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The rate of convergence of Hurst index estimate for the stochastic differential equation[☆]

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Abstract

We consider a stochastic differential equation involving a pathwise integral with respect to fractional Brownian motion. The estimates for the Hurst parameter are constructed according to first- and second-order quadratic variations of observed values of the solution. The rate of convergence of these estimates to the true value of a parameter is established when the diameter of interval partition tends to zero.

MSC: 60G22; 60H10

Keywords: Fractional Brownian motion; Stochastic differential equation; First- and second-order quadratic variations; Estimates of Hurst parameter; Rate of convergence

1. Introduction

Consider a stochastic differential equation

$$X_t = \xi + \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dB_s^H, \quad t \in [0,T], T > 0, \tag{1}$$

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where f and g are measurable functions, B^H is a fractional Brownian motion (fBm) with Hurst index $1/2 < H < 1$, ξ is a random variable. It is well-known that almost all sample paths of B^H have bounded p-variations for $p > 1/H$. Therefore it is natural to define the integral with respect to fractional Brownian motion as pathwise Riemann–Stieltjes integral (see, e.g., [27] for the original definition and [11] for the advanced results).

A solution to stochastic differential equation (1) on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F} = \{\mathcal{F}_t\}, t \in [0, T])$, with respect to the fixed fBm $(B^H, \mathbb{F})$, $1/2 < H < 1$ and with $\mathcal{F}_0$-measurable initial condition ξ is defined as an adapted and continuous process $X = \{X_t : 0 \leqslant t \leqslant T\}$ such that $X_0 = \xi$ a.s.,

$$\mathbf{P}\left(\int_0^t |f(X_s)|\,ds + \left|\int_0^t g(X_s)\,dB_s^H\right| < \infty\right) = 1 \quad \text{for every } 0 \leqslant t \leqslant T,$$

and its almost all sample paths satisfy (1).

For $0 < \alpha \leqslant 1$, $\mathcal{C}^{1+\alpha}(\mathbb{R})$ let denote the set of all continuously differentiable functions $g: \mathbb{R} \to \mathbb{R}$ such that

$$\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x - y|^\alpha} < \infty.$$

Let f be a Lipschitz continuous function and let $g \in \mathcal{C}^{1+\alpha}(\mathbb{R})$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Then there exists a unique solution to Eq. (1) having almost all sample paths in the space of continuous functions defined on $[0, T]$ with bounded p-variation for any $p > \frac{1}{H}$ (See [10,20,21,17]). Different, but similar in many features approach to the integration with respect to fractional Brownian motion based on the integration in Besov spaces and corresponding stochastic differential equations were studied in [26], see also [5,22], where different approaches to stochastic integration and to stochastic differential equations involving fractional Brownian motion are summarized.

The main goal of the present paper is to establish the rate of convergence of two estimates of the Hurst parameter to the true value of a parameter. The estimates are based on two types of quadratic variations of the observed solution to the stochastic differential equation. Equation involves the integral with respect to fractional Brownian motion and is considered on the fixed interval $[0, T]$. Note that the study of asymptotic behavior with probability 1 of power variations and other nonlinear functionals for Gaussian processes was originated by Gladyshev [12], Berman [3] and Kawada and Kôno [16] and was continued, e.g., by Jain and Monrad [15] and Norvaiša [23]. These results were applied for the estimation of Hurst index based on the observations of fBm, e.g., by Coeurjolly [8]. In the present paper, we apply the results concerning the convergence of nonlinear functionals of Gaussian processes with probability 1 from Benassi et al. [2] and Istas and Lang [14] and estimate Hurst index based on the observations of the solution of SDE involving fBm. It allows to establish the convergence of estimators of Hurst index with probability 1 as well as the rate of convergence. It should be pointed out that an estimator of the Hurst index for a pathwise solution of an SDE driven by a fBm was constructed and considered in [4], but in particular cases. The strong consistency and asymptotic normality were established. On the other hand, there is a number of results concerning the central limit theorem for weighted power variations for fractional Brownian motion and related processes; see, e.g., Nourdin [24] and Nourdin et al. [25] and the references therein. In the papers of Breton et al. [6] and Breton and Coeurjolly [7], an exact (non-asymptotic) confidence interval for Hurst index was derived with the help of concentration inequalities for quadratic forms of

Gaussian process. Since it is hard to generalize these inequalities to the solutions of SDE involving fBm, the same as it is difficult to obtain the rate of convergence applying the results on weak convergence of Nourdin et al., we choose another way and study analytically the asymptotic behavior with probability 1 for the quadratic variations of solutions of SDE. The paper is organized as follows: Section 2 contains the results on asymptotic behavior of the normalized first- and second-order quadratic variations of fractional Brownian motion. Section 3 describes the rate of convergence of the first- and second-order quadratic variations of the solution to the stochastic differential equation involving fBm. Section 4 contains the main result concerning the rate of convergence of the constructed estimates of Hurst index to its true value when the diameter of partitions of the interval $[0, T]$ tends to zero. Section 5 contains simulation results. In Appendix, the properties of p-variations and of the integrals with respect to the functions of bounded p-variations are given.

2. Asymptotic property of the first- and second-order quadratic variations of fractional Brownian motion

Consider the fractional Brownian motion (fBm) $B^H = \{B_t^H, t \in [0, T]\}$ with Hurst index $H \in (\frac{1}{2}, 1)$. Its sample paths are almost surely locally Hölder continuous up to order H. For any $0 < \gamma < H$ we have that $L_T^{H,\gamma} := \sup_{\substack{s \neq t \\ s,t \leqslant T}} \frac{|B_t^H - B_s^H|}{|t-s|^\gamma}$ is finite a.s. and moreover, $\mathbf{E}\left(L_T^{H,\gamma}\right)^k < \infty$ for any $k \geqslant 1$. Let $v_p(B^H; [s,t])$ denote the p-variation of B^H and $V_p(B^H; [s,t]) = v_p^{1/p}(B^H; [s,t])$. The following estimate for the p-variation of fBm is evident:

$$V_p\left(B^H; [s,t]\right) \leqslant L_T^{H,1/p} (t-s)^{1/p}, \tag{2}$$

where $s < t \leqslant T$, $p > 1/H$.

Let $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$, $T > 0$, be a sequence of uniform partitions of interval $[0, T]$ with $t_k^n = \frac{kT}{n}$ for all $n \in \mathbb{N}$ and all $k \in \{0, \dots, n\}$, and let X be some real-valued stochastic process defined on the interval $[0, T]$.

Definition 1. The normalized first- and second-order quadratic variations of X corresponding to the partitions $(\pi_n)_{n\in\mathbb{N}}$ and to the value $H \in (1/2, 1)$ are defined as

$$V_n^{(1)}(X, 2) = (nT^{-1})^{2H-1} \sum_{k=1}^{n} \left(\Delta_{k,n}^{(1)} X\right)^2, \quad \Delta_{k,n}^{(1)} X = X(t_k^n) - X(t_{k-1}^n),$$

and

$$V_n^{(2)}(X, 2) = (nT^{-1})^{2H-1} \sum_{k=1}^{n-1} \left(\Delta_{k,n}^{(2)} X\right)^2,$$

$$\Delta_{k,n}^{(2)} X = X\left(t_{k+1}^n\right) - 2X\left(t_k^n\right) + X\left(t_{k-1}^n\right).$$

For simplicity, we shall omit index n for points t_k^n of partitions π_n.

It is known (see, e.g., Gladyshev [12]) that $V_n^{(1)}(B^H, 2) \to T$ a.s. as $n \to \infty$. Also, it was proved in Benassi et al. [2] and Istas and Lang [14] that $V_n^{(2)}(B^H, 2) \to (4 - 2^{2H})T$ a.s. as $n \to \infty$.

Denote

$$V_n^{(1)}(B^H,2)_t = (nT^{-1})^{2H-1}\sum_{k=1}^{r(t)}(\Delta_{k,n}^{(1)}B^H)^2,$$

$$V_n^{(2)}(B^H,2)_t = (nT^{-1})^{2H-1}\sum_{k=1}^{r(t)-1}(\Delta_{k,n}^{(2)}B^H)^2, \quad t\in[0,T],$$

where $r(t)=\max\{k\colon t_k\le t\}=[\frac{tn}{T}]$. It is evident that

$$\mathbf{E}V_n^{(1)}(B^H,2)_t=\rho(t) \quad \text{and} \quad \mathbf{E}V_n^{(2)}(B^H,2)_t=(4-2^{2H})\rho(t), \tag{3}$$

where $\rho(t)=\max\{t_k\colon t_k\leqslant t\}$.

