JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **83**, 127–134 (1981)

A Bound on the Solutions of a Nonlinear Volterra Equation

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Submitted by J. P. LaSalle

We study the scalar, nonlinear Volterra integrodifferential equation (*), $x'(t) + \int_{[0,t]} g(x(t-s))\,d\mu(s) = f(t)$ $(t \geqslant 0)$. We let g be continuous, μ positive definite, and f integrable over $(0, \infty)$. The standard assumption on g which yields boundedness of the solutions of (*) prevents $g(x)$ from growing faster than an exponential as $x \to \infty$. Here we present a weaker condition on g, which does not restrict the growth rate of $g(x)$ as $x \to \infty$, but which still implies that the solutions of (*) are bounded. In particular, when g is nondecreasing and either nonnegative or odd, we get bounds which are independent of g.

1. INTRODUCTION

We study the scalar, nonlinear Volterra integrodifferential equation

$$x'(t) + \int_{[0,t]} g(x(t-s))\,d\mu(s) = f(t) \qquad (t \geqslant 0). \tag{1.1}$$

Throughout we suppose that g is continuous, and that it has an integral which is bounded from below:

$$g \in C(R), \quad \inf_{x \in R} G(x) > -\infty, \quad \text{where} \quad G(x) = G(0) + \int_0^x g(y)\,dy. \tag{1.2}$$

We let μ be a positive definite measure and f an integrable function on the interval $[0, T]$:

$$\int_0^t \varphi(\tau) \int_0^\tau \varphi(\tau - s)\,d\mu(s)\,d\tau \geqslant 0 \text{ whenever } 0 \leqslant t < T \text{ and } \varphi \in C[0, t], \tag{1.3}$$

$$f \in L^1(0, T). \tag{1.4}$$

Here $0 < T \leqslant \infty$.

0022-247X/81/090127-08$02.00/0

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If in addition g satisfies the growth condition

$$\limsup_{x\to\pm\infty} \frac{|g(x)|}{G(x)} < \infty, \tag{1.5}$$

then a well known argument [5, p. 572] shows that every solution x of (1.1) on $[0, T)$ satisfies

$$\sup_{0\leqslant t<T} G(x(t)) < \infty. \tag{1.6}$$

Clearly, if in addition

$$G(x) \to \infty \qquad (x \to \pm\infty), \tag{1.7}$$

then the solutions of (1.1) are bounded on $[0, T)$.

Essentially, (1.5) may be regarded as a restriction on the growth rate of g at $\pm\infty$. In particular, if (1.2), (1.5) hold, then $|g(x)|$ cannot grow faster than an exponential as $x \to \pm\infty$. Here we sharpen the above mentioned argument, and get sharper bounds, which permit $|g(x)|$ to grow arbitrarily fast as $x \to \pm\infty$. Some of our bounds (Corollaries 3.1 and 3.2 below) are actually independent of g, as long as g satisfies some general conditions.

In Section 4 we discuss a related boundedness result due to Levin.

2. A Basic Estimate

All our estimates are essentially applications of the following theorem:

THEOREM 2.1. (i) *Let* (1.2), (1.3), *and* (1.4) *hold for some* $T \leqslant \infty$. *For all* y *such that the set* $\{z \in R \mid G(z) \leqslant y\}$ *is nonempty, define*

$$u(y) = \sup\{|g(z)| \mid G(z) \leqslant y\}. \tag{2.1}$$

Then every solution x *of* (1.1) *on* $[0, T)$ *satisfies*

$$\int_{G(x(0))}^{G(x(t))} \frac{dy}{u(y)} \leqslant \int_0^t |f(\tau)|\, d\tau \qquad (0 \leqslant t < T). \tag{2.2}$$

(ii) *In addition to* (i), *suppose that*

$$\int_{G(x(0))}^{\infty} \frac{dy}{u(y)} > \int_0^T |f(\tau)| \, d\tau. \tag{2.3}$$

Then $G(x(t))$ *is bounded on* $[0, T)$. *In particular, this is true if the integral* $\int_{G(x(0))}^{\infty} (dy/u(y))$ *diverges.*

(iii) *If moreover* $G(x) \to \infty$ $(x \to \pm\infty)$, *then* x *is bounded on* $[0, T)$.

