ELSEVIER Discrete Mathematics 151 (1996) 19–38

DISCRETE MATHEMATICS

On a class of Hamiltonian laceable 3-regular graphs

Brian Alspach[a], C.C. Chen[b,*], Kevin McAvaney[c]

[a] *Department of Mathematics and Statistics, Simon Fraser University, Burnaby, BC, Canada V5A 1S6*
[b] *Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511*
[c] *Department of Computing and Mathematics, Deakin University, Geelong, Australia*

Received 6 January 1992; revised 24 May 1993

Abstract

Using the concept of brick-products, Alspach and Zhang showed in Alspach and Zhang (1989) that all cubic Cayley graphs over dihedral groups are Hamiltonian. It is also conjectured that all brick-products $C(2n, m, r)$ are Hamiltonian laceable, in the sense that any two vertices at odd distance apart can be joined by a Hamiltonian path. In this paper, we shall study the Hamiltonian laceability of brick-products $C(2n, m, r)$ with only one cycle (i.e. $m = 1$). To be more specific, we shall provide a technique with which we can show that when the chord length r is $3, 5, 7$ or 9, the corresponding brick-products are Hamiltonian laceable. Let $s = \gcd((r+1)/2, n)$ and $t = \gcd((r-1)/2, n)$. We then show that the brick-product $C(2n, 1, r)$ is Hamiltonian laceable if (i) st is even; (ii) s is odd and $rs \equiv r + 1 + 3s \pmod{4n}$; or (iii) t is odd and $rt \equiv r - 1 - 3t \pmod{4n}$. In general, when n is sufficiently large, say $n \geqslant r^2 - r + 1$, then the brick-product is also Hamiltonian laceable.

1. Introduction

Let G be a group and S a generating subset of G such that the identity element $1 \notin S$ and $x^{-1} \in S$ for each $x \in S$. The *Cayley graph* $X(G; S)$ on a group G has the elements of G as its vertices and edges joining g and gs for all $g \in G$ and $s \in S$. The question whether or not all Cayley graphs are Hamiltonian still remains unsolved, though it was shown in [3] that all Cayley graphs over Abelian groups which are neither bipartite nor isomorphic to a cycle are Hamiltonian-connected (in the sense that any two vertices can be joined by a Hamiltonian path). For the case when the groups involved are not Abelian, the problem seems extremely difficult. Even for dihedral groups D_n, it was only shown in [2] that the Cayley graphs are Hamiltonian when they are cubic. In the same paper, it was shown that when the Cayley graph is

* Corresponding author.

SSDI 0012-365X(94)00077-V

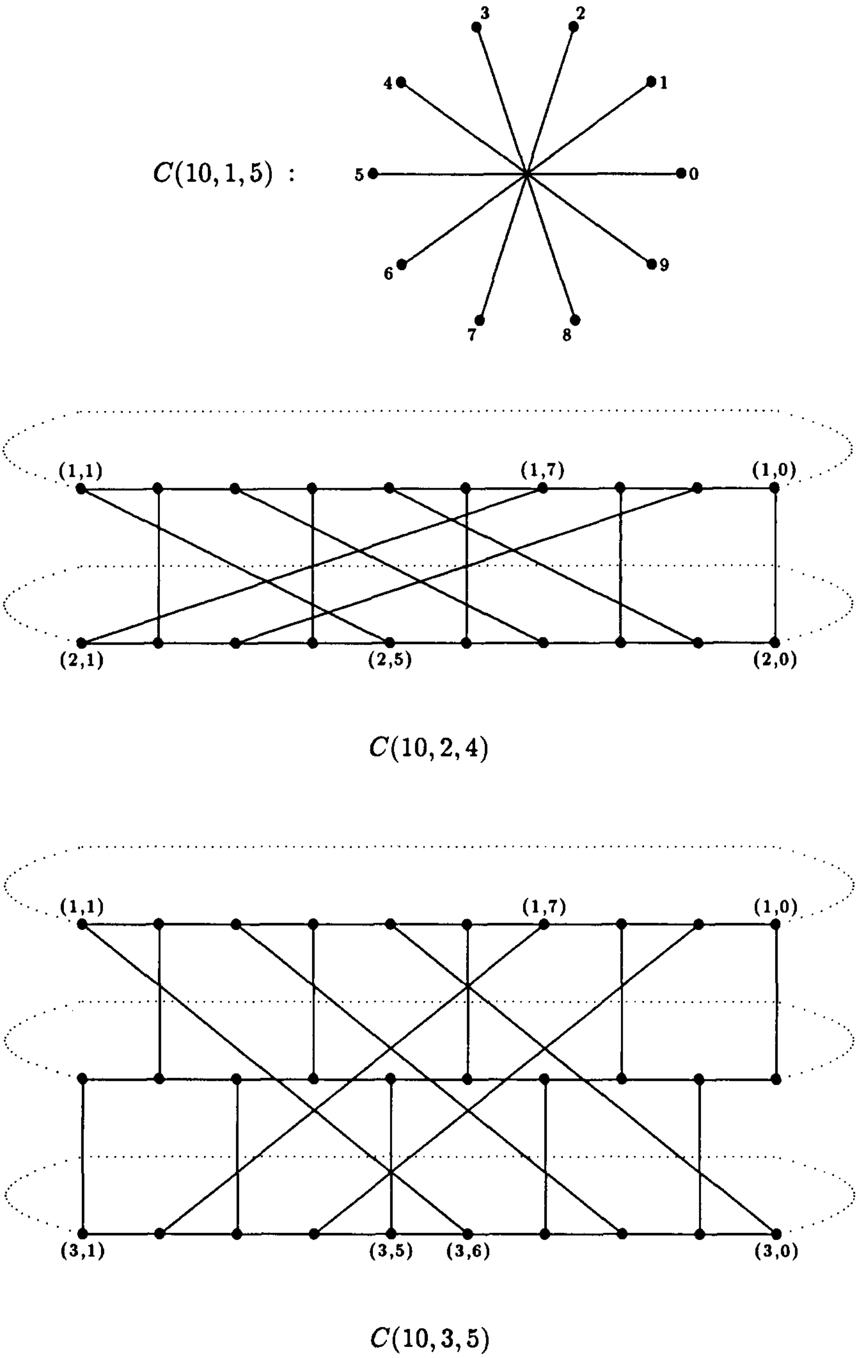

C(10,1,5) :
0
1
2
3
4
5
6
7
8
9
(1,1)
(1,7)
(1,0)
(2,1)
(2,5)
(2,0)
C(10,2,4)
(1,1)
(1,7)
(1,0)
(3,1)
(3,5)
(3,6)
(3,0)
C(10,3,5)

Fig. 1.

$X(D_n, S)$, where S is a minimal generating set of D_n consisting of three elements of order 2, it is isomorphic to a brick-product as defined below:

Definition. Let m, n and r be a positive integers. Let $C_{2n} = 012\dots(2n-1)0$ denote a cycle of order $2n$. The *(m, r)-brick-product of C_{2n}*, denoted by $C(2n, m, r)$, is defined in two cases as follows.

For $m = 1$, we require that r be odd and greater than 1. Then $C(2n, m, r)$ is obtained from C_{2n} by adding chords $2k(2k + r)$, $k = 1, \dots, n$, where the computation is performed modulo $2n$.