The following classical result will be used in the proof of Theorem 3.

Lemma 2 (*[19] Lévy–Octaviani inequality*)**.** *Let $X_1,\dots,X_n$ be independent random variables. Then for any fixed $t,s\geqslant 0$*

$$\mathbf{P}\left(\max_{1\leqslant i\leqslant n}\left|\sum_{j=1}^{i}X_j\right|>t+s\right)\leqslant\frac{\mathbf{P}\left(\left|\sum_{j=1}^{n}X_j\right|>t\right)}{1-\max\limits_{1\leqslant i\leqslant n}\mathbf{P}\left(\left|\sum_{j=i}^{n}X_j\right|>s\right)}.$$

Theorem 3. *The following asymptotic property holds for the first- and second-order quadratic variations of fractional Brownian motion:*

$$\sup_{t\leqslant T}\left|V_n^{(i)}(B^H,2)_t-\mathbf{E}V_n^{(i)}(B^H,2)_t\right|=\mathcal{O}\left(n^{-1/2}\ln^{1/2}n\right)\quad a.s., i=1,2. \tag{4}$$

Remark 4. It follows from (3) and (4) that $\sup_{t\leqslant T}\left|V_n^{(i)}(B^H,2)_t\right|=\mathcal{O}(1)$ a.s., $i=1,2$.

Proof. We can consider $V_n^{(i)}\left(B^H,2\right)$ as the square of the Euclidean norm of the n-dimensional Gaussian vector X_n with the components

$$(nT^{-1})^{H-1/2}\Delta_{k,n}^{(i)}B^H, \quad 1\leqslant k\leqslant n-(i-1).$$

Obviously, there exist nonnegative real numbers $(\lambda_{1,n}^{(i)},\dots,\lambda_{n-(i-1),n}^{(i)})$ and $n-(i-1)$-dimensional Gaussian vector Y_n with independent Gaussian $\mathcal{N}(0,1)$-components, satisfying the equality

$$V_n^{(i)}\left(B^H,2\right)=\sum_{j=1}^{n-(i-1)}\lambda_{j,n}^{(i)}\left(Y_n^{(j)}\right)^2.$$

The numbers $(\lambda_{1,n}^{(i)},\dots,\lambda_{n-(i-1),n}^{(i)})$ are the eigenvalues of the symmetric $n-(i-1)\times n-(i-1)$-matrix

$$\left((nT^{-1})^{2H-1}\mathbf{E}\left[\Delta_{j,n}^{(i)}B^H\Delta_{k,n}^{(i)}B^H\right]\right)_{1\leqslant j,k\leqslant n-(i-1)}.$$

Now we can apply Hanson and Wright's inequality (see Bégyn [1] or Hanson and Wright [13]), and it yields that for $\varepsilon > 0$

$$\mathbf{P}\left(\left|\sum_{j=k}^{n-(i-1)} \lambda_{j,n}^{(i)} \left[\left(Y_n^{(j)}\right)^2 - 1\right]\right| \geq \varepsilon\right) \leqslant 2\exp\left(-\min\left[\frac{C_1\varepsilon}{\lambda_{k,n}^{*(i)}}, \frac{C_2\varepsilon^2}{\sum\limits_{j=k}^{n-(i-1)} (\lambda_{j,n}^{(i)})^2}\right]\right)$$

$$\leqslant 2\exp\left(-\min\left[\frac{C_1\varepsilon}{\lambda_{n}^{*(i)}}, \frac{C_2\varepsilon^2}{\sum\limits_{j=1}^{n-(i-1)} (\lambda_{j,n}^{(i)})^2}\right]\right), \quad (5)$$

where C_1, C_2 are nonnegative constants, $\lambda_{k,n}^{*(i)} = \max_{k\leqslant j\leqslant n-(i-1)} \lambda_{j,n}^{(i)}$, $\lambda_n^{*(i)} = \max_{1\leqslant j\leqslant n-(i-1)} \lambda_{j,n}^{(i)}$.

Evidently,

$$\sum_{j=1}^{n-(i-1)} \lambda_{j,n}^{(i)} = \mathbf{E}V_n^{(i)}\left(B^H, 2\right).$$

Furthermore, it follows from (3) that the sequence $\mathbf{E}V_n^{(i)}\left(B^H, 2\right), n \geq 1$, is bounded. So, the sums $\sum_{j=1}^{n-(i-1)} \lambda_{j,n}^{(i)}$ are bounded as well. It is easy to check that

$$\sum_{j=1}^{n-(i-1)} (\lambda_{j,n}^{(i)})^2 \leqslant \lambda_n^{*(i)} \sum_{j=1}^{n-(i-1)} \lambda_{j,n}^{(i)}.$$

Therefore for any $0 < \varepsilon \leqslant 1$ inequality (5) can be rewritten as

$$\mathbf{P}\left(\left|\sum_{j=k}^{n-(i-1)} \lambda_{j,n} \left[\left(Y_n^{(j)}\right)^2 - 1\right]\right| \geq \varepsilon\right) \leqslant 2\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right), \quad (6)$$

where K is a positive constant.

Now we use the Lévy–Octaviani inequality (see Lemma 2) and evident inequality

$$\frac{x}{1-x} \leqslant 2x \quad \text{for } 0 < x \leqslant 1/2$$

to obtain the bound

$$\mathbf{P}\left(\max_{1\leqslant k\leqslant n-(i-1)}\left|\sum_{j=1}^{k} \lambda_{j,n}^{(i)} \left[\left(Y_n^{(j)}\right)^2 - 1\right]\right| > 2\varepsilon\right) \leqslant \frac{2\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right)}{1 - 2\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right)}$$

$$\leqslant 4\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right),$$

assuming that

$$\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right) \leqslant 1/4 \quad \text{and} \quad 0 < \varepsilon \leq 1.$$

So, for the values of parameters mentioned above,

$$\mathbf{P}\left((nT^{-1})^{2H-1} \max_{1\leqslant k\leqslant n-(i-1)} \left|\sum_{j=1}^{k}\left(\Delta_{j,n}^{(i)} B^H\right)^2 - \sum_{j=1}^{k}\mathbf{E}\left(\Delta_{j,n}^{(i)} B^H\right)^2\right| > 2\varepsilon\right)$$
$$\leqslant 4\exp\left(-\frac{K\varepsilon^2}{\lambda_n^{*(i)}}\right).$$

Furthermore,

$$\lambda_n^{*(i)} \leqslant K(nT^{-1})^{2H-1} \max_{1\leqslant k\leqslant n-(i-1)} \sum_{j=1}^{n-(i-1)} |d_{jkn}^{(i)}|,$$

where $d_{jkn}^{(i)} = \mathbf{E}\Delta_{j,n}^{(i)} B^H \Delta_{k,n}^{(i)} B^H$. From Bégyn [1] and Gladyshev [12], we get

$$\lambda_n^{*(i)} \leqslant Cn^{-1}. \tag{7}$$

Now we set

$$\varepsilon_n^2 = \frac{2C}{K} n^{-1} \ln n$$

and conclude that

$$\mathbf{P}\left((nT^{-1})^{2H-1} \max_{1\leqslant k\leqslant n-(i-1)} \left|\sum_{j=1}^{k}\left(\Delta_{j,n}^{(i)} B^H\right)^2 - \sum_{j=1}^{k}\mathbf{E}\left(\Delta_{k,n}^{(i)} B^H\right)^2\right| > 2\varepsilon_n\right)$$
$$\leqslant 4\exp(-2\ln n) = \frac{4}{n^2}.$$

It means that

$$\sum_{n=i}^{\infty}\mathbf{P}\left((nT^{-1})^{2H-1} \max_{1\leqslant k\leqslant n-(i-1)} \left|\sum_{j=1}^{k}\left(\Delta_{j,n}^{(i)} B^H\right)^2 - \sum_{j=1}^{k}\mathbf{E}\left(\Delta_{j,n}^{(i)} B^H\right)^2\right| > 2\varepsilon_n\right) < \infty.$$

Finally, we get the statement of the present theorem from the Borel–Cantelli lemma and the evident equality

$$\sup_{t\leqslant T}\left|V_n^{(i)}(B^H, 2)_t - \mathbf{E}V_n^{(i)}(B^H, 2)_t\right|$$
$$= (nT^{-1})^{2H-1} \max_{1\leqslant k\leqslant n-(i-1)} \left|\sum_{j=1}^{k}\left(\Delta_{j,n}^{(i)} B^H\right)^2 - \sum_{j=1}^{k}\mathbf{E}\left(\Delta_{j,n}^{(i)} B^H\right)^2\right|. \quad \square$$

3. The rate of convergence of the first- and second-order quadratic variations of the solution of the stochastic differential equation

First, we formulate the following result from [18] about the convergence of first- and second-order quadratic variation.