Clearly, the function u in (2.1) is nondecreasing, and $0 \leqslant u(y) \leqslant \infty$. If, e.g., $G(x) \to \infty$ $(x \to \pm\infty)$, then $u(y) < \infty$ for $y < \infty$. Interpret $1/0$ as ∞ and $1/\infty$ as 0 in the left hand side of (2.2).

The boundedness result mentioned in the Introduction is contained in Theorem 2.1. If (1.2), (1.5) hold, then we can, without loss of generality (add a constant to G), assume that $g(x) \leqslant CG(x)$ for some constant C. This implies $u(y) \leqslant Cy$; hence $\int_{G(x(0))}^{\infty} (dy/u(y)) \geqslant (1/C) \int_{G(x(0))}^{\infty} (dy/y) = \infty$, so Theorem 2.1 (ii) applies.

Proof of Theorem 2.1. Without loss of generality, take $\inf_{y \in R} G(y) = 0$ (i.e., replace $G(x)$ by $G(x) - \gamma$ and $u(y)$ by $u(y + \gamma)$, where $\gamma = \inf_{y \in R} G(y)$). Replace t by τ in (1.1), multiply by $g(x(\tau))$, and integrate over $(0, t)$ to obtain

$$\begin{aligned} G(x(t)) &+ \int_0^t g(x(\tau)) \int_0^\tau g(x(\tau - s)) \, d\mu(s) \, d\tau \\ &= G(x(0)) + \int_0^t g(x(\tau)) f(\tau) \, d\tau. \end{aligned} \tag{2.4}$$

Hence, by (1.3) and (2.1),

$$\begin{aligned} G(x(t)) &\leqslant G(x(0)) + \int_0^t g(x(\tau)) f(\tau) \, d\tau \\ &\leqslant G(x(0)) + \int_0^t u(G(x(\tau))) \, |f(\tau)| \, d\tau. \end{aligned} \tag{2.5}$$

The function u is nonnegative and nonincreasing, so if it is in addition continuous then one can apply Bihari's inequality [1, p. 83] to obtain (2.2). The discontinuous case can be reduced to the continuous case as follows. Fix $t \in [0, T)$. Without loss of generality, assume

$$\int_{G(x(0))}^{\infty} \frac{dy}{u(y)} > \int_0^t |f(\tau)| \, d\tau$$

(because otherwise (2.2) is trivially true). Then, for some $M < \infty$,

$$\int_{G(x(0))}^{M} \frac{dy}{u(y)} > \int_0^t |f(\tau)|\, d\tau.$$

This number M can be chosen so that $u(M) < \infty$, because $\int_{M_1}^{\infty} (dy/u(y)) = 0$, where $M_1 = \inf\{y \mid u(y) = \infty\}$. Pick $\varepsilon > 0$. Let v be a continuous, nondecreasing function on $[0, \infty)$ satisfying

$$u(y) \leqslant v(y) \qquad (0 < y \leqslant M),$$

and

$$\int_{G(x(0))}^{M} \frac{dy}{v(y)} > \max\left\{\int_0^t |f(\tau)|\, d\tau, \int_{G(x(0))}^{M} \frac{dy}{u(y)} - \varepsilon\right\}. \tag{2.6}$$

