Journal of Differential Equations **170**, 96–122 (2001)
doi:10.1006/jdeq.2000.3814, available online at http://www.idealibrary.com on IDEAL®

On the Roughness-Induced Effective Boundary Conditions for an Incompressible Viscous Flow

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We consider the laminar viscous channel flow with the lateral surface of the channel containing surface irregularities. It is supposed that a uniform pressure gradient is maintained in the longitudinal direction of the channel. After studying the corresponding boundary layers, we obtain rigorously the Navier friction condition. It is valid when the size and amplitude of the imperfections tend to zero. Furthermore, the coefficient in the law is determined through an auxiliary boundary-layer type problem, and the tangential drag force and the effective mass flow are determined up to order $O(\varepsilon^{3/2})$. The value of the effective coefficient is shown to be independent with respect to the position of the mean surface in the range of $O(\varepsilon)$. © 2001 Academic Press

Key Words: boundary with irregularities; roughness; boundary homogenization; Navier's boundary condition.

1. INTRODUCTION

The flow conditions at a solid wall are well-established. First, the fluid cannot penetrate the solid and the normal velocity is zero. For a viscous flow we should add more conditions, and it was observed in experiments that the tangential velocity is also zero. This experimental fact was not always accepted in the past, and another approach was to suppose that a layer of stagnant fluid existed close to the wall. Its thickness was assumed to be a function of the geometry, temperature, and fluid structure, and at the wall the fluid was allowed to slip. Navier claimed that the slip velocity should be proportional to the shear stress (see [16]). Navier's model can

[1] This paper was written during the sabbatical stay of A.M. at IWR, SFB 359, Universität Heidelberg, from October 1, 1998, to March 31, 1999.

0022-0396/01 $35.00

be confirmed, at least heuristically, by kinetic-theory calculations. However, the conclusion is that the proportionality constant in Navier's law is proportional to the mean free path divided by the continuum length (see [17]). Hence it is zero for most practical proposes.

Nevertheless, Navier's condition is used for simulations of flows in the presence of complex boundaries, e.g., as in geophysical fluid dynamics (see [18]). Using it, we reduce the rough boundary to a parameter in the effective boundary law and solve the Navier–Stokes system in a smooth domain. In general, the approach of replacing the no-slip condition at rough boundaries with the non-penetration condition plus a relation between the tangential velocity and the shear stress, is called the *wall laws*.

Rough boundaries cause boundary layers for the velocity gradient and we expect to get wall laws by corresponding multiscale expansions.

An extensive reference for the wall laws is the papers [1–3, 15] by O. Pironneau and his collaborators. The short note [1] presents a rigorous approach to the derivation of a wall law for the Laplace operator in an annular domain with rough perforations. Using a cell problem, an effective boundary condition of Robin's type was obtained. Paper [15] discusses wall laws used for simulation of viscous flows over a rough surface and over a wavy sea surface, respectively. Finally, References [2, 3] concern derivation of wall laws for incompressible viscous flows at high Reynolds numbers by a formal multiscale expansion. For such flow a Navier-type wall law and its nonlinear correction are obtained and the effective constants are calculated using semi-infinite cell problems.

It should be mentioned that the presence of a rough boundary influences the hydrodynamic drag. The influence of riblets on a longitudinal flow can be modeled by the Laplace equation in the transversal 2D section. For more details we refer the reader to the work [5] by Amirat and Simon. The same authors studied in [4] the Stokes flow between two infinite plates, one moving at a given velocity and other being at rest and having periodic asperities. Using a particular expansion, adapted to their problem, they have obtained the approximation for the velocity, pressure, and drag force to any order.

A somewhat related problem, studying the influence of the boundary layers on the effective behavior of the solution for the contact problems between a porous medium and a nonperforated domain under Dirichlet's conditions on the boundaries of the solid part, is studied in a number of recent papers by Jäger and Mikelić [11–13]. The article [13] is on the rigorous derivation of the effective boundary conditions at the contact interface between a porous medium and a viscous incompressible fluid. More precisely, in Beavers and Joseph [7] the effective law

$$\frac{\partial u_\tau}{\partial \nu} = \alpha (K^\varepsilon)^{-1/2} (u_\tau - u_\tau^D) \qquad \text{on } \Sigma,$$

was found, where Σ is the interface between media, ε is the characteristic pore size, α is a dimensionless constant depending only on the structure of the porous medium, K^ε is the scalar permeability tensor, and u_τ denotes the tangential effective velocity in the channel. The filtration velocity u^D satisfies the Darcy law

$$u^D = -\frac{K^\varepsilon}{\mu} \nabla p = -\varepsilon^2 \frac{K}{\mu} \nabla p,$$

where K depends only on the geometry of the porous medium. A theoretical approach to deriving the law of Beavers and Joseph at a physical level of rigor is made in Saffman [19]. Since the filtration velocity in the porous medium is much smaller than the effective velocity in the channel, Saffman deduced in [19] that the correct condition was

$$\frac{\partial u_\tau}{\partial \nu} = \alpha (K^\varepsilon)^{-1/2} u_\tau + O(\varepsilon) \qquad \text{on } \Sigma. \tag{1}$$

It is interesting that in the case of the flow across a rough boundary a wall law analogous to (1), but with different parameters and called in this situation *Navier's slip condition* or *Navier's friction condition*, is used.