Theorem 5. *Consider stochastic differential equation (1), where function f is Lipschitz continuous and $g \in \mathcal{C}^{1+\alpha}$ for some $0 < \alpha < 1$. Let X be its solution. Then*

$$V_n^{(i)}(X,2) \to c^{(i)} \int_0^T g^2(X(t))\,dt \quad \text{a.s. as } n \to \infty, \tag{8}$$

where

$$c^{(i)} = \begin{cases} 1 & \text{for } i = 1, \\ (4 - 2^{2H}) & \text{for } i = 2. \end{cases}$$

Second, we prove the following auxiliary result.

Lemma 6. *Let X be a solution of stochastic differential equation (1). Define a step-wise process X^π that is a discretization of process X:*

$$X_t^\pi = \begin{cases} X(t_k) & \text{for } t \in [t_k, t_{k+1}), k = 0, 1, \ldots, n-2, \\ X(t_{n-1}) & \text{for } t \in [t_{n-1}, t_n]. \end{cases}$$

Then for any $p > \frac{1}{H}$ we have that

$$\sup_{t \leqslant T} |X_t^\pi - X_t| = \mathcal{O}\left(n^{-1/p}\right) \text{ a.s..}$$

Proof. Consider $t \in [t_k, t_{k+1})$. We get immediately from the Love–Young inequality (A.8) that

$$\begin{aligned} |X_t^\pi - X_t| &= \left| \int_{\rho^n(t)}^t f(X_s)\,ds + \int_{\rho^n(t)}^t g(X_s)\,dB_s^H \right| \\ &\leqslant T n^{-1} \sup_{t_k \leqslant t \leqslant t_{k+1}} |f(X_t)| + C_{p,p} V_{p,\infty}(g(X); [t_k, t_{k+1}]) V_p(B^H; [t_k, t_{k+1}]). \end{aligned}$$

Further,

$$\sup_{t_k \leqslant t \leqslant t_{k+1}} |f(X_t)| \leqslant \sup_{t \leqslant T} |f(X_t) - f(\xi)| + |f(\xi)| \leqslant L V_p(X; [0,T]) + |f(\xi)|, \tag{9}$$

where L is the Lipschitz constant for f, and

$$\begin{aligned} V_{p,\infty}(g(X); [t_k, t_{k+1}]) &\leqslant |g'|_\infty V_p(X; [t_k, t_{k+1}]) + \sup_{t \leqslant T} |g(X_t) - g(\xi)| + |g(\xi)| \\ &\leqslant 2|g'|_\infty V_p(X; [0,T]) + |g(\xi)|. \end{aligned} \tag{10}$$

Recall that the solution of stochastic differential equation (1) belongs to $C\mathcal{W}_p([a,b])$. So we get the statement of the lemma from (9), (10) and inequality (2). □

Now we prove the main result of this section which specifies the rate of convergence in Theorem 5.

Theorem 7. *Let the conditions of Theorem 5 hold and, in addition, $\alpha > \frac{1}{H} - 1$. Then*

$$V_n^{(i)}(X,2) - c^{(i)} \int_0^T g^2(X(t))\,dt = \mathcal{O}\left(n^{-1/4} \ln^{1/4} n\right). \tag{11}$$

Proof. Decompose the left-hand side of (11) into three parts:

$$I_n^{(i)} := V_n^{(i)}(X,2) - c^{(i)} \int_0^T g^2\left(X(t)\right)\, dt = I_n^{(1,i)} + I_n^{(2,i)} + I_n^{(3,i)},$$

where

$$I_n^{(1,i)} = (nT^{-1})^{2H-1} \left(\sum_{k=1}^{n-(i-1)} \left(\Delta_{k,n}^{(i)} X\right)^2 - \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \left(\Delta_{k,n}^{(i)} B^H\right)^2 \right),$$

$$I_n^{(2,i)} = (nT^{-1})^{2H-1} \left(\sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \left(\Delta_{k,n}^{(i)} B^H\right)^2 - \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \mathbf{E} \left(\Delta_{k,n}^{(i)} B^H\right)^2 \right),$$

$$I_n^{(3,i)} = (nT^{-1})^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \mathbf{E} \left(\Delta_{k,n}^{(i)} B^H\right)^2 - c^{(i)} \int_0^T g^2(X_s) ds,$$

and $X_k = X(t_k)$. We start with the most simple term $I_n^{(3,i)}$ and by (A.2) we get immediately that for any $p > \frac{1}{H}$

$$|I_n^{(3,i)}| \leqslant c^{(i)} \sum_{k=1-(i-1)}^{n-(i-1)} \int_{t_{k-1+(i-1)}}^{t_{k+(i-1)}} \left| g^2(X(t_{k-1+(i-1)})) - g^2(X_s) \right| ds$$
$$\leqslant 2c^{(i)} T |g'|_\infty \sup_{t \leqslant T} |X_t^\pi - X_t| \cdot \left[|g'|_\infty V_p(X; [0,T]) + |g(\xi)| \right].$$

In order to estimate $I_n^{(2,i)}$, denote

$$S_t^{(i)} = (nT^{-1})^{2H-1} \sum_{k=1}^{r(t)-(i-1)} \left(\Delta_{k,n}^{(i)} B^H\right)^2, \quad t \in [0,T], i = 1, 2.$$

Then

$$(nT^{-1})^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \left(\Delta_{k,n}^{(i)} B^H\right)^2 = \int_0^T g^2(X_t)\, dS_t^{(i)}$$

and

$$(nT^{-1})^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \left[\left(\Delta_{k,n}^{(i)} B^H\right)^2 - \mathbf{E} \left(\Delta_{k,n}^{(i)} B^H\right)^2 \right]$$
$$= \int_0^T g^2(X_t)\, d\left[S_t^{(i)} - \mathbf{E} S_t^{(i)} \right].$$

Note that $1/p + 1/2 > 1$ for $\frac{1}{H} < p < 2$. Therefore, we obtain from the Love–Young inequality and from (A.5) to (A.6) that

$$|I_n^{(2,i)}| = \left| \int_0^T g^2(X_t)\, d\left[S_t^{(i)} - \mathbf{E} S_t^{(i)} \right] \right|$$
$$\leqslant C_{p,2} V_{p,\infty} \left(g^2(X); [0,T] \right) V_2 \left(S^{(i)} - \mathbf{E} S^{(i)}; [0,T] \right)$$

$$\leqslant C_{p,2}\left\{\mathrm{Osc}\left(S^{(i)}-\mathbf{E}S^{(i)};[0,T]\right)\right\}^{1/2} V_{p,\infty}\left(g^2(X);[0,T]\right)$$
$$\times V_1^{1/2}\left(S^{(i)}-\mathbf{E}S^{(i)};[0,T]\right)$$
$$\leqslant 2C_{p,2}\left(\sup_{t\leqslant T}\left|S_t^{(i)}-\mathbf{E}S_t^{(i)}\right|\right)^{1/2} V_{p,\infty}^2\left(g(X);[0,T]\right)$$
$$\times\left[(nT^{-1})^{2H-1}\sum_{k=1}^{n-(i-1)}\left(\Delta_{k,n}^{(i)}B^H\right)^2+c^{(i)}T\right]^{1/2}.$$

It follows from Theorem 3, Remark 4, and (A.3) that the rate of convergence of $I_n^{(2,i)}$ is $O(n^{-1/4}\ln^{1/4} n)$.

It still remains to estimate $I_n^{(1,i)}$. Consider only $i=2$, the proof for $i=1$ is similar.