Then

$$G(x(s)) \leqslant G(x(0)) + \int_0^s v(G(x(s)))\, |f(\tau)|\, d\tau \tag{2.7}$$

for as long as

$$G(x(s)) \leqslant M. \tag{2.8}$$

Apply Bihari's inequality to (2.7) to obtain

$$\int_{G(x(0))}^{G(x(s))} \frac{dy}{v(y)} \leqslant \int_0^s |f(\tau)|\, d\tau \tag{2.9}$$

for as long as (2.8) holds. But (2.6) and (2.9) imply that (2.8) holds on the whole interval $[0, t]$, and so (2.9) holds for $s = t$. This, together with (2.6) and the fact that ε can be chosen arbitrarily small, yields (2.2). Statements (ii) and (iii) follow trivially from (i). ∎

3. Further Bounds

If g is nondecreasing, then one can develop (2.2) further. The case when g does not change sign is simplest, so we discuss it first.

Suppose that g is nonnegative and nondecreasing. Define

$$\omega = \inf\{x \mid g(x) > 0\}. \tag{3.1}$$

The case $\omega = \infty$, i.e., $g \equiv 0$, is trivial, so we assume $-\infty \leqslant \omega < \infty$. Define $G(-\infty) = \lim_{x \to -\infty} G(x)$. Then G is defined and nondecreasing on $[-\infty, \infty)$,

$[-\infty, \infty)$, and strictly increasing on $[\omega, \infty)$. Let G_+ be the restriction of G to $[\omega, \infty)$. Then G_+^{-1} is continuous on $[G(\omega), \infty)$, and continuously differentiable on $(G(\omega), \infty)$, with

$$\frac{d}{dy} G_+^{-1}(y) = [g(G_+^{-1}(y))]^{-1} \qquad (y > G(\omega)). \tag{3.2}$$

Because of the monotonicity of g and G, the function u defined in (2.1) becomes

$$u(y) = g(G_+^{-1}(y)) \qquad (y > G(\omega)). \tag{3.3}$$

Substituting (3.2), (3.3) into the left hand side of (2.2) we obtain

$$\begin{aligned} \int_{G(x(0))}^{G(x(t))} \frac{dy}{u(y)} &= G_+^{-1}(G(x(t))) - G_+^{-1}(G(x(0))) \\ &= \max\{\omega, x(t)\} - \max\{\omega, x(0)\}. \end{aligned} \tag{3.4}$$

One could substitute this into (2.2) to obtain an upper bound on x, but a minor modification of the proof of Theorem 2.1 yields an even sharper extimate. In this case, as g is nonnegative, one can replace $|f(\tau)|$ in (2.5) by $\max\{0, f(\tau)\}$, and this means that the same substitution can be made in (2.2) and (2.3). Thus, we have almost proved the following corollary:

COROLLARY 3.1. *In addition to* (1.2), (1.3), *and* (1.4), *suppose that g is nonnegative and nondecreasing. Then every solution x of* (1.1) *on* $[0, T)$ *satisfies*

$$x(t) \leqslant x(0) + \int_0^t \max\{0, f(\tau)\}\, d\tau \qquad (0 \leqslant t < T). \tag{3.5}$$

Completion of Proof. Clearly, (3.5) follows from the preceding argument, provided $x(0) > \omega$. If $x(0) < \omega$ (in particular, if $g \equiv 0$ and $\omega = \infty$), then define $t_1 = \sup\{t \in [0, T) \mid x(s) \leqslant \omega \text{ in } [0, t]\}$. Clearly, (1.1) becomes the trivial equation $x'(t) = f(t)$ for $t \in [0, t_1)$, and this implies that (3.5) holds for $t < t_1$. In particular, if $t_1 = T$, then (3.5) holds. If $t_1 < T$, then $x(t_1) = \omega$, and $\int_0^{t_1} \max\{0, f(\tau)\}\, d\tau \geqslant \omega - x(0)$. Define $y(t) = x(t - t_1)$ for $0 \leqslant t < T - t_1$. Then $y(0) = \omega$, and y satisfies (1.1) with $f(t)$ replaced by $f(t + t_1)$. Apply (3.5) to obtain

$$\begin{aligned} y(t) &\leqslant \omega + \int_0^t \max\{0, f(\tau + t_1)\}\, d\tau \\ &= \omega + \int_{t_1}^{t_1 + t} \max\{0, f(\tau)\}\, d\tau. \end{aligned}$$