For $m > 1$, we require that $m + r$ be even. Then $C(2n, m, r)$ is obtained by first taking the disjoint union of m copies of C_{2n}, namely $C_{2n}(1), C_{2n}(2), \dots, C_{2n}(m)$, where for each $i = 1, 2, \dots, m$, $C_{2n}(i) = (i, 0)(i, 1)\dots(i, 2n)$. Next, for each odd $i = 1, 2, \dots, m-1$ and each even $k = 0, 1, \dots, 2n-2$, an edge (called a *brick edge*) is drawn to join (i, k) to $(i+1, k)$, whereas, for each even $i = 1, 2, \dots, m-1$ and each odd $k = 1, 2, \dots, 2n-1$, an edge (also called a *brick edge*) is drawn to join (i, k) to $(i+1, \mathrm{k})$. Finally, for each odd $k = 1, 2, \dots, 2n-1$, an edge (called a *hooking edge*) is drawn to join $(1, k)$ to $(m, k+r)$. An edge in $C(2n, m, r)$ which is neither a brick edge nor a hooking edge is called a *flat edge*.

Examples. The brick-products $C(10, 1, 5)$, $C(10, 2, 4)$ and $C(10, 3, 5)$ are given in Fig. 1.

Using the concept of brick-products, Alspach and Zhang show in [2] that all cubic Cayley graphs over dihedral groups are Hamiltonian. It is also conjectured that all brick-products $C(2n, m, r)$ are Hamiltonian laceable (in the sense that any two vertices at odd distance apart can be joined by a Hamiltonian path). In [1], it was shown that the conjecture is true for m even. In this paper, we shall study the Hamiltonian laceability of brick-products with only one cycle (i.e. $m = 1$). To be more specific, we shall show that when r is 3 or 5, the corresponding brick-products are Hamiltonian laceable. The technique employed can also be used to show the Hamiltonian laceability of brick-products with $r = 7$ or 9. Let $s = \gcd((r+1)/2, n)$ and $t = \gcd((r-1)/2, n)$. We then show that the brick-product $C(2n, 1, r)$ is Hamiltonian laceable if (i) st is even; (ii) s is odd and $rs \equiv r + 1 + 3s \pmod{4n}$; or (iii) t is odd and $rt \equiv r - 1 - 3t \pmod{4n}$. In general, when n is sufficiently large, say $n \geqslant r^2 - r + 1$, then the brick-product is also Hamiltonian laceable.

2. Cycles with chords of small length

Throughout this paper, we let $G = C(2n, 1, r)$ and denote the vertices of G by $\{1, 2, \dots, 2n\}$. Note that the edges of G are 12, 23, $\dots$, $(2n-1)(2n)$, $(2n)1$ and $(2k)(2k + r)$ for $k = 1, 2, \dots, n$ where $2k + r$ is computed modulo $2n$. We shall call each of the edges $(2k)(2k + r)$ a *chord edge* and r the *chord length* of G.

In this section, we shall show that if the chord length $r = 3$ or 5, then the brick-products involved are Hamiltonian laceable. The technique used here can also be applied to prove the Hamiltonian laceability of G when the chord length r is 7 or 9 (but not for $r \geqslant 11$). First, we need to introduce the following terminologies.

Let $m \leqslant 2n$. For each vertex x of G, we shall write:

$$xP[m] = x(x+1)(x+2)\cdots(x+m-1),$$

$$xP^{-1}[m] = x(x-1)(x-2)\cdots(x-m+1),$$

$$xJ = x(x+r) \quad \text{for even } x, \qquad xJ^{-1} = x(x-r) \quad \text{for odd } x.$$

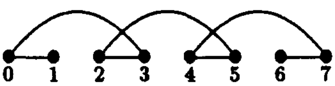
Note that $P[m]$ and $P^{-1}[m]$ are paths of order m, whereas, xJ and xJ^{-1} are chord edges. The symbols: $xP^2[m]$, $x(JP^{-1}[m])^3$, etc. are self-explanatory.

A path P in G from x to y is said to be *gapless* if there exists $a, b \in \mathbb{Z}$ with $a \leqslant b$, such that $V(P) = [a,b](=\{a, a+1, a+2, \ldots, b\})$ and each chord edge of P is of the form $e(e+r)$ with both $e, e+r \in [a,b]$. We shall call b the *extremal vertex* of P.

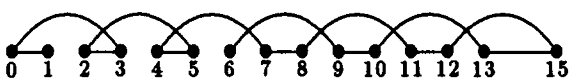
Note that a and b may be negative, as vertices of the form $2n-s$ may also be denoted by $-s$.

The following three lemmas are straightforward.

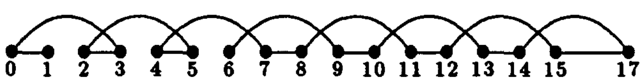
Lemma 1. *Let* $r = 3$. *Then* $1(P^{-1}[2]J)^k P^{-1}[2]$ *is a gapless path in* G *of order* $2k+2$ *from* 1 *to* $2k$, *where* $2k+2 \leqslant 2n$ (see the illustration in the following figure for $k = 3$).



Lemma 2. *Let* $r = 3$. *Then* $1(P^{-1}[2]J)^k(P[2]J)^m P^{-1}[3](J^{-1}P^{-1}[2])^{m-1}J^{-1}$ *is a gapless path in* G *of order* $2k+4m+2$ *from* 1 *to* $2k$, *where* $2k+4m+2 \leqslant 2n$ *and* $m \geqslant 0$ (see the illustration in the following figure for $k = 3$, $m = 2$).



Lemma 3. *Let* $r = 3$. *Then* $1(P^{-1}[2]J)^k(P[2]J)^m P[3](J^{-1}P^{-1}[2])^m J^{-1}$ *is a gapless path in* G *of order* $2k+4m+4$ *from* 1 *to* $2k$, *where* $2k+4m+4 \leqslant 2n$ *and* $m \geqslant 0$ (see the illustration in the following figure for $k = 3$, $m = 2$).



Combining the above three lemmas, we see that there exists a Hamiltonian path in G from 1 to $2k$ for each $k \leqslant n$. However, as G is a Cayley graph and so vertex-transitive, we have the following theorem.

Theorem 4. *$C(2n, 1, 3)$ is Hamiltonian laceable.*

We shall now consider the case when the chord length r is 5.

Lemma 5. *Let c, d be vertices of G with c odd and d even. If there exists a gapless path P of order m from c to d in G, then there exists a gapless path from c to d in G of order $m + k(r-1)$, for each non-negative integer k with $m + k(r-1) \leqslant 2n$.*

Proof. Let b be the extremal vertex of P. Then P must contain the edge $b(b-1)$. Now, replacing the edge $b(b-1)$ in P by the path $b(P[r-1]J)^k P^{-1}[r](J^{-1}P^{-1}[r-1])^{k-1}J^{-1}(b-1)$ yields a gapless path in G from c to d of order $m + 2k(r-1)$. On the other hand, replacing the edge $b(b-1)$ in P by the path $b(P[r-1]J)^k P[r](J^{-1}P^{-1}[r-1])^k J^{-1}(b-1)$ yields a gapless path in G from c to d of order $m + (2k+1)(r-1)$. □

Lemma 6. *If there exists a gapless path Q of order m from 1 to $2u$ in G with extremal vertex $2s+1$, where $2s+1-2u < r$, then for each k with $m + k(r+1) \leqslant 2n$, there exists a gapless path in G of order $m + k(r+1)$ from 1 to $2u + k(r+1)$.*

Proof. Let $q = r - (2s + 1 - 2u)$. Then the required gapless path is

$$Q(JP^{-1}[q]JP^{-1}[2s+2-2u])^k. \quad \square$$

Let $2u, 2v$ be integers. The pair $(2u, 2v)$ is said to be an *attainable pair* if there exists a gapless path of order $2u$ in G from 1 to $2v$. Thus, by definition, if $(2u, 2v)$ is attainable and the order of G is $2u$, then G contains a Hamiltonian path from 1 to $2v$.