Denote

$$J_k^1=\int_{t_k}^{t_{k+1}}[f(X_s)-f(X_k)]\,ds-\int_{t_{k-1}}^{t_k}[f(X_s)-f(X_k)]\,ds,$$
$$J_k^2=\int_{t_{k-1}}^{t_k}\left(g(X_k)-g(X_s)-\int_s^{t_k}g'(X_k)f(X_k)\,du-\int_s^{t_k}g'(X_k)g(X_k)\,dB_u^H\right)dB_s^H,$$
$$J_k^3=\int_{t_k}^{t_{k+1}}\left(g(X_s)-g(X_k)-\int_{t_k}^{s}g'(X_k)f(X_k)\,du-\int_{t_k}^{s}g'(X_k)g(X_k)\,dB_u^H\right)dB_s^H,$$
$$J_k^4=g'(X_k)f(X_k)\left(\int_{t_k}^{t_{k+1}}(s-t_k)\,dB_s^H+\int_{t_{k-1}}^{t_k}(t_k-s)\,dB_s^H\right),$$
$$J_k^5=\frac{1}{2}g'(X_k)g(X_k)\left(\left(\Delta_{k,n}^{(1)}B^H\right)^2+\left(\Delta_{k+1,n}^{(1)}B^H\right)^2\right),\qquad J_k^6=g(X_k)\Delta_{k,n}^{(2)}B^H.$$

Equalities

$$\int_{t_{k-1}}^{t_k}\left(\int_s^{t_k}g'(X_k)g(X_k)\,dB_u^H\right)dB_s^H=\frac{1}{2}g'(X_k)g(X_k)\left(\Delta_k^{(1)}B^H\right)^2,$$
$$\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{s}g'(X_k)g(X_k)\,dB_u^H\right)dB_s^H=\frac{1}{2}g'(X_k)g(X_k)\left(\Delta_{k+1,n}^{(1)}B^H\right)^2,$$

(see Proposition 9) imply

$$\Delta_{k,n}^{(2)}X=\sum_{l=1}^{6}J_k^l.$$

Taking into account the Lipschitz continuity of f and Lemma 6, we can conclude that

$$|f(X_t)-f(X_k)|\leqslant L\sup_{t\leqslant T}|X_t^\pi-X_t|=\mathcal{O}\left(n^{-1/p}\right).$$

Therefore

$$\sum_{k=1}^{n-1}(J_k^1)^2 \leqslant 2Tn^{-1}\sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}}[f(X_s)-f(X_k)]^2\,ds$$
$$+2Tn^{-1}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}[f(X_s)-f(X_k)]^2\,ds$$
$$\leqslant 4Tn^{-1}L^2\left(\sup_{t\leqslant T}|X_t^\pi - X_t|\right)^2$$
$$=\mathcal{O}\left(n^{-1-2/p}\right).$$

Consider J_k^2. It follows from equality (A.11) that for any fixed $t \in [t_{k-1}, t_k]$

$$g(X_k)-g(X_t)=\int_t^{t_k} g'(X_s)f(X_s)\,ds+\int_t^{t_k} g'(X_s)g(X_s)\,dB_s^H. \tag{12}$$

Substituting equality (12) into J_k^2 we get

$$|J_k^2| \leqslant \left|\int_{t_{k-1}}^{t_k}\int_s^{t_k}\left[g'(X_u)f(X_u)-g'(X_k)f(X_k)\right]du\,dB_s^H\right|$$
$$+\left|\int_{t_{k-1}}^{t_k}\int_s^{t_k}\left[g'(X_u)g(X_u)-g'(X_k)g(X_k)\right]dB_u^H\,dB_s^H\right|. \tag{13}$$

Transforming identically the first term in the right-hand side of (13) and applying to it Love–Young inequality (A.8), we conclude that for any $p > \frac{1}{H}$

$$\left|\int_{t_{k-1}}^{t_k}\int_s^{t_k}\left[g'(X_u)f(X_u)-g'(X_k)f(X_k)\right]du\,dB_s^H\right|$$
$$\leqslant C_{p,1}V_1\left(\int_\cdot^{t_k}\left[g'(X_u)f(X_u)-g'(X_k)f(X_k)\right]du;[t_{k-1},t_k]\right)V_p\left(B^H;[t_{k-1},t_k]\right)$$
$$\leqslant C_{p,1}V_p\left(B^H;[t_{k-1},t_k]\right)\int_{t_{k-1}}^{t_k}\left|g'(X_u)f(X_u)-g'(X_k)f(X_k)\right|du. \tag{14}$$

Henceforth we consider the following interval of the values of p: $\frac{1}{H} < p < 1+\alpha$. Then it follows from inequality (A.10) that the second term in the right-hand side of (13) admits the bound:

$$\left|\int_{t_{k-1}}^{t_k}\int_s^{t_k}\left[g'(X_u)g(X_u)-g'(X_k)g(X_k)\right]dB_u^H\,dB_s^H\right|$$
$$\leqslant C_{p,p/\alpha}V_{p/\alpha}\left(\int_\cdot^{t_k}\left[g'(X_u)g(X_u)-g'(X_k)g(X_k)\right]dB_u^H;[t_{k-1},t_k]\right)$$
$$\times V_p\left(B^H;[t_{k-1},t_k]\right)$$
$$\leqslant 2C_{p,p/\alpha}^2V_{p/\alpha}\left(g'(X)g(X);[t_{k-1},t_k]\right)V_p^2\left(B^H;[t_{k-1},t_k]\right). \tag{15}$$

We conclude from (13)–(15) that

$$|J_k^2| \leqslant Tn^{-1}C_{p,1}V_{p/\alpha}\left(g'(X)f(X);[t_k,t_{k+1}]\right)V_p\left(B^H;[t_k,t_{k+1}]\right)$$
$$+2C_{p,p/\alpha}^2V_{p/\alpha}\left(g'(X)g(X);[t_k,t_{k+1}]\right)V_p^2\left(B^H;[t_k,t_{k+1}]\right).$$

Applying inequalities (A.6) and (A.7) we immediately obtain that

$$\begin{aligned}\sum_{k=1}^n\left(J_k^2\right)^2 &\leqslant 2T^2C_{p,1}^2n^{-2}\\ &\quad\times\max_{0\leqslant k\leqslant n-1}\left[V_{p/\alpha}\left(g'(X)f(X);[t_k,t_{k+1}]\right)V_p\left(B^H;[t_k,t_{k+1}]\right)\right]\\ &\quad\times V_{p/\alpha}\left(g'(X)f(X);[0,T]\right)V_p\left(B^H;[0,T]\right)\\ &\quad+4C_{p,p/\alpha}^4\max_{0\leqslant k\leqslant n-1}\left[V_{p/\alpha}\left(g'(X)g(X);[t_k,t_{k+1}]\right)V_p^3\left(B^H;[t_k,t_{k+1}]\right)\right]\\ &\quad\times V_{p/\alpha}\left(g'(X)g(X);[0,T]\right)V_p\left(B^H;[0,T]\right)\\ &\leqslant 2T^2C_{p,1}^2C_{p/\alpha}n^{-2}\max_{0\leqslant k\leqslant n-1}\left[V_p\left(B^H;[t_k,t_{k+1}]\right)\right]V_{p/\alpha,\infty}^2\left(g'(X);[0,T]\right)\\ &\quad\times V_{p,\infty}^2\left(f(X);[0,T]\right)V_p\left(B^H;[0,T]\right)\\ &\quad+4C_{p,p/\alpha}^4C_{p/\alpha}\max_{0\leqslant k\leqslant n-1}\left[V_p^3\left(B^H;[t_k,t_{k+1}]\right)\right]\\ &\quad\times V_{p/\alpha,\infty}^2\left(g'(X);[0,T]\right)V_{p/\alpha,\infty}^2\left(g(X);[0,T]\right)V_p\left(B^H;[0,T]\right).\end{aligned}$$

It follows from inequalities (A.1)–(A.4) that the values of the variations

$$V_{p,\infty}\left(f(X);[0,T]\right),\quad V_{p/\alpha,\infty}\left(g(X);[0,T]\right),\quad\text{and}\quad V_{p/\alpha,\infty}\left(g'(X);[0,T]\right)$$

are finite. Therefore we get from (2) that

$$\sum_{k=1}^n\left(J_k^2\right)^2=\mathcal{O}(n^{-2-1/p})+\mathcal{O}(n^{-3/p})=\mathcal{O}(n^{-3/p}).$$