Thus, for $t \geqslant t_1$ we have

$$x(t) \leqslant \omega + \int_{t_1}^{t} \max\{0, f(\tau)\} \leqslant x(0) + \int_{0}^{t} \max\{0, f(\tau)\}\, d\tau.$$

so in this case, too, (3.5) holds. ∎

If g, instead of being nonnegative, is nonpositive, then the same argument yields

$$x(t) \geqslant x(0) - \int_{0}^{t} \min\{0, f(\tau)\}\, d\tau. \tag{3.6}$$

Finally, suppose that g is nondecreasing and changes sign. Define ω as in (3.1), and let α be the number

$$\alpha = \sup\{x \mid g(x) < 0\}. \tag{3.7}$$

Then $-\infty < \alpha \leqslant \omega < \infty$, and G is strictly decreasing on $(-\infty, \alpha]$. Define G_+^{-1} as before, and let G_-^{-1} be the inverse of the restriction of G to $(-\infty, \alpha]$. Then

$$\frac{d}{dy} G_-^{-1}(y) = [g(G_-^{-1}(y))]^{-1} \qquad (y > G(\alpha)), \tag{3.8}$$

and the function u becomes

$$u(y) = \max\{|g(G_-^{-1}(y))|, g(G_+^{-1}(y))\}. \tag{3.9}$$

In particular, if g is odd, then $u(y) = |g(G_+^{-1}(y))|$, and the same argument as that used when g did not change sign yields the following corollary:

COROLLARY 3.2. *In addition to* (1.2), (1.3), *and* (1.4), *suppose that g is nondecreasing and odd. Then every solution x of* (1.1) *on* $[0, T)$ *satisfies*

$$|x(t)| \leqslant |x(0)| + \int_{0}^{t} |f(\tau)|\, d\tau. \tag{3.10}$$

If g is not odd, then in many cases it is still true that either $g(G_+^{-1}(y))$ dominates $|g(G_-^{-1}(y))|$ in the sense that

$$|g(G_-^{-1}(y))| \leqslant q\, g(G_+^{-1}(y)) \qquad (y > G(\alpha)) \tag{3.11}$$

for some positive constant q, or $|g(G_-^{-1}(y))|$ dominates $g(G_+^{-1}(y))$ in the same sense:

$$g(G_+^{-1}(y)) \leqslant r\, |g(G_-^{-1}(y))| \qquad (y > G(\omega)). \tag{3.12}$$

By (3.2), (3.8), this is equivalent to requiring that either $|(d/dy)\, G_-^{-1}(y)|$ dominates $(d/dy)\, G_+^{-1}(y)$, or vice versa. Of course, it may happen that both (3.11) and (3.12) hold simultaneously. If (3.11) is true, then u satisfies

$$u(y) \leqslant \max\{q, 1\}\; g(G_+^{-1}(y)),$$

and this together with (2.2), (3.2) yields

$$G_+^{-1}(G(x(t))) \leqslant G_+^{-1}(G(x(0))) + \max\{1, q\} \int_0^t |f(\tau)|\, d\tau.$$

This inequalty can be slightly sharpened, if one redefines u so that $u(y) = g(G_+^{-1}(y))$, and replaces $|f(\tau)|$ in (2.5) by $\max\{f(\tau), -qf(\tau)\}$. This leads to the same substitution in (2.2). Thus, we have the following corollary:

COROLLARY 3.3. *In addition to* (1.2), (1.3), *and* (1.4), *suppose that g is nondecreasing, changes sign, and satisfies* (3.11) *for some* $q > 0$. *Then every solution x of* (1.1) *on* $[0, T)$ *satisfies*

$$G_+^{-1}(G(x(t))) \leqslant G_+^{-1}(G(x(0))) + \int_0^t \max\{f(\tau), -qf(\tau)\}\, d\tau \quad (0 \leqslant t < T). \tag{3.13}$$

In particular, x is bounded on $[0, T)$.