The rigorous justification of the interface constitutive law (1) in [13] was based on the boundary layers constructed in [11], and it was possible not only to prove the convergence of the homogenization process but also to determine the constants. In this paper we are going to derive the Navier slip condition rigorously from the first principles. Our approach will follow [11] and [13] but the corresponding asymptotic expansions have analogies with those from [1–3]. However, it should be noted that we have a semi-infinite cell problem with a transmission-type condition at the interface, aiming to correct the shear stress created by the zeroth order approximation. In [1] one has a finite cell problem.

This paper concludes with an approximation for the tangential drag force, but we leave the interpretation of the result and its extension to a general profile for a forthcoming paper. Similarly, the case of a flow at higher Reynolds numbers from [2] and [3] is not considered here. As precisely stated in Section 2, we concentrate *on the laminar channel flow at moderate Reynolds numbers*.

We start by fixing the problem setting.

We consider the laminar viscous two-dimensional incompressible flow through a domain Ω^ε consisting of the channel $\Omega = (0, b) \times (0, h)$, the interfaces $\Sigma_1 = (0, b) \times \{0\}$ and $\Sigma_2 = (0, b) \times \{h\}$, and the layers of roughness Ω_j^ε, $j = 1, 2$.

We assume that the structure of the layers is periodic and generated by translations of a hump domain $Y^\varepsilon = \varepsilon(Y - (0, 1))$, where Y is a domain inside the standard cell, $Z = (0, 1)^2$, with a continuous boundary. It is assumed that $|\partial Y \cap \{y_2 = 1\}| > 0$ and that points $(0, 1)$ and $(1, 1)$ are from the boundary of Y (i.e., they are among the highest, resp., lowest, points of the hump). $\partial Y \in C^2$ in some neighborhood of $\{y_2 = 1\}$.

Let χ be the characteristic function of $Y - (0, 1)$, extended by periodicity in y_1 to $\mathbb{R} \times (0, 1)$. We set $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, $x \in \mathbb{R} \times (-1, 0)$, and define Ω_1^ε by $\Omega_1^\varepsilon = \{x \in (0, b) \times (-\varepsilon, 0) \mid \chi^\varepsilon(x) = 1\}$. Ω_2^ε is defined analogously, with a periodic but in general different structure of impurities, as a subset of $(0, b) \times (h, h + \varepsilon_1)$ with $\varepsilon_1 = C_0 \varepsilon$. Now, the flow region is $\Omega^\varepsilon = \Omega \cup \Sigma_1 \cup \Omega_1^\varepsilon \cup \Sigma_2 \cup \Omega_2^\varepsilon$. It is assumed that b/ε and $b/(C_0 \varepsilon) \in \mathbb{N}$.

Therefore, our rough boundary is assumed to consist of a large number of periodically distributed humps of characteristic length and amplitude ε, small compared with a characteristic length of the macroscopic domain.

A uniform pressure gradient is maintained in the longitudinal direction in Ω^ε. More precisely, for a fixed $\varepsilon > 0$ we define $\{u^\varepsilon, p^\varepsilon\}$ by the equations of motion and mass conservation,

$$-\mu \triangle u^\varepsilon + (u^\varepsilon \nabla) u^\varepsilon + \nabla p^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{2}$$

$$\operatorname{div} u^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{3}$$

$$u^\varepsilon = 0 \qquad \text{on} \quad \partial\Omega^\varepsilon \backslash \partial\Omega, \tag{4}$$

$$p^\varepsilon = p_0 \quad \text{on } \{0\} \times (0, h) \qquad \text{and} \qquad p^\varepsilon = p_b \quad \text{on } \{b\} \times (0, h), \tag{5}$$

$$u_2^\varepsilon = 0 \qquad \text{on} \quad (\{0\} \cup \{b\}) \times (0, h), \tag{6}$$

where $\mu > 0$ is the viscosity and p_0 and p_b are given constants.

Now we want to study the effective behavior of the velocities u^ε and pressures p^ε as $\varepsilon \to 0$, i.e., when the characteristic size of the irregularities tends to zero.

It is clear that in Ω the flow continues to be governed by the Navier–Stokes system. The presence of the irregularities would only contribute to the effective boundary conditions at the lateral boundary and the main goal of this paper is finding the effective behavior of $\{u^\varepsilon, p^\varepsilon\}$ on Σ in the limit $\varepsilon \to 0$.