The similar reasonings lead to the similar bound for J_k^3, and we conclude that

$$\sum_{k=0}^{n-1}[J_k^2+J_k^3]^2=\mathcal{O}\left(n^{-3/p}\right).$$

Consider J_k^4. It consists of two terms that can be estimated in a similar way. Applying inequalities (2) and (A.8), we obtain the following bound for the first term:

$$\begin{aligned}&\sum_{k=1}^{n-1}\left[g'(X_k)f(X_k)\right]^2\left(\int_{t_k}^{t_{k+1}}(s-t_k)\,dB_s^H\right)^2\\ &\quad\leqslant C_{p,1}^2\sum_{k=1}^{n-1}\left[g'(X_k)f(X_k)\right]^2(t_{k+1}-t_k)^2V_p^2\left(B^H;[t_k,t_{k+1}]\right)\\ &\quad\leqslant n^{-1}T^2C_{p,1}^2\max_{1\leqslant k\leqslant n-1}\left[g'(X_k)f(X_k)V_p\left(B^H;[t_k,t_{k+1}]\right)\right]^2=\mathcal{O}\left(n^{-1-2/p}\right).\end{aligned}$$

As a consequence,

$$\sum_{k=1}^{n-1}\left(J_k^4\right)^2=\mathcal{O}\left(n^{-1-2/p}\right).$$

Furthermore, note that under our assumptions $\sup_{s\in[0,T]}|g(X_s)|<\infty$ and $\sup_{s\in[0,T]}|g'(X_s)| < \infty$ a.s. Therefore we have for the first term in J_k^5 that

$$\begin{aligned}\sum_{k=1}^{n-1}\left[g'(X_k)g(X_k)\left(\Delta_{k+1,n}^{(1)}B^H\right)^2\right]^2 &\leqslant \max_{1\leqslant k\leqslant n-1}\left[g'(X_k)g(X_k)\right]^2\sum_{k=0}^{n-1}\left(\Delta_{k+1,n}^{(1)}B^H\right)^4\\ &=\mathcal{O}\left(n^{1-4/p}\right).\end{aligned}$$

The second term is bounded in a similar way, and we conclude that

$$\sum_{k=1}^{n-1}\left(J_k^5\right)^2=\mathcal{O}\left(n^{1-4/p}\right).$$

Thus

$$\left(\sum_{l=1}^{5}J_k^l\right)^2=\mathcal{O}\left(n^{-1-2/p}\vee n^{-3/p}\vee n^{1-4/p}\right)=\mathcal{O}\left(n^{1-4/p}\right).$$

At last,

$$\begin{aligned}(nT^{-1})^{2H-1}\sum_{k=1}^{n-1}\left[\Delta_{k,n}^{(2)}X-g(X_k)\Delta_{k,n}^{(2)}B^H\right]^2 &=\mathcal{O}\left(n^{1-4/p+2H-1}\right)\\ &=\mathcal{O}\left(n^{-4/p+2H}\right) \qquad (16)\end{aligned}$$

for any $\frac{1}{H}<p<1+\alpha$. Set $1/p=H-\varepsilon$ for $\varepsilon<(H/2-1/16)\wedge(H-\frac{1}{1+\alpha})$. Then

$$\begin{aligned}V_n^{(2)}(X,2)-c^{(2)}\int_0^T g^2(X_s)\,ds &=\mathcal{O}\left(n^{-4/p+2H}\right)+\mathcal{O}\left(n^{-1/4}\ln^{1/4}n\right)+\mathcal{O}\left(n^{-1/p}\right)\\ &=\mathcal{O}\left(n^{-2H+4\varepsilon}\right)+\mathcal{O}\left(n^{-1/4}\ln^{1/4}n\right)+\mathcal{O}\left(n^{-H+\varepsilon}\right)\\ &=\mathcal{O}\left(n^{-1/4}\ln^{1/4}n\right). \quad \square \qquad (17)\end{aligned}$$

4. The rate of convergence of estimators of Hurst index

Consider the following statistics:

$$R_n^{(i)}=\frac{\sum_{k=1}^{2n-(i-1)}(\Delta_{k,2n}^{(i)}X)^2}{\sum_{k=1}^{n-(i-1)}(\Delta_{k,n}^{(i)}X)^2}$$

and construct the following estimate of Hurst index H:

$$\widehat{H}_n^{(i)}=\left(\frac{1}{2}-\frac{1}{2\ln 2}\ln R_n^{(i)}\right)\mathbf{1}_{\widetilde{C}_n},$$

where

$$\widetilde{C}_n = \left\{2^{-1}\left(1 - 2n^{-1/4}(\ln n)^{1/4+\beta}\right) \leqslant R_n^{(i)} \leqslant 1 + 2n^{-1/4}(\ln n)^{1/4+\beta}\right\}, \quad \beta > 0.$$

Further, introduce the following notation: $g^{(i)}(T) = c^{(i)} \int_0^T g^2(X_s)ds$.

Theorem 8. *Let the conditions of Theorem 5 hold with* $\alpha > \frac{1}{H} - 1$. *Also, let X be a solution of (1) and assume that random variable* $g^{(i)}(T)$ *is separated from zero: there exists a constant* $c_0 > 0$ *such that* $g^{(i)}(T) \geqslant c_0$ *a.s.,* $i = 1, 2$. *Then* $\widehat{H}_n^{(i)}$ *is a strongly consistent estimator of the Hurst index H and the following rate of convergence holds:*

$$|\widehat{H}_n^{(i)} - H| = \mathcal{O}\left(n^{-1/4}(\ln n)^{1/4+\beta}\right) \quad a.s.,$$

for any $\beta > 0$.

Proof. Consider a sequence $1 > \delta_n \downarrow 0$ as $n \to \infty$. It will be specified later on. Introduce the events

$$C_n = \left\{\frac{1}{2}(1 - \delta_n) \leqslant R_n^{(i)} \leqslant 1 + \delta_n\right\}.$$

Also, introduce the notations

$$A_n^{(i)} = V_{2n}^{(i)}(X, 2) \quad \text{and} \quad B_n^{(i)} = V_n^{(i)}(X, 2)$$

and note that $2^{2H-1}R_n^{(i)} = \frac{A_n^{(i)}}{B_n^{(i)}}$. Then

$$C_n = \left\{2^{2H-2}(1 - \delta_n) \leqslant \frac{A_n^{(i)}}{B_n^{(i)}} \leqslant 2^{2H-1}(1 + \delta_n)\right\},$$

and estimate $\widehat{H}_n^{(i)}$ has a form

$$\widehat{H}_n^{(i)} = \left(\frac{1}{2} - \frac{1}{2\ln 2}\ln R_n^{(i)}\right)\mathbf{1}_{C_n}.$$

It is easy to see that $\overline{C}_n := \Omega \setminus C_n$ has a form

$$\begin{aligned}\overline{C}_n &= \left\{\frac{A_n^{(i)}}{B_n^{(i)}} < 2^{2H-2}(1 - \delta_n)\right\} \bigcup \left\{\frac{A_n^{(i)}}{B_n^{(i)}} > 2^{2H-1}(1 + \delta_n)\right\} \\ &\subset \left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}} - 1\right| > \delta_n\right\}. \qquad (18)\end{aligned}$$

We transform the estimate $\widehat{H}_n^{(i)}$ in the following way:

$$\begin{aligned}\widehat{H}_n^{(i)} &= \left(\frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{A_n^{(i)}}{B_n^{(i)}} + \frac{\ln 2^{2H-1}}{2\ln 2}\right)\mathbf{1}_{C_n} \\ &= H\mathbf{1}_{C_n} - \frac{1}{2\ln 2}\ln\frac{A_n^{(i)}}{B_n^{(i)}}\mathbf{1}_{C_n}.\end{aligned}$$

The latter representation implies that

$$\left|\widehat{H}_n^{(i)} - H\right| \leqslant H\mathbf{1}_{\left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}}-1\right|>\delta_n\right\}} + \frac{1}{2\ln 2}\left|\ln \frac{A_n^{(i)}}{B_n^{(i)}}\right| \mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 1+\delta_n\right\}}$$
$$- \left(\frac{1}{2\ln 2}\ \ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{2^{2H-2}(1-\delta_n)\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1-\delta_n\right\}}$$
$$+ \left(\frac{1}{2\ln 2}\ \ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{1+\delta_n< \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 2^{2H-1}(1+\delta_n)\right\}} := \sum_{l=1}^{4} L_n^l. \tag{19}$$

In what follows we need an elementary inequality: $-\ln(1-x) \leq 2\ln(1+x) \leqslant 2x$ provided that $0 \leqslant x \leqslant 1/2$.