As an immediate consequence of Lemma 5, we have the following corollary.

Corollary 7. *If $(2u, 2v)$ is attainable, then $(2u + k(r-1), 2v)$ is also attainable for any non-negative integer k such that $2u + k(r-1) \leqslant 2n$.*

Also, as an immediate consequence of Lemma 6, we have the following corollary.

Corollary 8. *If there exists a gapless path of order m from 1 to $2u$ in G with extremal vertex $2s+1$, where $2s+1-2u < r$, then for each k with $m + k(r+1) \leqslant 2n$ $(m + k(r+1), 2u + k(r+1))$ is attainable.*

Theorem 9. *$G = C(2n, 1, 5)$ is Hamiltonian laceable.*

Proof. As G is vertex transitive, we need only to show that there exists a Hamiltonian path in G from 1 to $2k$ for any $k \leqslant n$. By virtue of the corollaries to Lemmas 5 and 6, we need only to establish the following nine claims.

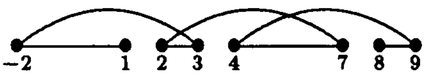
Claim 1. *There exists a Hamiltonian path in G from 1 to 2. Indeed, a required path is* $1(2n)(2n-1)\cdots 32$.

Claim 2. *There exists a Hamiltonian path in G from* 1 *to* 6. *Indeed, a required path is* $12345(2n)(2n-1)\cdots 76$.

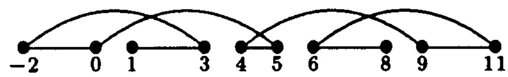
Claim 3. $(8,4)$ *is attainable. Indeed, a required path is* $123(-2)(-1)054$.

Claim 4. $(10,4)$ *is attainable. Indeed, a required path is* $10(-1)(-2)327654$.

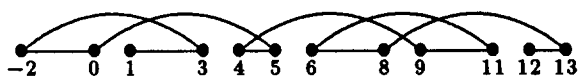
Claim 5. $(12,8)$ *is attainable. Indeed, a required path is* $1P^{-1}[4]JP^{-1}[2]JP^{-1}[4]JP^{-1}[2]$ *(as shown below).*



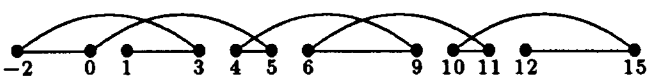
Claim 6. $(14,8)$ *is attainable. Indeed, a required path is* $1P[3]J^{-1}P[3]JP^{-1}[2]JP[3]J^{-1}P[3]$ *(as shown below).*



Claim 7. $(16,12)$ *is attainable. Indeed, a required path is* $1P[3]J^{-1}P[3]JP^{-1}[2]JP[3]J^{-1}P[3]JP^{-1}[2]$ *(as shown below).*



Claim 8. $(18,12)$ *is attainable. Indeed, a required path is* $1P[3]J^{-1}P[3]JP^{-1}[2]JP^{-1}[4]JP^{-1}[2]JP^{-1}[4]$ *(as shown below).*



Claim 9. *There exists a Hamiltonian path in G from* 1 *to* $2n-2$.

Again, by virtue of the corollary to Lemma 6, we need only to establish the following four subclaims.

Subclaim 1. $(8,6)$ *is attainable. Indeed, a required path is* 10543276.

Subclaim 2. $(10,8)$ *is attainable. Indeed, a required path is* 1056723498.

Subclaim 3. (18, 16) *is attainable. Indeed, a required path is* 1056723498(13)(14)(15)(10)(11)(12)(17)(16).

Subclaim 4. *If G is of order* 12, *then there exists a Hamiltonian path in G from* 1 *to* $2n-2$. *Indeed, a required path is* $123450(-1)(-6)(-5)(-4)(-3)(-2)$. □

By using a similar argument as that for Theorem 9, we can continue to show that $C(2n,1,7)$ and $C(2n,1,9)$ are Hamiltonian laceable. However, the number of cases we need to consider increases considerably. Also, the same approach is no longer valid when the chord length is more than 9. In the next two sections, we shall study the Hamiltonian laceability of $C(2n,1,r)$ when the chord length r is larger than 5.

3. Decomposable brick-products

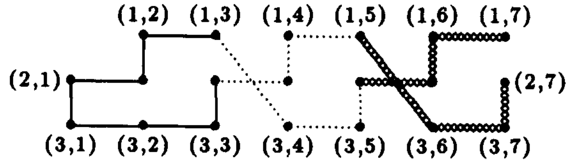
Let $G = C(2n,1,r)$ whose vertices are labeled $1,2,\dots,2n$, as mentioned before. Let G' and G'' denote, respectively, the sequence $12(2+r)(2+r+1)(2+2r+1)\times(2+2r+2)\cdots(2+(n-1)r+(n-2))(2+(n-1)r+(n-1))$ and the sequence $1(1-r)(2-r)(2-2r)\cdots(n-(n-1)r)(n-nr)$. If G' and G'' are both Hamiltonian cycles of G, then we say the graph G is *indecomposable*. Otherwise, G is said to be *decomposable.*

Lemma 10. *Let* $G = C(2n,3,1)$. *Then, for any odd* y *with* $1<y<2n$, *the subgraph of* G *induced by* $G(y)=\{(x,k)\mid 1\leqslant k\leqslant y\}-\{(1,1)\}$ *contains a Hamiltonian path from* $(2,y)$ *to* $(1,y)$.

Proof. If $y=3$, then a required path is

$$P(3)=(2,3)(3,3)(3,2)(3,1)(2,1)(2,2)(1,2)(1,3).$$

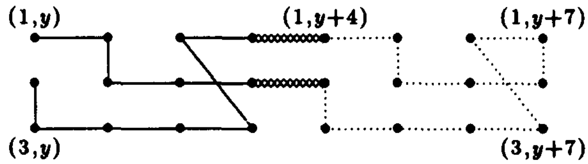
If $y=5$, then a required path is $P(5)=(2,5)(3,5)(3,4)(P(3))^{-1}(2,4)(1,4)(1,5)$. If $y=7$, then a required path is $P(7)=(2,7)(3,7)(3,6)(P(5))^{-1}(2,6)(1,6)(1,7)$. The result now follows by induction on y (see the illustration in the figure below). □



Lemma 11. *Let* $G=C(2n,3,1)$. *For any odd* y *with* $1<y<2n$ *and any positive integer* k *with* $y+4k\leqslant 2n+1$, *the subgraph of* G *induced by* $\{(i,j)\mid i=1,2,3;\ y\leqslant j<y+4k\}$ *contains a Hamiltonian path from* $(1,y)$ *to* $(2,y)$.

Proof. We shall prove by induction on k. If $k = 1$, a required path is $(1,y)(1,y+1)(2,y+1)(2,y+2)(2,y+3)(1,y+3)(1,y+2)(3,y+3)(3,y+2)(3,y+1)(3,y)(2,y)$.