As in the case of the contact between a channel flow and a porous medium, the main difficulty comes from the appearance of the boundary layers in the neighbourhoods of the contact surface, where the gradient of the solution differs greatly from the behavior inside the interiors of the domains and, as in the paper [13], the crucial role is played by an auxiliary problem. It reads as follows:

Find $\{\beta^{\rm bl}, \omega^{\rm bl}\}$ that solve

$$-\triangle_y \beta^{\rm bl} + \nabla_y \omega^{\rm bl} = 0 \qquad \text{in} \quad Z^+ \cup (Y - (0,1)) \tag{7}$$

$$\operatorname{div}_y \beta^{\rm bl} = 0 \qquad \text{in } Z_{\rm bl} \tag{8}$$

$$[\beta^{\rm bl}]_S(\cdot, 0) = 0 \qquad \text{on } S \tag{9}$$

$$[\{\nabla_y \beta^{\rm bl} - \omega^{\rm bl} I\}\, e_2]_S(\cdot, 0) = e_1 \qquad \text{on } S \tag{10}$$

$$\beta^{\rm bl} = 0 \quad \text{on } (\partial Y - (0,1)), \qquad \{\beta^{\rm bl}, \omega^{\rm bl}\} \text{ is } y_1\text{-periodic}, \tag{11}$$

where $S = (0,1) \times \{0\}$, $Z^+ = (0,1) \times (0, +\infty)$, and $Z_{\rm bl} = Z^+ \cup S \cup (Y - (0,1))$.

Let $V = \{z \in L^2_{\rm loc}(Z_{\rm BL})^2 : \nabla_y z \in L^2(Z_{\rm BL})^4;\ z = 0$ on $((\partial Y \backslash \partial Z) - (0,1))$; $\operatorname{div}_y z = 0$ in $Z_{\rm BL}$ and z is y_1-periodic$\}$. Then the Lax–Milgram lemma implies the existence of a variational solution $\{\beta^{\rm bl}, \omega^{\rm bl}\} \in V \cap C^\infty(Z^+ \cup (Y - (0,1)))^2 \times C^\infty(Z^+ \cup (Y - (0,1)))$ to (7)–(11), where $\beta^{\rm bl}$ is unique and $\omega^{\rm bl}$ is unique up to a constant.

In the neighborhood of S we have $\beta^{\rm bl} - ((y_2 - y_2^2/2)\, e^{-y_2} H(y_2), 0)) \in W^{2,q}$ and $\omega^{\rm bl} \in W^{1,q}$, $\forall q \in [1, \infty)$.

Then we have

Lemma 1 [11]. *Any solution* $\{\beta^{\rm bl}, \omega^{\rm bl}\}$ *satisfies*

$$\begin{aligned}
&\int_0^1 \beta_2^{\rm bl}(y_1, a)\, dy_1 = 0, && \forall a \in (0, +\infty),\\
&\int_0^1 \omega^{\rm bl}(y_1, a_1)\, dy_1 = \int_0^1 \omega^{\rm bl}(y_1, a_2)\, dy_1, && \forall a_1 > a_2 \geqslant 0,\\
&\int_0^1 \beta_1^{\rm bl}(y_1, a_1)\, dy_1 = \int_0^1 \beta_1^{\rm bl}(y_1, a_2)\, dy_1, && \forall a_1 > a_2 \geqslant 0,\\
&C_1^{\rm bl} = \int_0^1 \beta_1^{\rm bl}(y_1, 0)\, dy_1 = -\int_{Z_{\rm BL}} |\nabla \beta^{\rm bl}(y)|^2\, dy.
\end{aligned} \tag{12}$$

Lemma 2. *Let* $a > 0$ *and let* $\beta^{a,\rm bl}$ *be the solution for* (7)–(11) *with* S *replaced by* $S_a = (0,1) \times \{a\}$ *and* Z^+ *replaced by* $Z_a^+ = (0,1) \times (a, +\infty)$. *Then we have*

$$C_1^{a,\rm bl} = \int_0^1 \beta_1^{a,\rm bl}(y_1, a)\, dy_1 = C_1^{\rm bl} - a. \tag{13}$$

Proof. By (12),

$$C_1^{a,\rm bl} = \int_0^1 \beta_1^{a,\rm bl}(y_1, c)\, dy_1, \qquad \forall c \geqslant a.$$

Let $0 \leqslant c_1 < a < c_2$. Integration of the first component of (7),

$$\operatorname{div}\{\nabla \beta_1^{a,\,\mathrm{bl}} - \omega^{a,\,\mathrm{bl}} e_1\} = 0,$$

over (c_1, c_2) gives

$$\int_0^1 \left\{ \frac{\partial \beta_1^{a,\,\mathrm{bl}}}{\partial y_2}(y_1, c_2) - \frac{\partial \beta_1^{a,\,\mathrm{bl}}}{\partial y_2}(y_1, a+0) \right. \\ \left. + \frac{\partial \beta_1^{a,\,\mathrm{bl}}}{\partial y_2}(y_1, a-0) - \frac{\partial \beta_1^{a,\,\mathrm{bl}}}{\partial y_2}(y_1, c_1) \right\} dy_1 = 0.$$