Consider L_n^2. We divide it in two parts. As to the first part, it is obvious that

$$\left(\ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1\right\}} = \left(\ln\left[1-\left(1-\frac{A_n^{(i)}}{B_n^{(i)}}\right)\right]\right)\mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1\right\}},$$

and

$$1-\delta_n \leqslant \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \quad \text{implies that } 0 < 1-\frac{A_n^{(i)}}{B_n^{(i)}} \leqslant \delta_n.$$

Applying inequality $-\ln(1-x) \leqslant 2x$, $0 \leqslant x \leqslant 1/2$, we deduce that for $\delta_n \leqslant 1/2$

$$\left(-\ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1\right\}} \leqslant 2\left(1-\frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1\right\}} \leqslant 2\delta_n\mathbf{1}_{\left\{1-\delta_n\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1\right\}}.$$

As to the second part,

$$\left(\ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{1\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 1+\delta_n\right\}} = \left(\ln\left[1+\left(\frac{A_n^{(i)}}{B_n^{(i)}}-1\right)\right]\right)\mathbf{1}_{\left\{1\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 1+\delta_n\right\}}$$
$$\leqslant \left(\frac{A_n^{(i)}}{B_n^{(i)}}-1\right)\mathbf{1}_{\left\{1\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 1+\delta_n\right\}} \leqslant \delta_n\mathbf{1}_{\left\{1\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}\leqslant 1+\delta_n\right\}}.$$

Consider L_n^3. From here we easily deduce that

$$-\left(\frac{1}{2\ln 2}\ \ln \frac{A_n^{(i)}}{B_n^{(i)}}\right)\mathbf{1}_{\left\{2^{2H-2}(1-\delta_n)\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1-\delta_n\right\}}$$
$$\leqslant -\frac{1}{2\ln 2}\left[\ln\left(2^{2H-2}(1-\delta_n)\right)\right]\mathbf{1}_{\left\{2^{2H-2}(1-\delta_n)\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1-\delta_n\right\}}$$
$$\leqslant \left((1-H)-\frac{\ln(1-\delta_n)}{2\ln 2}\right)\mathbf{1}_{\left\{2^{2H-2}(1-\delta_n)\leqslant \frac{A_n^{(i)}}{B_n^{(i)}}<1-\delta_n\right\}}$$

$$\leqslant \left(1 - H + \frac{\delta_n}{\ln 2}\right) \mathbf{1}_{\left\{2^{2H-2}(1-\delta_n) \leqslant \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\right\}}$$
$$\leqslant (1 + 2\delta_n)\mathbf{1}_{\left\{\frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\right\}}.$$

The term L_n^4 is estimated similarly as L_n^3. Thus we get

$$\left(\frac{1}{2\ln 2} \ln \frac{A_n^{(i)}}{B_n^{(i)}}\right) \mathbf{1}_{\left\{1+\delta_n < \frac{A_n^{(i)}}{B_n^{(i)}} \leqslant 2^{2H-1}(1+\delta_n)\right\}} \leqslant (1 + \delta_n)\mathbf{1}_{\left\{\frac{A_n^{(i)}}{B_n^{(i)}} > 1+\delta_n\right\}}.$$

Summarizing, we conclude that

$$|\widehat{H}_n^{(i)} - H| \leqslant (1 + 2\delta_n)\mathbf{1}_{\left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}} - 1\right| > \delta_n\right\}} + 2\delta_n \mathbf{1}_{\left\{1-\delta_n \leqslant \frac{A_n^{(i)}}{B_n^{(i)}} \leqslant 1+\delta_n\right\}}$$
$$\leqslant (1 + 2\delta_n)\mathbf{1}_{\left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}} - 1\right| > \delta_n\right\}} + 2\delta_n.$$

Now, let $\beta > 0$. Note that

$$\left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}} - 1\right| > \delta_n\right\} \subset \left\{\left|\frac{A_n^{(i)}}{B_n^{(i)}} - 1\right| > \delta_n,\, B_n^{(i)} \geqslant (\ln n)^{-\beta}\right\} \bigcup \left\{B_n^{(i)} < (\ln n)^{-\beta}\right\}$$
$$= \left\{|A_n^{(i)} - B_n^{(i)}| > \delta_n B_n^{(i)},\, B_n^{(i)} \geqslant (\ln n)^{-\beta}\right\} \cup \left\{B_n^{(i)} < (\ln n)^{-\beta}\right\}$$
$$\subset \left\{\left|A_n^{(i)} - B_n^{(i)}\right| > \delta_n (\ln n)^{-\beta}\right\} \cup \left\{B_n^{(i)} < (\ln n)^{-\beta}\right\}.$$

Therefore

$$|\widehat{H}_n^{(i)} - H| \leqslant (1 + 2\delta_n)\mathbf{1}_{\{|A_n^{(i)} - B_n^{(i)}| > \delta_n (\ln n)^{-\beta}\} \cup \{B_n^{(i)} < (\ln n)^{-\beta}\}} + 2\delta_n.$$

It follows from (17) that

$$|A_n^{(i)} - B_n^{(i)}| = O\left(n^{-1/4} \ln^{1/4} n\right)$$

and

$$\left|B_n^{(i)} - c^{(i)} \int_0^T g^2(X_s)ds\right| = O\left(n^{-1/4} \ln^{1/4} n\right).$$

Obviously, for any $n > \exp\{\left(\frac{2}{c_0}\right)^{\frac{1}{\beta}}\}$ we have that $g^{(i)}(T) \geqslant c_0 \geq 2(\ln n)^{-\beta}$ a.s. and

$$\left\{B_n^{(i)} < (\ln n)^{-\beta}\right\} = \left\{B_n^{(i)} < (\ln n)^{-\beta},\, g^{(i)}(T) \geqslant 2(\ln n)^{-\beta}\right\}.$$

Now, let $\delta_n < (\ln n)^{-\beta}$. Then it is not hard to deduce that

$$\left\{B_n^{(i)} < (\ln n)^{-\beta},\, g^{(i)}(T) \geqslant 2(\ln n)^{-\beta}\right\}$$
$$= \left\{B_n^{(i)} < (\ln n)^{-\beta},\, g^{(i)}(T) \geqslant 2(\ln n)^{-\beta},\, B_n^{(i)} < g^{(i)}(T) - \delta_n\right\}$$
$$\subset \left\{|B_n^{(i)} - g^{(i)}(T)| > \delta_n\right\}.$$

Table 1
$|\widehat{H}_n^{(1)} - H|$.

H	n points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,06581	0,07794	0,03056	0,01651	0,00898	0,00426	0,00288	0,00151
0.6	0,05332	0,04579	0,01768	0,01722	0,0086	0,00485	0,00218	0,00258
0.65	0,0617	0,04571	0,02336	0,00988	0,00682	0,00528	0,00289	0,0014
0.7	0,08991	0,06428	0,03157	0,02309	0,00966	0,00445	0,00313	0,00262
0.75	0,07045	0,04574	0,02525	0,02059	0,00923	0,00818	0,0042	0,00227
0.8	0,04889	0,06025	0,03721	0,02402	0,01462	0,00963	0,00677	0,00544
0.85	0,04552	0,03142	0,02588	0,02041	0,01305	0,01073	0,00644	0,00526
0.9	0,05481	0,06175	0,04591	0,0422	0,03635	0,03267	0,02861	0,02601
0.95	0,03198	0,03079	0,02905	0,02755	0,02527	0,02447	0,02307	0,02224

Therefore,

$$\left\{B_n^{(i)} < (\ln n)^{-\beta}\right\} \subset \left\{|B_n^{(i)} - g^{(i)}(T)| > \delta_n\right\}$$

if $n > \exp\{\left(\frac{2}{c_0}\right)^{\frac{1}{\beta}}\}$.

Finally, specify δ_n. More precisely, set $\delta_n = n^{-1/4}(\ln n)^{1/4+2\beta}$, $\beta > 0$. Note that $\delta_n < (\ln n)^{-\beta}$ for sufficiently large n and, moreover,

$$\frac{\mathcal{O}\left(n^{-1/4}\ln^{1/4} n\right)}{\delta_n(\ln n)^{-\beta}} = \frac{\mathcal{O}\left(n^{-1/4}\ln^{1/4} n\right)}{n^{-1/4}(\ln n)^{1/4+\beta}} \longrightarrow 0 \quad \text{a.s. as } n \to \infty.$$

The latter relation together with Theorem 7 implies that for any $\omega \in \Omega'$ with $P(\Omega') = 1$ there exists $n_0 = n_0(\omega)$ such that for any $n > n_0$

$$\mathbf{1}_{\left\{|A_n^{(i)} - B_n^{(i)}| > \delta_n(\ln n)^{-\beta}\right\} \cup \left\{B_n^{(i)} < (\ln n)^{-\beta}\right\}} = 0 \quad \text{a.s.},$$

and we obtain the proof. □

5. Simulation results

Consider stochastic differential equations

$$dX_t = X_t\,dt + (2 + \sin(X_t))\,dB_t^H, \qquad dX_t = \sin(X_t)\,dt + (2 + \cos(X_t))\,dB_t^H,$$
$$X_0 = 1,$$

with the step 0.05 and for increasing (in the logarithmic scale) number n of points from $n = 10^2$ to $n = 2.5 \cdot 10^5$. Tables 1–8 are given for simulation results of the first and second equations, respectively. Tables 1, 3, 5 and 7 present the values of the difference $|\widehat{H}_n^{(i)} - H|$, $i = 1, 2$, for the values of H from 0.55 to 0.95. We can conclude that the difference $|\widehat{H}_n^{(i)} - H|$ decreases rapidly in n. Tables 2, 4, 6 and 8 demonstrate that the rate of convergence agrees with Theorem 8, at least, for $\beta = 0.05$ and $\widehat{H}_n^{(2)}$. The results for $\widehat{H}_n^{(1)}$ are of their own accord with Theorem 8 as well, however they are worse, comparatively to $\widehat{H}_n^{(2)}$.