Assume that the result hold for $k = N$ and consider the case when $k = N+1$. By inductive hypothesis, there exists a Hamiltonian path P in the subgraph of G induced by $\{(i,j)\,|\,i = 1,2,3;\ y+4 \leqslant j < y+4k\}$ from $(1,y+4)$ *to* $(2,y+4)$. *Then*, $(1,y)(1,y+1)(2,y+1)(2,y+2)(2,y+3)P^{-1}(1,y+3)(1,y+2)(3,y+3)(3,y+2)(2,y+1)(3,y)(2,y)$ will be a required Hamiltonian path (see the figure shown below for $k = 2$). □

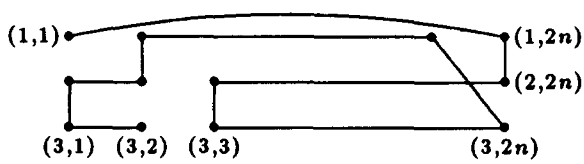


Lemma 12. *$C(2n,2m+1,1)$ is Hamiltonian laceable.*

Proof. We shall first show that the result is true for $m = 1$. As $C(2n,3,1)$ is a Cayley graph and so vertex transitive, we need only to show that there exists a Hamiltonian path from $x = (1,1)$ in the first cycle $C_{2n}(1)$ to any other vertex y at odd distance apart from x.

Case 1: y is in the third cycle $C_{2n}(3)$.

In this case, $y = (3,k)$ where k is even. If $k = 2$ then a required Hamiltonian path is $(1,1)(1,2n)(2,2n)(1,2n-1)\cdots(2,3)(3,3)(3,4)\cdots(3,2n)(1,2n-1)(1,2n-2)\cdots(1,2)(2,2)(2,1)(3,1)(3,2)$ (see the illustration in the following figure).

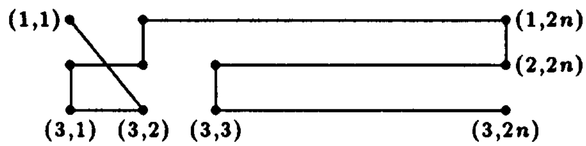


If $k = 2n$, then a required Hamiltonian path is

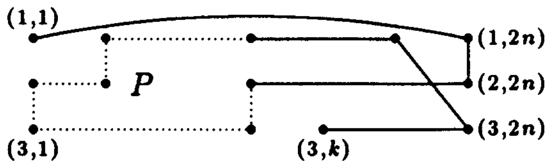
$$(1,1)(3,2)(3,1)(2,1)(2,2)(1,2)(1,3)\cdots(1,2n)(2,2n)(2,2n-1)\cdots$$

$$(2,3)(3,3)(3,4)\cdots(3,2n)$$

(see the illustration in the following figure).



Now, let $2 < k < 2n$. By Lemma 10, there exists a Hamiltonian path P in the induced subgraph $G(k-1)$ from $(2, k-1)$ to $(1, k-1)$. We then have a Hamiltonian path in G from $(1, 1)$ to $(3, k)$ given by $(1,1)(1,2n)(2,2n)(2,2n-1)\cdots(2,2k-1)\times P(1,k-1)(1,k)\cdots(1,2n-1)(3,2n)\ (3,2n-1)\cdots(3,k)$ (see the illustration in the following figure).



Case 2: y is in the first cycle $C_{2n}(1)$.

In this case, $y = (1, k)$ where k is even.

If $k = 2$, then a required Hamiltonian path is $(1,1)(1,2n)(2,2n)(2,2n-1)\cdots(2,1)$ $(3,1)(3,2)\cdots(3,2n)(1,2n-1)(1,2n-2)\cdots(1,2)$.

If $k = 2n$, then a required Hamiltonian path is $(1,1)(3,2)(3,1)(3,2n)(3,2n-1)\cdots$ $(3,3)(2,3)(2,4)\cdots(2,2n)(2,1)(2,2)(1,2)(1,3)\cdots(1,2n)$.

Now, let $2 < k < 2n$. Then by Lemma 10, the induced subgraph $G(k-1)$ contains a Hamiltonian path P from $(2, k-1)$ to $(1, k-1)$. Then a required Hamiltonian path in G from $(1, 1)$ to $(1, k)$ is given by $(1,1)(1,2n)(2,2n)(2,2n-1)\cdots(2,k-1)$ $P(1,k-1)(3,k)(3,k+1)\cdots(3,2n)(1,2n-1)(1,2n-2)\cdots(1,k)$.

Case 3. y lies in the second cycle $C_{2n}(2)$.

In this case $y = (2, k)$ where k is odd.

If $k = 1$, then a required Hamiltonian path is $(1,1)(3,2)(3,1)(3,2n)(3,2n-1)\cdots$ $(3,3)(2,3)(2,4)\cdots(2,2n)(1,2n)(1,2n-1)\cdots(1,2)(2,2)(2,1)$.

If $k = 3$, then a required Hamiltonian path is

$$(1,1)(3,2)(3,3)(3,4)(1,3)(1,2)(2,2)(2,1)(3,1)(3,2n)(3,2n-1)\cdots$$

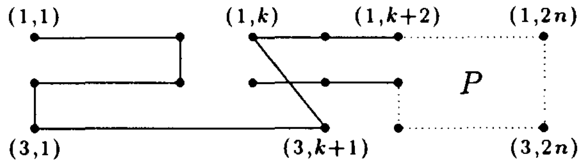
$$\cdots(3,5)(2,5)(2,6)\cdots(2,2n)(1,2n)(1,2n-1)\cdots(1,4)(2,4)(2,3).$$

If $k = 5$, then a required Hamiltonian path is

$$(1,1)(1,2n)(1,2n-1)\cdots(1,4)(2,4)(2,3)(3,3)(3,2)(3,1)(3,2n)(3,2n-1)\cdots$$

$$(3,4)(1,3)(1,2)(2,2)(2,1)(2,2n)(2,2n-1)\cdots(2,5).$$

Hence we may now assume that $k > 5$. Now, if $2n - k - 1$ is divisible by 4, then by Lemma 11, there exists a Hamiltonian path P from $(1, k+2)$ to $(2, k+2)$ in the subgraph of G induced by $\{(i,j) \mid i = 1,2,3;\ k+2 \leqslant j \leqslant 2n\}$. A required Hamiltonian path in G is: $(1,1)(1,2)\cdots(1,k-1)(2,k-1)(2,k-2)\cdots(2,1)(3,1)(3,2)\cdots(3,k+1)\times(1,k)(1,k+1)\ P(2,k+1)(2,k)$ (see the illustration in the following figure).



Finally, if $2n - k - 3$ is divisible by 4, then again by Lemma 11, there exists a Hamiltonian path P in the subgraph of G induced by $\{(i,j) \mid i = 1,2,3;\ k+2 \leqslant j \leqslant 2n-2\}$ from $(1, k+2)$ to $(2, k+2)$. A required Hamiltonian path in G is $(1,1)(1,2n)(1,2n-1)(3,2n)(3,2n-1)(2,2n-1)(2,2n)(2,1)(3,1)(3,2)(3,3)(3,4)(1,3)(1,2)(2,2)(2,3)(2,4)(1,4)(1,5)\cdots(1,k-1)(2,k-1)(2,k-2)\cdots(2,5)(3,5)(3,6)\cdots(3,k+1)(1,k)(1,k+1)P(2,k+1)(2,k)$.

This proves that the result is true for $m = 1$.