Hence from (10) and Lemma 1

$$\frac{d}{dy_2} \int_0^1 \beta_1^{a,\,\mathrm{bl}}(y_1, y_2)\, dy_1 = -1, \qquad \text{for} \quad c_1 < y_2 < a,$$

and

$$\int_0^1 \beta_1^{a,\,\mathrm{bl}}(y_1, y_2)\, dy_1 = a - y_2 + C_1^{a,\,bl}, \qquad \text{for} \quad 0 \leqslant y_2 \leqslant a. \tag{14}$$

The variational equation for $\beta^{a,\,\mathrm{bl}} - \beta^{\mathrm{bl}}$ reads

$$\int_{Z_{\mathrm{BL}}} \nabla(\beta^{a,\,\mathrm{bl}} - \beta^{\mathrm{bl}})\, \nabla\varphi\, dy = -\int_0^1 (\varphi_1(y_1, a) - \varphi_1(y_1, 0))\, dy_1, \qquad \forall \varphi \in V.$$

Testing with $\varphi = \beta^{a,\,\mathrm{bl}} - \beta^{\mathrm{bl}}$ and using (14), we obtain

$$\int_{Z_{\mathrm{BL}}} |\nabla(\beta^{a,\,\mathrm{bl}} - \beta^{\mathrm{bl}})|^2\, dy = -\int_0^1 (\beta_1^{a,\,\mathrm{bl}}(y_1, a) - \beta_1^{a,\,\mathrm{bl}}(y_1, 0))\, dy_1 = a.$$

On the other hand,

$$\int_{Z_{\mathrm{BL}}} |\nabla(\beta^{a,\,\mathrm{bl}} - \beta^{\mathrm{bl}})|^2\, dy = \int_{Z_{\mathrm{BL}}} |\nabla\beta^{a,\,\mathrm{bl}}|^2\, dy + \int_{Z_{\mathrm{BL}}} |\nabla\beta^{\mathrm{bl}}|^2\, dy \\ -2\int_{Z_{\mathrm{BL}}} \nabla\beta^{a,\,\mathrm{bl}}\, \nabla\beta^{\mathrm{bl}}\, dy = C_1^{\mathrm{bl}} - C_1^{a,\,\mathrm{bl}},$$

and (13) is proved. ∎

This simple result will imply the invariance of the obtained law on the position of the interface Σ.

In the next step we determine the decay in Z^+ by reduction to the Laplace operator. We choose the free constant in the pressure field in the way that $\int_0^1 \omega^{\mathrm{bl}}(y_1, 0)\, dy_1 = 0$. We note that in such a situation the decay can be obtained by using Tartar's lemma from [14], but because of (12) the averages with respect to y_1 are always constant or zero. Then, by using the separation of variables, we obtain

LEMMA 3.

$$\begin{aligned} |D^\alpha \operatorname{curl}_y \beta^{\mathrm{bl}}(y_1, y_2)| &\leqslant Ce^{-2\pi y_2}, && y_2 > 0, \quad \alpha \in \mathbb{N}^2 \cup (0,0),\\ |\beta^{\mathrm{bl}}(y_1, y_2) - (C_1^{\mathrm{bl}}, 0)| &\leqslant C(\delta)\, e^{-\delta y_2}, && y_2 > 0, \quad \forall \delta < 2\pi,\\ |D^\alpha \beta^{\mathrm{bl}}(y_1, y_2)| &\leqslant C(\delta)\, e^{-\delta y_2}, && y_2 > 0, \quad \alpha \in \mathbb{N}^2, \quad \forall \delta < 2\pi,\\ |\omega^{\mathrm{bl}}(y_1, y_2)| &\leqslant Ce^{-2\pi y_2}, && y_2 > 0, \end{aligned} \tag{15}$$

Following the approach from [13], the Navier friction condition should correspond to taking into account the next-order corrections for the velocity. Then formally we get

$$u^\varepsilon = v^0 - \varepsilon \beta^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right)\frac{\partial v_1^0}{\partial x_2} + \varepsilon C_1^{\mathrm{bl}}\left(\frac{\partial v_1^0}{\partial x_2}\mathbf{e}_1 + d^1\right) H(x_2) + O(\varepsilon^2),$$

where v^0 is the Hagen–Poiseuille velocity in Ω and d^1 corresponds to the counterflow generated by the boundary condition $d^1 = -(\partial v_1^0/\partial x_2)\,\mathbf{e}_1$ on Σ. Then on the interface Σ,

$$\frac{\partial u_1^\varepsilon}{\partial x_2} = \frac{\partial v_1^0}{\partial x_2}\left(1 - \frac{\partial \beta_1^{\mathrm{bl}}}{\partial y_2}\left(\frac{x}{\varepsilon}\right)\right) + O(\varepsilon) \qquad \text{and} \qquad \frac{1}{\varepsilon} u_1^\varepsilon = -\beta_1^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right)\frac{\partial v_1^0}{\partial x_2} + O(\varepsilon).$$