The boundedness of x follows from the fact that in Corollary 3.3, $G(x) \to (x \to \pm\infty)$; hence (3.13) acts as a two-sided bound for x. Observe that $G_+^{-1}(G(x(t))) = x(t)$ whenever $x(t) \geqslant \omega$.

If g satisfies (3.12) rather than (3.11), then (3.13) becomes

$$G_-^{-1}(G(x(t))) \geqslant G_-^{-1}(G(x(0))) - \int_0^t \min\{f(\tau), -rf(\tau)\}\, d\tau \qquad (0 \leqslant t < T). \tag{3.14}$$

If $x(t) \leqslant \alpha$, then $G_-^{-1}(G(x(t))) = x(t)$.

Let us illustrate the conclusion of Corollary 3.3 with an example. Let g be continuous and nondecreasing on $[0, \infty)$, with $g(0) = 0$ and $g(x) > 0$ $(x > 0)$ (e.g., $g(x) = x^p$ for some $p > 0$ will do). Extend g to $(-\infty, \infty)$ by $g(x) = -kg(k|x|)$ $(x < 0)$, where k is some positive constant. Then $G_-(-x) = G_+(kx)$ $(x > 0)$, and $g(G_-^{-1}(y)) = -kg(G_-^{-1}(y))$ $(y \geqslant G(0))$. In particular, (3.11) is true with $q = k$, and (3.12) is true with $r = 1/k$. In this case (3.13) and (3.14) yield the same bound, namely,

$$\max\{x(t), -kx(t)\} \leqslant \max\{x(0), -kx(0)\} + \int_0^t \max\{f(\tau), -kf(\tau)\}\, d\tau \qquad (0 \leqslant t < T).$$

When $k = 1$, g is odd, and we get the old bound (3.10).

4. A Bound Due to Levin

In [3] Levin studies the integrated version

$$x(t) + \int_0^t a(t-s)\, g(x(s))\, ds = h(t) \qquad (t \geqslant 0) \tag{4.1}$$

of (1.1). Equations (1.1) and (4.1) are eqivalent, if one defines $a(t) = \mu([0, t])$, and $h(t) = x(0) + \int_0^t f(\tau)\, d\tau$. Theorem 1 in [3] reads as follows:

THEOREM A. *Let g be continuous, $xg(x) \geqslant 0$ $(-\infty < x < \infty)$, $a \geqslant 0$, a nonincreasing on $[0, \infty)$, $a(0) < \infty$, and $h \in C[0, \infty) \cap BV[0, \infty)$. Then there exists a continuous solution of* (1.1) *on* $[0, \infty)$. *Moreover, any continuous solution x of* (1.1) *satisfies*

$$\sup_{0 \leqslant t < \infty} |x(t)| \leqslant V(h) + \sup_{0 \leqslant t < \infty} |h(t)|, \tag{4.2}$$

where $V(h)$ is the total variation of h on $[0, \infty)$.

Theorem A was later improved (see [2, 4]). In particular, in [4] Levin replaces $\sup |h(t)|$ in (4.2) by $\inf |h(t)|$. If h is absolutely continuous, then $V(h) = \int_0^\infty |h'(\tau)|\, d\tau$, and substituting $h'(\tau) = f(\tau)$ one gets a bound which is very close to (3.10) (observe that $h(0) = x(0)$). Levin's assumption on a is stronger than our assumption on μ, because if a is nonnegative and nonincreasing with $a(0) < \infty$, then μ satisfies (1.3). On the other hand, contrary to Corollary 3.2, Levin does not require that g be nondecreasing or odd. Levin's proof is completely different from ours.

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