\bar{a} \in Arr$ is such that $\bar{a}^2 \neq 0$, then $\overline{ab} = 0$ for all $\bar{b} \in Arr$, $\bar{b} \neq \bar{a}$.
(4) If $\bar{a}, \bar{b} \in Arr$ with $\bar{a} \neq \bar{b}$ and $\overline{ab} \neq 0$, then $\overline{ac} = 0 = \overline{bc}$ for all $\bar{c} \in Arr$ with $\bar{c} \neq \bar{a}, \bar{b}$.
(5) For each $\bar{a}, \bar{b} \in Arr$, not necessarily distinct, if $\overline{ab} \neq 0$ then $f(\overline{ab}) = 1$.

Proof. The assumption that the simple A-modules are one dimensional implies there is a surjection $\pi\colon KQ \to A$ such that Q is the quiver of A and that $\ker(\pi)$ is an admissible ideal. Let $f\colon A \to K$ be a linear form obtained from A being symmetric.

Let $Arr = \pi(Q_1)$. It follows that *Arr* is linearly independent over K and generates $\operatorname{rad}(A)$. Moreover *Arr* satisfies property (2). If $\bar{a}$ and $\bar{b}$ are in *Arr*, then we set $\gamma_{a,b} = f(\overline{ab})$.

Let $Y = \operatorname{Span}_K(Arr)$. Note that *Arr* is a K-basis of Y. We begin by making a series of linear changes of bases starting with the basis *Arr* of Y. Suppose that there is an element $\bar{a} \in Arr$ such that $\bar{a}^2 \neq 0$. Then consider the change of basis with $\bar{a}$ remaining unchanged and if $\bar{b} \in Arr$ with $\bar{b} \neq \bar{a}$, replace $\bar{b}$ by $\bar{b} - \frac{\gamma_{a,b}}{\gamma_{a,a}}\bar{a}$. Note that after this change of basis, if $\bar{b} \neq \bar{a}$, $f(\bar{a}(\bar{b} - \frac{\gamma_{a,b}}{\gamma_{a,a}}\bar{a})) = 0$ and hence, in the new basis, $\overline{ab} = 0$ by Lemma 5.4. By abuse of notation, we still call the new basis *Arr*.

If there is another $\bar{b} \in Arr$ such that $\bar{b}^2 \neq 0$, perform the same change of basis for $\bar{b}$ instead of $\bar{a}$. Note that under this change of basis, $\bar{a}$ remains unchanged since $\gamma_{a,b} = 0$. Continuing in this fashion, we arrive at a basis, again called *Arr*, such that if $\bar{a}$ is an element in *Arr* and $\bar{a}^2 \neq 0$, then, for all $\bar{b} \neq \bar{a}$, we have $\overline{ba} = 0$ and $\overline{ab} = 0$.

Now let $\bar{a}$ be an element in Arr with $\bar{a}^2 = 0$. Then by Lemma 5.2 there must be an element $\bar{b}$ in Arr, $\bar{b} \neq \bar{a}$, such that $\overline{ab} \neq 0$. Note that $\bar{b}^2 = 0$, since if not, $\overline{ab}$ would equal 0. Consider the change of basis that leaves $\bar{a}$ and $\bar{b}$ unchanged, and where if $\bar{c}$ is an element in Arr different from $\bar{a}$ and $\bar{b}$, we replace $\bar{c}$ by $\bar{c} - \frac{\gamma_{b,c}}{\gamma_{a,b}}\bar{a} - \frac{\gamma_{a,c}}{\gamma_{a,b}}\bar{b}$. Applying f to the new basis, we see that $\overline{ab} \neq 0$, $\overline{ac} = 0 = \overline{bc}$ for all $\bar{c}$ different from $\bar{a}$ and $\bar{b}$. Note that if $\bar{c}$ is an element of Arr with $\bar{c}^2 = 0$, then $\bar{c}$ remains unchanged since $\gamma_{a,c} = 0 = \gamma_{b,c}$. Continuing in this fashion, we obtain a new basis of Y, which we call again Arr satisfying properties (3) and (4).

For each pair $\bar{a}, \bar{b}$ satisfying (4) above, choose either $\bar{a}$ or $\bar{b}$ and call it a *chosen element*. We make one final change of basis of Y. For each $\bar{a} \in Arr$ such that $\bar{a}^2 \neq 0$, replace $\bar{a}$ by $(1/(\gamma_{a,a})^{\frac{1}{2}})\bar{a}$. For each pair $\bar{a}, \bar{b}$ satisfying (4) above, replace the chosen element, say $\bar{a}$, by $(1/\gamma_{a,b})\bar{a}$ and leave $\bar{b}$ unchanged. We then obtain a basis of Y, which we call again Arr, satisfying properties (1)–(5) and we take this to be the desired set. □

We remark that it follows from the proof of Proposition 5.5 that the canonical surjection $\pi : KQ \to A$ maps the arrows of Q bijectively to Arr.

We now present the final result needed to finish the proof of Theorem 5.1.

Theorem 5.6. *Let K be an algebraically closed field and let $A = KQ/I$ be a finite dimensional basic indecomposable K-algebra. Suppose that A is symmetric and that $\mathrm{rad}^3(A) = 0$ but $\mathrm{rad}^2(A) \neq 0$. Then A is isomorphic to a Brauer configuration algebra.*

Proof. Let Arr be a set satisfying properties (1)–(5) in Proposition 5.5. Recall that if $x \in KQ$ then $\bar{x}$ will denote the image of x in A under the canonical surjection $KQ \to A$ where the arrows of Q are mapped bijectively to Arr. We show that property (M) holds. Let a be an arrow in Q and $\bar{a}$ its image in Arr. First we show that there is at most one arrow b such that $ab \notin I$. If $a^2 \notin I$, that is if $\bar{a}^2 \neq 0$, then Proposition 5.5(3) yields the result. If $a^2 \in I$, that is, if $\overline{a}^2 = 0$, then Proposition 5.5(4) shows that if $ab \notin I$ for some arrow b, then $ad \in I$ for all $d \neq a$ or b. But $a^2 \in I$ and hence there is at most one arrow b such that $ab \notin I$.

Given an arrow a of Q, by Lemma 5.4 we have that $ab \notin I$ for some arrow b if and only if $ba \notin I$. By the first part of the proof above b is unique if it exists. Therefore, it follows directly that if $ab \notin I$ then $ba \notin I$ and there is no other arrow c with $c \neq b$ such that $ca \notin I$. Note that this also holds if b is equal to a.

Thus (M) holds and hence A is a symmetric special multiserial algebra. The result then follows from Theorem 4.1. □

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