After averaging we obtain the familiar form of the Navier slip condition,

$$u_1^{\mathrm{eff}} = -\varepsilon C_1^{\mathrm{bl}} \frac{\partial u_1^{\mathrm{eff}}}{\partial x_2} \qquad \text{on } \Sigma, \tag{NFC}$$

where u^{eff} is the average over the impurities and C_1^{bl} is defined by (12). The higher order terms are neglected.

The rest of the paper contains the rigorous justification of the interface law (the Navier Friction condition). In order to cover realistic flows from the physical literature (see [18, 19] and the references therein), we give the proof for the Navier–Stokes equations (2)–(6) rather than for the simplified model from [11]. The solvability of the system (2)–(6) and the uniform a priori estimates are given in Section 2. In Section 3, we first construct an approximation for the velocity and pressure field in Ω^ε and an outer boundary layer. Then we define a correction of order $O(\varepsilon^{3/2})$ for the velocity. This result enables us to establish the justification of the Navier friction condition (NFC) in Theorem 2 and to compare the physical velocity with the effective one. Finally, the effective mass flow and the tangential drag force are determined with an error of order $O(\varepsilon^{3/2})$.

2. SOLVABILITY OF THE ε-PROBLEM AND UNIFORM A PRIORI ESTIMATES

In this section we first address the existence of solutions for the ε-problem (2)–(6). The existence for fixed ε and a small pressure difference is proven in [8]. Similar results, but with the dynamic pressure $p^\varepsilon + \frac{1}{2}|u^\varepsilon|^2$ given instead of p^ε, were obtained in the exhaustive paper [10]. For the equations (2)–(6) in general geometries global existence results for arbitrary data do not seem to be known. As pointed out in [6], the non-homogeneous boundary conditions for the pressure lead to cubic terms in velocity which cannot easily be estimated. Consequently, we can, in general, expect existence only for small $|p_b - p_0|$.

Let us note that for the somewhat similar problem of nonstationary incompressible flow through the filter, considered in [12], it was possible to prove the global existence, uniqueness, and regularity, regardless of the size of the pressure at the boundary. As in the case of the porous filter, here the rough boundary is expected to considerably slow down the flow and help us to obtain the global existence. But since we are interested in the channel flow with Reynolds numbers corresponding to the laminar flow, it is natural to consider the system (2)–(6) with $|p_b - p_0|$ not too large.

Since we need not only existence for a given ε, but also the a priori estimates independent of ε, we give a direct proof of existence and uniqueness, leading to uniform a priori estimates. It should be noted that in fact our result implies the nonlinear stability of the Poiseuille flow with respect to the perturbation of the boundary by the small impurities. Our proof follows the corresponding one from [13].

First, we observe that the classic Poiseuille flow in Ω, satisfying the no-slip conditions at $\Sigma_1 \cup \Sigma_2$, is given by

$$\begin{aligned} v^0 &= \left(\frac{p_b - p_0}{2b\mu} x_2(x_2 - h), 0\right) \qquad \text{for} \quad 0 \leqslant x_2 \leqslant h, \\ p^0 &= \frac{p_b - p_0}{b} x_1 + p_0 \qquad \text{for} \quad 0 \leqslant x_1 \leqslant b. \end{aligned} \tag{16}$$

For $|p_b - p_0| \leqslant C(b, h)\,\mu^2$, (16) defines the unique solution to (2)–(7) between all those lying in the ball

$$B_0 = \{z \in H^1(\Omega_1)^2 \mid \|z\|_{L^4(\Omega_1)^2} \leqslant C(b, h)\,\mu\}.$$

We extend it to $\Omega^\varepsilon \setminus \Omega$ by setting $v^0 = 0$ and keeping the same form of p^0. The idea is to construct the solution to (2)–(6) as a small perturbation to the Hagen–Poiseuille flow (16). Before the existence result, we prove an auxiliary lemma:

LEMMA 4. *Let $\varphi \in H^1(\Omega^\varepsilon \setminus \Omega)$ be such that $\varphi = 0$ on $\partial\Omega^\varepsilon \setminus \partial\Omega$. Then we have*

$$\|\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)} \leqslant C\varepsilon\, \|\nabla\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)^2}, \tag{17}$$

$$\|\varphi\|_{L^2(\Sigma_1 \cup \Sigma_2)} \leqslant C\varepsilon^{1/2}\, \|\nabla\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)^2}, \tag{18}$$

$$\int_0^b (|\varphi(x_1, 0)| + |\varphi(x_1, h)|)\, dx_1 \leqslant C\sqrt{\varepsilon}\, \left\| \frac{\partial\varphi}{\partial x_2} \right\|_{L^2(\Omega^\varepsilon \setminus \Omega)}. \tag{19}$$

Proof. The estimate (17) is well known. For the estimate (18) we refer the reader to [11].