As in [1], Hamiltonian paths in $C(2n,3,1)$ can be extended to those in $C(2n,2m+1,1)$ for any positive integer m (see the illustrations in Figs. 2–7, where Hamiltonian paths in Figs. 2–4 are extended to Hamiltonian paths in Figs. 5–7, respectively). Hence $C(2n,2m+1,1)$ is Hamiltonian laceable. □

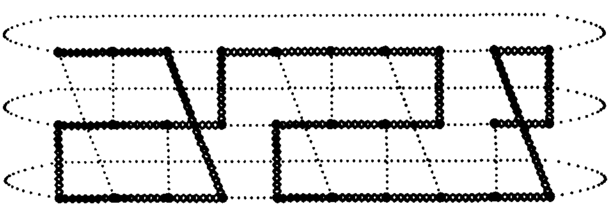


Fig. 2. A Hamiltonian path in $C(10,3,1)$ from $(1,1)$ to $(2,9)$.

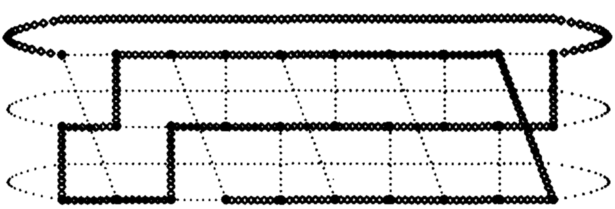


Fig. 3. A Hamiltonian path in $C(10,3,1)$ from $(1,1)$ to $(3,4)$.

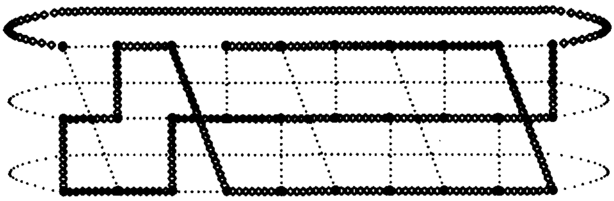


Fig. 4. A Hamiltonian path in $C(10,3,1)$ from $(1,1)$ to $(1,4)$.

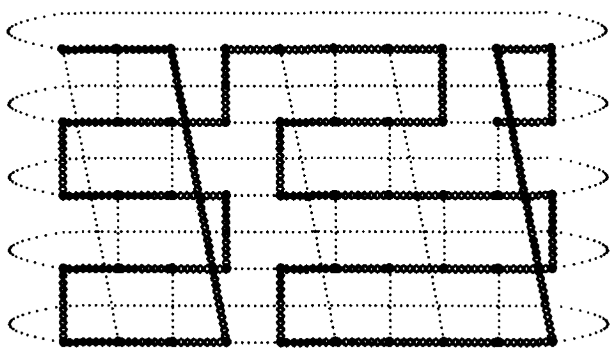


Fig. 5. A Hamiltonian path in $C(10,5,1)$ from $(1,1)$ to $(2,9)$.

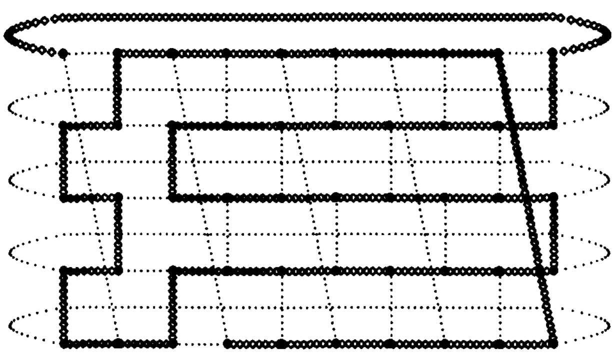


Fig. 6. A Hamiltonian path in $C(10,5,1)$ from $(1,1)$ to $(5,4)$.

It is easy to see that a brick-product $C(2n,1,r)$ is decomposable if and only if $\gcd((r+1)/2,n) > 1$ or $\gcd((r-1)/2,n) > 1$. We have

Theorem 13. *Let G be the brick-product $C(2n,1,r)$. Let $s = \gcd((r+1)/2,n)$ and $t = \gcd((r-1)/2,n)$. If st is even, then G is Hamiltonian laceable.*

Proof. Without loss of generality, we may assume that s is even. Then $C = 12(2+r)(2+r+1)(2+2r+1)(2+2r+2)\cdots(2+kr+k-1)$ is a cycle, where $k = n/s$. From the construction of C and the fact that the mapping f from G to

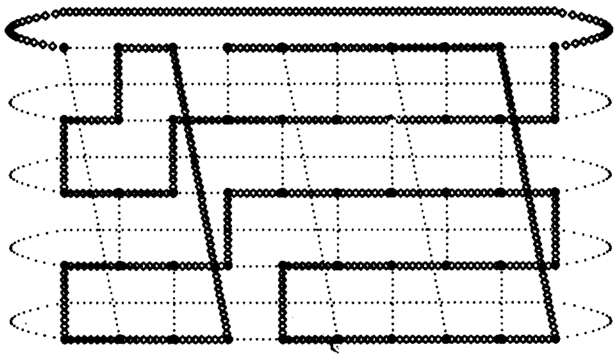


Fig. 7. A Hamiltonian path in $C(10,5,1)$ from $(1,1)$ to $(1,4)$.

G defined by $f(x) = x + 2$ is an automorphism of G, $2h + C$ is a cycle which is either disjoint with C or identical to C, for each positive integer h. There are exactly s such cycles. In fact, G is isomorphic to $C(2k, s, r_1)$ for some even r_1, and is thus Hamiltonian laceable by a result in [1], since s is even. □

Theorem 14. *Let G be the brick-product $C(2n, 1, r)$. Let $s = \gcd((r+1)/2, n)$ and $t = \gcd((r-1)/2, n)$. Then G is Hamiltonian laceable if any one of the following condition holds*:

(i) *s is odd and $rs \equiv r + 1 + 3s \pmod{4n}$*;

(ii) *t is odd and $rt \equiv r - 1 - 3t \pmod{4n}$*.

Proof. From the given conditions, it is easy to see that both s, t are greater than 1. Using the automorphism f of G as given in the proof of Theorem 13, it can be shown that if (i) is satisfied, then G can be decomposed into s cycles and $G \cong C(2n/s, s, r_1)$ with $(r-3)s \equiv (r+1)r_1 \pmod{4n}$. However, as $rs \equiv r + 1 + 3s \pmod{4n}$, we have $r_1 = 1$. Hence G is Hamiltonian laceable, by Lemma 12. Similarly, if (ii) is satisfied, then $G \cong C(2n/t, t, r_1)$ with $(r+3)t \equiv (r-1)r_1 \pmod{4n}$. Again, the condition $rt \equiv r - 1 - 3t \pmod{4n}$ ensures that $r_1 = 1$ and so G is Hamiltonian laceable, by Lemma 12. □

Corollary 15. *All $C(2n, 1, r)$ are Hamiltonian laceable when n is even.*

Proof. This follows from Theorem 13 and the fact that either $(r-1)/2$ or $(r+1)/2$ is even. □

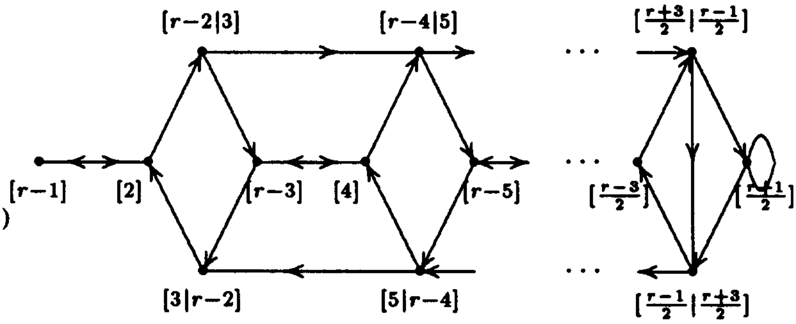
4. Hamiltonian laceability of sufficiently large brick-products

A path of the form $xJP^{-1}[m]$ is called a *loop* and will be denoted by $[m]$, whereas, a path of the form $xJP[m]J^{-1}P[k]$, with $m + k = r + 1$ is called an (k, m) – *twist* (or simply a *twist*) and will be denoted by $[k|m]$.