Table 2
$|\widehat{H}_n^{(1)} - H| \cdot n^{0.25}(\ln n)^{-0.3}$.

H	*n* points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,13162	0,18561	0,09622	0,06298	0,04615	0,02672	0,02464	0,01583
0.6	0,10663	0,10904	0,05569	0,06568	0,04416	0,03046	0,01866	0,0271
0.65	0,1234	0,10887	0,07357	0,03769	0,03503	0,03314	0,0247	0,01467
0.7	0,17981	0,15308	0,09943	0,08808	0,04963	0,02795	0,02673	0,02749
0.75	0,14089	0,10893	0,07951	0,07854	0,04741	0,05139	0,0359	0,02382
0.8	0,09778	0,14348	0,11717	0,09164	0,07512	0,06047	0,0578	0,05712
0.85	0,09104	0,07483	0,08149	0,07786	0,06704	0,06735	0,05502	0,0552
0.9	0,10961	0,14706	0,14456	0,16098	0,18671	0,20513	0,24443	0,27308
0.95	0,06396	0,07333	0,09148	0,10509	0,12981	0,15363	0,19712	0,23346

Table 3
$|\widehat{H}_n^{(2)} - H|$.

H	*n* points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,14101	0,10612	0,06521	0,03262	0,01482	0,00931	0,00495	0,00221
0.6	0,2171	0,0836	0,03467	0,02301	0,01508	0,01294	0,00634	0,00372
0.65	0,21082	0,11821	0,03606	0,03042	0,01397	0,01017	0,00345	0,00252
0.7	0,11664	0,11457	0,04514	0,01502	0,01344	0,00663	0,00434	0,00252
0.75	0,11062	0,08693	0,0531	0,03037	0,01391	0,00708	0,00516	0,00203
0.8	0,14089	0,09838	0,04047	0,03186	0,01586	0,00942	0,00493	0,00222
0.85	0,1271	0,08784	0,036151	0,02734	0,016685	0,00938	0,00601	0,00168
0.9	0,1039	0,08231	0,06022	0,01845	0,0118	0,0072	0,00398	0,00258
0.95	0,11859	0,04503	0,03472	0,03221	0,01669	0,00759	0,00464	0,00246

Table 4
$|\widehat{H}_n^{(2)} - H| \cdot n^{0.25}(\ln n)^{-0.3}$.

H	*n* points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,28201	0,25273	0,20535	0,12442	0,07614	0,05847	0,04228	0,02317
0.6	0,43419	0,19911	0,10919	0,08779	0,07745	0,08127	0,0542	0,0391
0.65	0,42165	0,28153	0,11356	0,11603	0,07176	0,06382	0,02948	0,02642
0.7	0,23329	0,27285	0,14214	0,05729	0,06903	0,04162	0,03708	0,02649
0.75	0,22124	0,20704	0,16723	0,11587	0,07143	0,04446	0,04406	0,02133
0.8	0,28178	0,2343	0,12744	0,12152	0,08146	0,05913	0,04213	0,02332
0.85	0,2542	0,2092	0,11385	0,10429	0,08571	0,05892	0,05133	0,01761
0.9	0,20779	0,19602	0,18965	0,0704	0,06061	0,04518	0,03399	0,02708
0.95	0,23718	0,10723	0,10935	0,12286	0,08575	0,04767	0,03966	0,02581

For the simulation of B^H we use the circulant matrix embedding method, as described in Coeurjolly [9] and the references therein. For the simulation of the solution of the stochastic differential equation we use the Euler approximation.

Table 5
$|\widehat{H}_n^{(1)} - H|$.

H	n points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,0873	0,04143	0,0243	0,0189	0,00865	0,00747	0,00151	0,00139
0.6	0,06177	0,07075	0,02148	0,01426	0,0085	0,00437	0,00268	0,00202
0.65	0,07241	0,04409	0,02211	0,01886	0,00785	0,00445	0,00205	0,00168
0.7	0,06757	0,03122	0,02208	0,01605	0,00827	0,00248	0,00226	0,00188
0.75	0,07199	0,0793	0,03303	0,01902	0,01302	0,00866	0,00401	0,00307
0.8	0,07166	0,04787	0,02728	0,01615	0,01123	0,00696	0,00413	0,00288
0.85	0,05945	0,04555	0,02251	0,02379	0,01387	0,01039	0,00735	0,00579
0.9	0,05687	0,05154	0,0369	0,03336	0,02649	0,02324	0,01872	0,01588
0.95	0,04979	0,04558	0,03356	0,02747	0,02379	0,02098	0,01817	0,0164

Table 6
$|\widehat{H}_n^{(1)} - H| \cdot n^{0.25} (\ln n)^{-0.3}$.

H	n points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,17452	0,09866	0,07654	0,07209	0,04443	0,04691	0,01291	0,01463
0.6	0,12353	0,16849	0,06763	0,05438	0,04367	0,02743	0,02292	0,02122
0.65	0,14482	0,105	0,06964	0,07194	0,04033	0,02794	0,01752	0,01758
0.7	0,13514	0,07435	0,06955	0,06124	0,04248	0,01557	0,01929	0,01977
0.75	0,14399	0,18887	0,10402	0,07255	0,06688	0,05438	0,03428	0,03218
0.8	0,14331	0,11402	0,08591	0,06161	0,05767	0,04372	0,03525	0,03023
0.85	0,1189	0,10848	0,07089	0,09075	0,07127	0,06523	0,06276	0,06077
0.9	0,11374	0,12275	0,11621	0,12724	0,13605	0,14588	0,1599	0,16671
0.95	0,09958	0,10855	0,10567	0,10477	0,12218	0,13171	0,15528	0,17223

Table 7
$|\widehat{H}_n^{(2)} - H|$.

H	n points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,121	0,15167	0,05507	0,02752	0,01441	0,00842	0,007	0,00289
0.6	0,19793	0,11052	0,04291	0,03615	0,01105	0,01066	0,00271	0,00416
0.65	0,09991	0,09147	0,03938	0,02468	0,01921	0,008	0,00322	0,00267
0.7	0,11982	0,0694	0,0527	0,02754	0,019	0,009	0,00471	0,00332
0.75	0,10525	0,05726	0,05048	0,01907	0,0159	0,00675	0,00431	0,00275
0.8	0,16145	0,08377	0,04853	0,03326	0,0107	0,00607	0,00347	0,00354
0.85	0,10618	0,08441	0,06081	0,04251	0,01198	0,0098	0,00397	0,00338
0.9	0,13429	0,06519	0,03618	0,01873	0,01027	0,00664	0,0047	0,00331
0.95	0,11862	0,1011	0,02994	0,02461	0,01268	0,00825	0,00318	0,00244

Appendix. The functions of bounded p-variation

First, we mention some information concerning p-variation and the functions of bounded p-variation. It is contained, e.g., in [11,27]. Let interval $[a, b] \subset \mathbb{R}$. Consider the following class

Table 8
$|\widehat{H}_n^{(2)} - H| \cdot n^{0.25} (\ln n)^{-0.3}$.