It remains to prove (19). Using the boundary conditions we get

$$\int_0^b |\varphi(x_1, 0)|\, dx_1 \leqslant \int_{\Omega^\varepsilon \setminus \Omega} \left| \frac{\partial\varphi}{\partial x_2} \right| dx \leqslant C\sqrt{\varepsilon}\, \left\| \frac{\partial\varphi}{\partial x_2} \right\|_{L^2(\Omega^\varepsilon \setminus \Omega)},$$

implying (19). ∎

Now we are in a position to prove the desired non-linear stability result:

PROPOSITION 1. *There exists a constant $C(b, h)$ such that for $|p_b - p_0|/b\mu \leqslant C(b, h)\,\mu$ and $\varepsilon \leqslant \varepsilon_0$ the problem (2)–(6) has a solution $\{u^\varepsilon, p^\varepsilon\} \in H^2(\Omega^\varepsilon)^2 \times H^1(\Omega^\varepsilon)$ satisfying*

$$\|\nabla(u^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} \leqslant C\sqrt{\varepsilon}. \tag{20}$$

Moreover, after a possible modification of $C(b, h)$, all solutions lying in the ball

$$B = \{z \in H^1(\Omega_1)^2 \mid \|z\|_{L^4(\Omega_1)^2} \leqslant C(b, h)\,\mu\}$$

are equal to $\{u^\varepsilon, p^\varepsilon\}$.

Proof. We search u^ε in the form $u^\varepsilon = v^0 + w^\varepsilon$. Let

$$\mathscr{X}^\varepsilon = \{z \in H^1(\Omega^\varepsilon)^2 : z = 0 \text{ on } \partial\Omega^\varepsilon \setminus \partial\Omega;\ z_2 = 0 \text{ on } (\{0\} \cup \{b\}) \times (0, h)\}.$$

Then we are looking for $w^\varepsilon \in \mathscr{W}^\varepsilon = \{\varphi \in \mathscr{X}^\varepsilon : \operatorname{div} \varphi = 0 \text{ in } \Omega^\varepsilon\}$ such that

$$\mu \int_{\Omega^\varepsilon} \nabla w^\varepsilon \nabla \varphi + \int_{\Omega^\varepsilon} (w^\varepsilon \nabla)\, w^\varepsilon \varphi + \int_{\Omega^\varepsilon} v_1^0 \frac{\partial w^\varepsilon}{\partial x_1} \varphi + \int_{\Omega^\varepsilon} w_2^\varepsilon \frac{\partial v_1^0}{\partial x_2} \varphi_1$$
$$= \mu \frac{\partial v_1^0}{\partial x_2}(0) \int_{\Sigma_1} \varphi_1 - \mu \frac{\partial v_1^0}{\partial x_2}(h) \int_{\Sigma_2} \varphi_1 - \frac{p_b - p_0}{b} \int_{\Omega^\varepsilon \setminus \Omega} \varphi_1, \qquad \forall \varphi \in \mathscr{W}^\varepsilon. \tag{21}$$

The proof of solvability for (21) consists of several steps:

(a) Let $\mathscr{H} = \{\varphi \in H^1(\Omega)^2 : \varphi^2 = 0 \text{ on } (\{0\} \cup \{b\}) \times (0, h)\}$ and let $\varpi(\psi, \varphi)$ be a bilinear form on $\mathscr{H} \times \mathscr{H}$ given by

$$\varpi(\psi, \varphi) = \mu \int_\Omega \nabla\psi \nabla\varphi + \int_\Omega v_1^0 \frac{\partial \psi}{\partial x_1} \varphi + \int_\Omega \psi_2 \frac{\partial v_1^0}{\partial x_2} \varphi_1. \tag{22}$$

For the last two terms in (22) we have

$$\int_\Omega v_1^0 \frac{\partial \psi}{\partial x_1} \psi + \int_\Omega \psi_2 \frac{\partial v_1^0}{\partial x_2} \psi_1$$
$$= \int_\Omega v_1^0 \psi_2 \operatorname{curl} \psi + \int_\Omega v_1^0 \psi_1 \left(\frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_2}{\partial x_2} \right), \qquad \forall \psi \in \mathscr{H}. \tag{23}$$