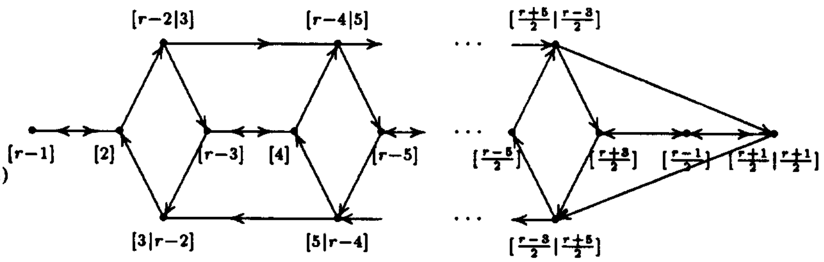
The *tail-digraph* $T[r]$ is defined as follows.

There are $r-2$ vertices, labeled by $[2],[4],\dots,[r-1]$ and $[r-2|3]$, $[r-4|5],\dots,[3|r-2]$. Every vertex labeled by $[u]$ is joined to a vertex labeled by $[v]$ if $u+v=r+1$; each vertex labeled by $[u]$ is joined to a vertex labeled by $[w|v]$ if $u+w=r$; each vertex labeled by $[u|v]$ is joined to a vertex labeled by $[w]$ if $v+w=r$; whereas, each vertex labeled by $[u|v]$ is joined to a vertex labeled by $[w|y]$ if $u-w=2$ (or $y-v=2$) (see the following figures).

$T[r]$:
$r\equiv 3 \pmod 4$

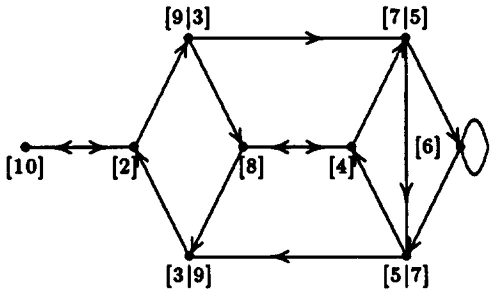


$T[r]$:
$r\equiv 1 \pmod 4$

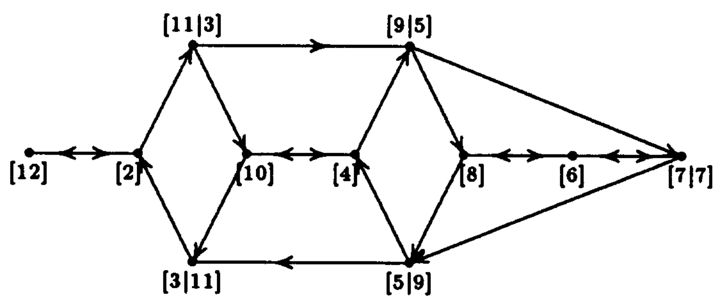


Remark. As shown in the figures above, there are essentially two kinds of tail-digraphs $T[r]$, depending on whether r is congruent to 1 or 3 modulo 4. For example, the tail-digraphs $T[11]$ and $T[13]$ are given in the following figures.

$T[11]$:



$T[13]$:



The *weight* $w(x)$ of each vertex x in $T[r]$ is defined by

$$w(x) = \begin{cases} m & \text{if } x = [m], \\ r+1 & \text{otherwise (i.e. if } x = [a|b] \text{ for some } a, b). \end{cases}$$

A sequence $\mathscr{S} = s_1 s_2 \dots s_m$ of vertices of $T[r]$ is called a *trail* if each s_i is joined to s_{i+1} by an arc for each $i = 1, 2, \dots, m-1$. The *length* $\ell(\mathscr{S})$ of the trail $\mathscr{S}$ is defined as

$$\ell(\mathscr{S}) = \sum_{i=1}^{m} w(s_i).$$

By interpreting each vertex $[a|b]$ in $\mathscr{S}$ as a twist and each vertex $[m]$ either as $P^{-1}[m]$ if $[m] = s_1$ or as a loop $JP^{-1}[m]$ otherwise, it is easy to see that a trail $\mathscr{S}$ of length $k \leqslant 2n$ will induce, in the natural way, a gapless path $x\mathscr{S}$ of order k in G with initial vertex x, where x is any odd vertex in G. For example, let $\mathscr{S} = [2][11|3][10][3|11][2][11|3][9|5][8]$ be a trail in $T[13]$. Then the length of $\mathscr{S}$ is $2 + 14 + 10 + 14 + 2 + 14 + 14 + 8 = 78$ and the path induced by $\mathscr{S}$ with initial vertex 13 is

$$(13)(12)(25)(26)(27)(14)(15)\cdots(24)(37)(36)\cdots$$
$$\cdots(28)(41)(42)\cdots(51)(38)(39)(40)(53)(52)(65)(66)(67)(54)(55)\cdots$$
$$\cdots(64)(77)(78)\cdots(81)(68)(69)\cdots(76)(89)(88)\cdots(82).$$

Note that, except for the first vertex $[2]$, the trail $\mathscr{S}$ can be uniquely determined by the symbol

$$[2]\,tltlttl$$

where each t is a twist and each l is a loop. We shall adopt this abbreviation throughout the following discussion.

Lemma 16. *For any two vertices $[a]$ and $[b]$ in $T[r]$ and any even nonnegative integer $i \leqslant r-1$, there exists a trail $\mathscr{S} = [a]s_1 s_2 \dots s_k[b]$ in $T[r]$ such that the length ℓ of $s_1 s_2 \dots s_k$ is at most $(r-3)(r+1)+4$ and $\ell \equiv i \pmod{r+1}$.*

Proof. We shall only consider the case $r \equiv 3 \pmod 4$. The other case where $r \equiv 1 \pmod 4$ is similar.

Let $e = (r-3)/2$, $L = \{2, 4, \dots, e\}$ and $R = \{e+2, e+4, \dots, r-1\}$. For any element a in $L \cup R$, we shall write $a\uparrow$ (resp. $a\downarrow$) to mean that $a \in L$ (resp. $a \in R$). Also, for any $a, b \in L \cup R$, we shall write $(a \to b)$ if any of the following conditions hold:

(1) $a\uparrow, b\uparrow$ and $a \leqslant b$;
(2) $a\uparrow, b\downarrow$ and $a \leqslant r-1-b$;
(3) $a\downarrow, b\uparrow$ and $r-1-a < b$;
(4) $a\downarrow, b\downarrow$ and $b \leqslant a$.

Furthermore, for any $a, b \in L \cup R$ with $(a \to b)$, we shall define $\|a \to b\|$ as follows:

(1) if $a\uparrow, b\uparrow$, and $(a \to b)$, then $\|a \to b\| = \frac{b}{2} - \frac{a}{2} + 1$;

(2) if $a\uparrow, b\downarrow$, and $(a \to b)$, then $\|a \to b\| = \frac{r-1-b}{2} - \frac{a}{2} + 1$;

(3) if $a\downarrow, b\uparrow$, and $(a \to b)$, then $\|a \to b\| = \frac{b}{2} - \frac{r-1-a}{2} + 1$;

(4) if $a\downarrow, b\downarrow$, and $(a \to b)$, then $\|a \to b\| = \frac{a}{2} - \frac{b}{2} + 1$.