H	*n* points							
	100	250	1000	2500	10^4	$2.5 \cdot 10^4$	10^5	$2.5 \cdot 10^5$
0.55	0,242	0,36122	0,17343	0,10498	0,07402	0,05284	0,05977	0,03031
0.6	0,39585	0,2632	0,13512	0,1379	0,05678	0,06693	0,02312	0,04367
0.65	0,19981	0,21784	0,124	0,09415	0,09869	0,04992	0,02747	0,02802
0.7	0,23963	0,16528	0,16596	0,10506	0,09758	0,05651	0,04025	0,03481
0.75	0,21049	0,13637	0,15898	0,07273	0,08169	0,04235	0,03678	0,02883
0.8	0,32289	0,1995	0,15283	0,12697	0,05497	0,03812	0,0296	0,03721
0.85	0,21235	0,20103	0,19149	0,16215	0,06156	0,06154	0,03389	0,03543
0.9	0,26857	0,15525	0,11394	0,07144	0,05277	0,04168	0,04016	0,03474
0.95	0,23723	0,24077	0,0943	0,09386	0,06512	0,05182	0,02719	0,02566

of functions:

$$\mathcal{W}_p([a,b]) := \left\{ f : [a,b] \to \mathbb{R} : \; v_p(f;[a,b]) < \infty \right\},$$

where

$$v_p(f;[a,b]) = \sup_{\pi} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p .$$

Here $\pi = \{x_i : \; i = 0,\dots,n\}$ stands for any finite partition of $[a,b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. Denote $\Pi([a,b])$ the class of such partitions. We say that function f has bounded p-variation on $[a;b]$ if $v_p(f;[a,b]) < \infty$. Denote by $C\mathcal{W}_p([a,b])$ the space of continuous functions on $[a,b]$, with bounded p-variation.

Let $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}(f;[a,b])$. Then for any fixed f we have that $V_p(f)$ is a non-increasing function of p. It means that for any $0 < q < p$ the relation $V_p(f) \leqslant V_q(f)$ holds.

Let $a < c < b$ and let $f \in \mathcal{W}_p([a,b])$ for some $p \in (0,\infty)$. Then

$$v_p(f;[a,c]) + v_p(f;[c,b]) \leqslant v_p(f;[a,b]),$$
$$V_p(f;[a,b]) \leqslant V_p(f;[a,c]) + V_p(f;[c,b]).$$

Denote

$$|A|_\infty = \sup_{x\in\mathbb{R}} |A(x)|, \qquad |A|_\alpha = \sup_{x,y\in\mathbb{R}} \frac{|A(x)-A(y)|}{|x-y|^\alpha}.$$

Let F be a Lipschitz continuous function with Lipschitz constant L and $G \in \mathcal{C}^{1+\alpha}(\mathbb{R})$ with $0 < \alpha \leqslant 1$. Also, let $1 \leqslant p < 1+\alpha$. Then for any $h \in \mathcal{W}_p([a,b])$

$$\begin{aligned} V_{p,\infty}(F(h);[a,b]) &\leqslant LV_p(h;[a,b]) + \sup_{a\leqslant x\leqslant b} |F(h(x)) - F(h(a))| + |F(h(a))| \\ &\leqslant 2LV_p(h;[a,b]) + |F(h(a))|, \end{aligned} \tag{A.1}$$

$$\sup_{a\leqslant x\leqslant b} |G(h(x))| \leqslant |G'|_\infty V_p(h;[a,b]) + |G(h(a))| \tag{A.2}$$

$$V_{p/\alpha,\infty}(G(h);[a,b]) \leqslant V_{p,\infty}(G(h);[a,b]) \leqslant 2|G'|_\infty V_p(h;[a,b]) + |G(h(a))|, \tag{A.3}$$

and

$$V_{p/\alpha,\infty}\left(G'(h);[a,b]\right) \leqslant 2|G'|_\alpha V_p^\alpha(h;[a,b]) + |G'(h(a))|, \tag{A.4}$$

where $V_{p,\infty}(f;[a,b]) = V_p(f;[a,b]) + \sup_{a\leqslant x\leqslant b}|f(x)|$.

Let $f \in \mathcal{W}_p([a,b])$ and $p_1 > p > 0$. Then

$$V_{p_1}(f;[a,b]) \leqslant \operatorname{Osc}(f;[a,b])^{(p_1-p)/p_1} V_p^{p/p_1}(f;[a,b]), \tag{A.5}$$

where $\operatorname{Osc}(f;[a,b]) = \sup\{|f(x)-f(y)|: x,y\in[a,b]\}$.

Take functions $f_1, f_2 \in \mathcal{W}_p([a,b])$, $0<p<\infty$. Then $f_1 f_2 \in \mathcal{W}_p([a,b])$ and

$$V_p(f_1 f_2;[a,b]) \le V_{p,\infty}(f_1 f_2;[a,b]) \leqslant C_p V_{p,\infty}(f_1;[a,b])\, V_{p,\infty}(f_2;[a,b]). \tag{A.6}$$

Let $f_1 \in \mathcal{W}_q([a,b])$ and $f_2 \in \mathcal{W}_p([a,b])$. Then it follows from Young's version of Hölder's inequality that for any partition $\pi \in \Pi([a,b])$ and for any $p^{-1}+q^{-1}\geqslant 1$

$$\sum_i V_q(f_1;[x_{i-1},x_i])\, V_p(f_2;[x_{i-1},x_i]) \leqslant V_q(f_1;[a,b])\, V_p(f_2;[a,b]). \tag{A.7}$$

Second, we state some facts from the theory of the Riemann–Stieltjes integration.

Let $f\in\mathcal{W}_q([a,b])$ and $h\in\mathcal{W}_p([a,b])$ with $0<p<\infty$, $q>0$, $1/p+1/q>1$. If f and h have no common discontinuities then the Riemann–Stieltjes integral $(RS)\int_a^b f\,dh$ exists and for any $y\in[a,b]$ the following inequality holds:

$$\left|\int_a^b f\,dh - f(y)\,[h(b)-h(a)]\right| \leqslant C_{p,q} V_q(f;[a,b])\, V_p(h;[a,b]), \tag{A.8}$$

whence

$$V_p\left(\int_a^{\cdot} f\,dh;[a,b]\right) \leqslant C_{p,q} V_{q,\infty}(f;[a,b])\, V_p(h;[a,b]). \tag{A.9}$$

Here $C_{p,q} = \zeta(p^{-1}+q^{-1})$ and $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ is the Riemann zeta function. Further, for any $y\in[a,b]$

$$\begin{aligned} V_p&\left(\int_a^{\cdot}[f(x)-f(y)]\,dh(x);[a,b]\right)\\ &\leqslant C_{p,q}\left[V_q(f;[a,b]) + \sup_{a\leqslant x\leqslant b}|f(x)-f(y)|\right] V_p(h;[a,b])\\ &\leqslant 2C_{p,q} V_q(f;[a,b])\, V_p(h;[a,b]). \end{aligned} \tag{A.10}$$

Proposition 9. *Let $f:[a,b]\to\mathbb{R}$ be such a function that for some $1\leqslant p<2$ we have $f\in\mathcal{CW}_p([a,b])$. Also, let $F:\mathbb{R}\to\mathbb{R}$ be a differentiable function with locally Lipschitz derivative F'. Then composition $F'(f)$ is Riemann–Stieltjes integrable with respect to f and*

$$F(f(b)) - F(f(a)) = (RS)\int_a^b F'(f(x))\,df(x).$$

Furthermore, the following substitution rule holds.

Proposition 10. *Let f_1, f_2 and f_3 be functions from $\mathcal{CW}_p([a,b])$, $1 \leqslant p < 2$. Then*

$$(RS)\int_a^b f_1(x)\,d\left((RS)\int_a^x f_2(y)\,df_3(y)\right) = (RS)\int_a^b f_1(x)f_2(x)\,df_3(x).$$

Finally, assume that

$$F_1(x) = (R)\int_a^x f_1(y)\,dy \quad \text{and} \quad F_2(x) = (RS)\int_a^x f_2(y)\,df_3(y),$$

where f_1 is a continuous function, symbol (R) stands for the Riemann integral, funktions $f_2, f_3 \in \mathcal{CW}_p([a,b])$ for some $1 \leqslant p < 2$, and Q is a differentiable function with locally Lipschitz derivative q. It follows from Propositions 9 and 10 that

$$\begin{aligned} Q(F_1(x)+F_2(x)) - Q(0) &= \int_a^x q(F_1(y)+F_2(y))\,d(F_1(y)+F_2(y)) \\ &= \int_a^x q(F_1(y)+F_2(y))f_1(y)\,dy \\ &\quad + \int_a^x q(F_1(y)+F_2(y))f_2(y)\,df_3(y). \end{aligned} \tag{A.11}$$

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