Let $m = \frac{1}{2b} \int_0^b (\psi_1(x_1, 0) + \psi_1(x_1, h))\, dx_1$. Using (23), we conclude that there is a constant $C = C(b, h)$ such that for $|p_b - p_0|/b\mu \leqslant C(b, h)\, \mu$ we have

$$\frac{\mu}{2} \int_\Omega |\nabla\psi|^2 \geqslant -\int_\Omega v_1^0 \psi_2 \operatorname{curl} \psi - \int_\Omega v_1^0 (\psi_1 - m) \left(\frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_2}{\partial x_2} \right), \qquad \forall \psi \in \mathscr{H}. \tag{24}$$

The estimate (24) implies, under the same condition on the pressure difference,

$$\varpi(\psi, \psi) \geqslant \frac{\mu}{2} \int_\Omega |\nabla\psi|^2 - m \int_\Omega v_1^0 \left(\frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_2}{\partial x_2} \right), \qquad \forall \psi \in \mathscr{H}. \tag{25}$$

(b) Now let $w^k \in \mathscr{W}^\varepsilon$, $\|w^k\|_{H^1(\Omega^\varepsilon)^2} \leqslant R$. We consider the problem

$$\varpi(w^{k+1}, \varphi) + \mu \int_{\Omega^\varepsilon \setminus \Omega} \nabla w^{k+1} \nabla \varphi + \int_{\Omega^\varepsilon} (w^k \nabla)\, w^{k+1} \varphi$$
$$= \mu \frac{\partial v_1^0}{\partial x_2}(0) \int_{\Sigma_1} \varphi_1 - \mu \frac{\partial v_1^0}{\partial x_2}(h) \int_{\Sigma_2} \varphi_1 - \frac{p_b - p_0}{b} \int_{\Omega^\varepsilon \setminus \Omega} \varphi_1, \qquad \forall \varphi \in \mathscr{W}^\varepsilon. \tag{26}$$

We want to prove the unique solvability of (26) in $\mathscr{W}^\varepsilon$. Obviously, it is enough to have a convenient estimate for the third term on the left hand side. Using Poincaré's inequality in $\Omega^\varepsilon \backslash \Omega$ and interpolation we obtain for $\varphi \in \mathscr{W}^\varepsilon$,

$$\left| \int_{\Omega^\varepsilon \backslash \Omega} (w^k \nabla) \, \psi \varphi \right| \leqslant CR \sqrt{\varepsilon} \, \|\nabla \psi\|_{L^2(\Omega^\varepsilon \backslash \Omega)^4} \, \|\nabla \varphi\|_{L^2(\Omega^\varepsilon \backslash \Omega)^4};$$

$$\left| \int_{\Omega} (w^k \nabla) \, \psi \varphi \right| \leqslant \left| \int_{\Omega} (w^k \nabla) \, \psi (\varphi - m \mathbf{e}_1) \right| + |m| \left| \int_{\Omega} (w^k \nabla) \, \psi_1 \right| \tag{27}$$

$$\leqslant CR(1 + \sqrt{\varepsilon}) \, \|\nabla \psi\|_{L^2(\Omega^\varepsilon)^4} \, \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}.$$

Therefore, for $R \leqslant \bar{C}(b, h) \, \mu$ and $\varepsilon \leqslant \varepsilon_0 = C\mu^4$, we have

$$\varpi(\varphi, \varphi) + \mu \int_{\Omega^\varepsilon \backslash \Omega} |\nabla \varphi|^2 + \int_{\Omega^\varepsilon} (w^k \nabla) \, \varphi \varphi \geqslant \frac{\mu}{4} \int_{\Omega^\varepsilon} |\nabla \varphi|^2 \qquad \forall \varphi \in \mathscr{W}^\varepsilon, \tag{28}$$

and the problem (26) has a unique solution $w^{k+1} \in \mathscr{W}^\varepsilon$.

(c) Now we define a nonlinear mapping $\mathscr{T}$ by

$$\mathscr{T}(w^k) = w^{k+1}. \tag{29}$$

Let us check if $\mathscr{T}$ is a continous map, $\mathscr{T}: \mathscr{W}^\varepsilon \to \mathscr{W}^\varepsilon$.

Let $z^j \in \mathscr{W}^\varepsilon$, $j = 1, 2$, and let $w^j = \mathscr{T} z^j$, $j = 1, 2$. Furthermore, let $z = z^1 - z^2$ and $w = w^1 - w^2$. Then we have

$$-\int_{\Omega^\varepsilon} (z \nabla) \, w^2 w = \varpi(w, w) + \mu \int_{\Omega^\varepsilon \backslash \Omega} |\nabla w|^2 + \int_{\Omega^\varepsilon} (z^1 \nabla) \, ww \geqslant \frac{\mu}{4} \int_{\Omega^\varepsilon} |\nabla w|^2,$$

leading to

$$\int_{\Omega^\varepsilon} |\nabla w|^2 \leqslant C \, \|z\|_{L^4(\Omega^\varepsilon)^2} \, \|w\|_{L^4(\Omega^\varepsilon)^2} \, \|\nabla w\|_{L^4(\Omega^\varepsilon)^2}, \tag{30}$$

which proves the continuity.