Now, let $a, b, i \in L \cup R$. We shall give below a required trail in $T[r]$ from $[a]$ to $[b]$ satisfying the given conditions, for each of the following possible cases.

Case 1: $a\uparrow, b\uparrow$ and $i\uparrow$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|i\to e\|}[i]t^{\|i\to e\|}t^{\|b\to e\|}[b].$$

Case 2.1: $a\uparrow, b\uparrow$, $i\downarrow$, $(a \to i)$ and $(b \to i)$. Required trail:

$$[a]t^{\|a\to i\|}[i]t^{\|b\to i\|}[b].$$

Case 2.2: $a\uparrow, b\uparrow$, $i\downarrow$, $(a \to i)$ and $(i \to b)$. Required trail:

$$[a]t^{\|a\to (i-2)\|}[i-2]t^{\|2\to (i-2)\|}[2]t^{\|2\to e\|}t^{\|b\to e\|}[b].$$

Case 2.3: $a\uparrow, b\uparrow$, $i\downarrow$, $(i \to a)$ and $(b \to i)$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|2\to e\|}[2]t^{\|2\to (i-2)\|}[i-2]t^{\|b\to (i-2)\|}[b].$$

Case 2.4: $a\uparrow, b\uparrow$, $i\downarrow$, $(i \to a)$, $(i \to b)$ and $i \neq (r+5)/2$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|2\to e\|}[2]t^{\|2\to (i-4)\|}[i-4]t^{\|2\to (i-4)\|}[2]t^{\|2\to e\|}t^{\|b\to e\|}[b].$$

Case 2.5: $a\uparrow, b\uparrow$, $i\downarrow$, $(i \to a)$, $(i \to b)$ and $i = (r+5)/2$. Required trail:

$$[a]t^{\|a\to e\|}[i-2]t^{\|2\to e\|}[2]t^{\|2\to e\|}t^{\|b\to e\|}[b].$$

Case 3.1: $a\uparrow, b\downarrow$, $i\uparrow$ and $(i \to b)$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|i\to e\|}[i]t^{\|i\to b\|}[b].$$

Case 3.2: $a\uparrow, b\downarrow$, $i\uparrow$ and $(b \to i)$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|(i-2)\to e\|}[i-2]t^{\|(i-2)\to e\|}t^{\|2\to e\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|a\to e\|}t^{\|2\to e\|}[i][b] \quad \text{if } i = 2 \text{ and } b = r-1.$$

Case 4.1: $a\uparrow, b\downarrow$, $i\downarrow$, $(a \to i)$ and $i \neq (r+1)/2$. Required trail:

$$[a]t^{\|a\to (i-2)\|}[i-2]t^{\|2\to (i-2)\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|a\to(i-2)\|}[i-2]t^{\|2\to(i-2)\|}[2][b] \quad \text{if } b=r-1.$$

Case 4.2: $a\uparrow, b\downarrow, i\downarrow$, $(a\to i)$ and $i=(r+1)/2$. Required trail:

$$[a]t^{\|a\to e\|}t[i-2]tt^{\|2\to e\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|a\to e\|}t[i-2]tt^{\|2\to e\|}[2][b] \quad \text{if } b=r-1.$$

Case 4.3: $a\uparrow, b\downarrow, i\downarrow$, $(i\to a)$ and $i\neq(r+5)/2$. Required trail:

$$[a]t^{\|a\to e\|}t^{\|2\to e\|}[2]t^{\|2\to(i-4)\|}[i-4]t^{\|2\to(i-4)\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|a\to e\|}t^{\|2\to e\|}[2]t^{\|2\to(i-4)\|}[i-4]t^{\|2\to(i-4)\|}[2][b] \quad \text{if } b=r-1.$$

Case 4.4: $a\uparrow, b\downarrow, i\downarrow$, $(i\to a)$ and $i=(r+5)/2$. Required trail:

$$[a]t^{\|a\to e\|}[i-2]t^{\|2\to e\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|a\to e\|}[i-2]t^{\|2\to e\|}[2][b] \quad \text{if } b=r-1.$$

Case 5.1: $a\downarrow, b\uparrow, i\uparrow$ and $(i\to a)$. Required trail:

$$[a]t^{\|i\to a\|}[i]t^{\|i\to e\|}t^{\|b\to e\|}[b].$$

Case 5.2: $a\downarrow, b\uparrow, i\uparrow$ and $(a\to i)$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|(i-2)\to e\|}[i-2]t^{\|(i-2)\to e\|}t^{\|b\to e\|}[b]$$

or

$$[a][2]t^{\|2\to e\|}t^{\|(i-2)\to e\|}[i-2]t^{\|(i-2)\to e\|}t^{\|b\to e\|}[b] \quad \text{if } a=r-1,\ i\neq 2;$$

or

$$[a][2]t^{\|2\to e\|}t^{\|b\to e\|}[b] \quad \text{if } i=2.$$

Case 6.1: $a\downarrow, b\uparrow, i\downarrow$, $(b\to i)$ and $i\neq(r+1)/2$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to(i-2)\|}[i-2]t^{\|b\to(i-2)\|}[b].$$

Case 6.2: $a\downarrow, b\uparrow, i\downarrow$, $(b\to i)$ and $i=(r+1)/2$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t[i-2]tt^{\|b\to e\|}[b].$$

Case 6.3: $a\downarrow, b\uparrow, i\downarrow$ and $(i\to b)$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to(i-4)\|}[i-4]t^{\|2\to(i-4)\|}[2]t^{\|2\to e\|}t^{\|b\to e\|}[b]$$

or

$$[a]t^{\|2\to a\|}[2]t^{\|2\to(i-2)\|}[i-2][b] \quad \text{if } b=r-i+3;$$

or

$$[a]t^{\|2\to a\|}[2]t^{\|2\to(i-2)\|}[i-2]t[b] \quad \text{if } b=r-i+1.$$

Case 7.1: $a\downarrow, b\downarrow, i\uparrow$, $(i \to a)$ and $i \neq 2$. Required trail:

$$[a]t^{\|(i-2)\to a\|}[i-2]t^{\|(i-2)\to e\|}t^{\|2\to e\|}[2]t^{\|2\to b\|}[b].$$

Case 7.2: $a\downarrow, b\downarrow, i\uparrow$, $(i \to a)$ and $i = 2$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to b\|}[b].$$

Case 7.3: $a\downarrow, b\downarrow, i\uparrow$, $(a \to i)$ and $(i \to b)$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|(i-2)\to e\|}[i-2]\,t^{\|(i-2)\to b\|}[b]$$

Case 7.4: $a\downarrow, b\downarrow, i\uparrow$, $(a \to i)$, $(b \to i)$ and $i \neq 4$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|(i-4)\to e\|}[i-4]t^{\|(i-4)\to e\|}t^{\|2\to e\|}[2]t^{\|2\to b\|}[b]$$

or

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|(i-2)\to e\|}[i-2]t[b] \quad \text{if } b = r+1-i.$$

Case 7.5: $a\downarrow, b\downarrow, i\uparrow$, $(a \to i)$, $(b \to i)$ and $i = 4$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|2\to e\|}[2]t^{\|2\to b\|}[b].$$

Case 8.1: $a\downarrow, b\downarrow, i\downarrow$ and $i > (r+5)/2$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to (i-4)\|}[i-4]t^{\|2\to (i-4)\|}[2]t^{\|2\to b\|}[b].$$

Case 8.2: $a\downarrow, b\downarrow, i\downarrow$ and $i \leqslant (r+5)/2$. Required trail:

$$[a]t^{\|2\to a\|}[2]t^{\|2\to e\|}t^{\|(i-4)\to e\|}[i-4]t^{\|(i-4)\to e\|}t^{\|2\to e\|}[2]t^{\|2\to b\|}[b]. \qquad \square$$

Remark. Note that when $a = r-1$, $b = 2$ and $i = 4$, then the required trial is

$$[r-1][2]t^{\|2\to e\|}t^{\|2\to e\|}[2]t^{\|2\to e\|}t^{\|2\to e\|}[2],$$

which attains the maximal length $(r-3)(r+1)+4$ as stated in Lemma 16.

By virtue of Lemma 16 and the fact that each trail in $T[r]$ induces a gapless path in G with a given initial vertex, we have the following corollary.

Corollary 17. *For any two paths $xP^{-1}[m]$ and $yP[k]$ in G, where $m, k \in L \cup R$, and x is an odd integer and y an even integer such that $y - x \geqslant (r-3)(r+1)+4$, there exists a gapless path P in G with initial vertex x, terminal vertex y and $V(P) = \{x-m+1, x-m+2, \ldots, y+k-1\}$.*

Theorem 18. *Let r be an odd integer with $r \geqslant 7$. If $n \geqslant r^2 - r + 1$, then $G = C(2n, 1, r)$ is Hamiltonian laceable.*

Proof. We first introduce some sequences of paths and vertices in G as follows.

Let $s \in \{-3, -5, \ldots, -r+2\}$. We define, recursively, three sequences of paths P_i, P'_i, P''_i, $i = 1, 2, \ldots$, as follows:

$$P_1 = 0P[2],$$

$$P'_1 = P_1 J^{-1} P[s+r],$$

$$P''_1 = P'_1 J^{-1} P[-s-1];$$

and for $k > 1$, we put

$$P_k = P''_{k-1} J^{-1} P[2],$$

$$P'_k = P_k J^{-1} P[s+r],$$

$$P''_k = P'_k J^{-1} P[-s-1].$$

Next, for each $k = 1, 2, \ldots$, we let

$$x_k = \text{end vertex of } P_k,$$

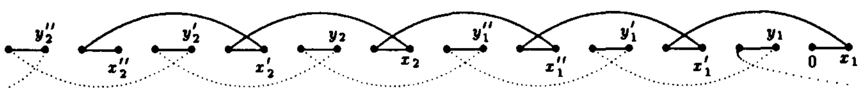
$$x'_k = \text{end vertex of } P'_k,$$

$$x''_k = \text{end vertex of } P''_k,$$

$$y_k = x_k - 2,$$

$$y'_k = x'_k - s - r,$$

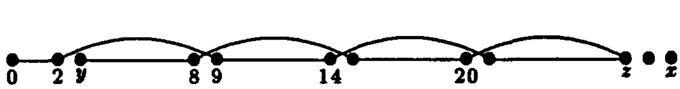
$$y''_k = x''_k + s + 1.$$



It can be shown that for any negative odd integer $x \geqslant -2n$, there exist s and i such that $x \in \{x_i, x'_i, x''_i, y_i, y'_i, y''_i\}$.

As G is vertex transitive, we need only to show that, for any odd vertex x in G, there exists a Hamiltonian path from 0 to x. We have the following two cases to consider.

Case 1: $0 < x \leqslant n$. In this case, by the argument used in the proof of Lemma 5, there exists a gapless path P from 0 to some positive odd vertex $y < r$ such that $V(P) = \{0, 1, \ldots, z\}$, where z is an odd positive vertex with $x - z < r$ (see the illustration in the following figure, where $r = 7$, $x = 29$, $y = 3$, $z = 27$).



By the corollary to Lemma 16, there exists a path Q from x to $2n + y - r$, where $V(Q) = \{z + 1, z + 2, \ldots, 2n - 1\}$. Then PQ^{-1} will be a Hamiltonian path in G from 0 to x. (Note here that $(2n + y - r) - x \geqslant (2n + y - r) - n = n + y - r \geqslant r^2 - r + 1 + y - \mathrm{r} > r^2 - 2\mathrm{r} + 1 = (\mathrm{r} - 3)(\mathrm{r} + 1) + 4$.)

Case 2: $-n \leqslant x < 0$. As mentioned above, there exists x_1, s and i such that $x \in \{x_i, x_i', x_i'', y_i, y_i', y_i''\}$. If $x = x_i$, then let P be the gapless path

$$P_{i-1}''J^{-1}P^{-1}[-s](JP^{-1}[s + r]JP^{-1}[2]JP^{-1}[-s - 1])^{i-1}$$

from 0 to $x_1' + 1$ with $V(P) = \{x_i' + 1, x_i' + 2, \ldots, x_1\}$. By the corollary to Lemma 16, G contains a gapless path Q from $x_1' + 1 + r$ to $y_i + 2 - r$ with $V(Q) = \{x_1 + 1, x_1 + 2, \ldots, 2n + x_i'\}$. Then $PQ(y_i + 2)(y_i + 3) \cdots x_i$ will be a Hamiltonian path in G from 0 to x_i.

The cases $x = x_i'$ or x_i'' can be settled in like manner.

If $x = y_i$, then let P be the gapless path

$$P_iJ^{-1}P[s + r + 1](JP^{-1}[s + r]JP^{-1}[2]JP^{-1}[-s - 1])^{i-1}$$

from 0 to $x_1' + 1$ with $V(P) = \{y_i' + 1, y_i' + 2, \ldots, x_1\}$. By the corollary to Lemma 16, G contains a gapless path Q from $x_1' + 1 + r$ to $x_i' + 2 - r$ with $V(Q) = \{x_1 + 1, x_1 + 2, \ldots, 2n + y_i'\}$. Then $PQ(x_i' + 2)(x_i' + 3) \cdots y_i$ will be a Hamiltonian path in G from 0 to y_i.

The cases $x = y_i'$ or y_i'' can be settled in like manner. □

5. A final remark

Although the results in the paper show that most $C(2n, 1, r)$ are Hamiltonian laceable, with which it is possible to deduce the Hamiltonian laceability of some classes of $C(2n, t, r)$ for odd t, the general problem whether or not all brick-products are Hamiltonian laceable remains open.

Acknowledgements

The authors wish to thank the referee for the many invaluable suggestions, which certainly enhance the presentation of this paper. The second author also wishes to express his thanks to the Australian Vice-Chancellor Committee (AVCC) for the AVCC Fellowship Award, and also to the Department of Computing and Mathematics, Deakin University, Geelong, Australia for allowing him to use its facility for a period of two months from 1st of June till 31st of July 1991 during which this paper was written.

References

[1] B. Alspach and C.C. Chen, On the Hamiltonian laceability of brick products of even cycles, Research Report No. 468, May 1991, Lee Kong Chian Centre for Mathematical Research.
[2] B. Alspach and C.-Q. Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, Ars Combinatoria 28 (1989) 101–108.
[3] C.C. Chen and N. Quimpo, On strongly Hamiltonian abelian group graphs, Lecture Notes in Mathematics, Vol. 884 (Springer, Australia, 1980) 23–34; Proc. of the Internat. Conf. in Combinatorial Theory, Canberra.