(d) It remains to check that $\mathscr{T}(B_R) \subset B_R$ for $R \leqslant \bar{C}\mu$, as a map from $\mathscr{L}(\mathscr{W}^\varepsilon, \mathscr{W}^\varepsilon)$. By Lemma 4, we have

$$\left| \int_{\Sigma} \varphi_1 \right| \leqslant C \sqrt{\varepsilon} \, \|\nabla \varphi\|_{L^2(\Omega^\varepsilon \backslash \Omega)^4}, \qquad \forall \varphi \in \mathscr{Z}^\varepsilon. \tag{31}$$

Consequently, (26) and (28) imply

$$\frac{\mu}{4} \|\nabla w^{k+1}\|_{L^2(\Omega^\varepsilon)^4} \leqslant C\sqrt{\varepsilon}$$

and, for $\varepsilon \leqslant \varepsilon_0$, $\mathcal{T}(B_R) \subset B_{C\sqrt{\varepsilon}} \subset B_R$.

Therefore, by (31), for $\varepsilon \leqslant \varepsilon_0$, $\mathcal{T}$ has a fixed point $w^\varepsilon \in \mathcal{W}^\varepsilon$ satisfying the estimate (20). Uniqueness in B and regularity are obvious. ∎

PROPOSITION 2. *For the solution to* (2)–(6), *satisfying* (20), *we have the a priori estimates*

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon \backslash \Omega)^2} \leqslant C\varepsilon\sqrt{\varepsilon}, \tag{32}$$

$$\|u^\varepsilon\|_{L^2(\Sigma_1 \cup \Sigma_2)^2} \leqslant C\varepsilon, \tag{33}$$

$$\|u^\varepsilon - v^0\|_{L^2(\Omega)^2} \leqslant C\varepsilon, \tag{34}$$

$$\left| \int_{\Sigma_1} \frac{\partial u_1^\varepsilon}{\partial x_2}(x_1, 0)\, dx_1 + \frac{p_b - p_0}{2\mu} h \right| + \left| \int_{\Sigma_2} \frac{\partial u_1^\varepsilon}{\partial x_2}(x_1, h)\, dx_1 - \frac{p_b - p_0}{2\mu} h \right| \leqslant C\varepsilon, \tag{35}$$

$$\|p^\varepsilon - p^0\|_{L^2(\Omega^\varepsilon)} \leqslant C\sqrt{\varepsilon}. \tag{36}$$

Proof. The inequalities (32) and (33) are direct consequences of Poincaré's inequality and the trace inequality in $\Omega^\varepsilon \backslash \Omega$, respectively.

In order to get the estimates (34) and (36) we note that $w^\varepsilon = u^\varepsilon - v^0$ and $\pi^\varepsilon = p^\varepsilon - p^0$ satisfy the system

$$\begin{aligned}
&-\triangle w^\varepsilon + \nabla \pi^\varepsilon + v_1^0 \frac{\partial w^\varepsilon}{\partial x_1} + w_2^\varepsilon \frac{\partial v_1^0}{\partial x_2} e_1 + (w^\varepsilon \nabla) w^\varepsilon = 0 \qquad \text{in } \Omega,\\
&\operatorname{div} w^\varepsilon = 0 \qquad \text{in } \Omega,\\
&w^\varepsilon = \xi^\varepsilon \quad \text{on } \Sigma_1 \cup \Sigma_2,\\
&w_2^\varepsilon = 0 \quad \text{and} \quad \pi^\varepsilon = 0 \qquad \text{on } (\{0\} \cup \{b\}) \times (0, h),
\end{aligned} \tag{37}$$

where $|\xi^\varepsilon|_{L^2(\Sigma_1 \cup \Sigma_2)^2} \leqslant C\varepsilon$, by (33).

The theory of the very weak solutions for the Stokes system was developed in [9]. By using the analogous very weak variational formulation for Oseen's problem (37), we get (34). The estimate (36) follows from the first equation in (37) and Nečas' inequality in Ω.

It remains to prove (35). We test the expression for the tangential drag by $\varphi = (\varphi_1(x_2), 0)$, where $\varphi_1 = (1 - 2x_2/h)^2$ for $0 \leqslant x_2 \leqslant h/2$. Then, by using (32)–(34), we get

$$\left| \int_{\Sigma_1} \frac{\partial(u_1^\varepsilon - v_1^0)}{\partial x_2} (x_1, 0)\, dx_1 \right|$$

$$= \left| \int_0^b \int_0^{h/2} \operatorname{div}((\nabla(u_1^\varepsilon - v_1^0) - (p^\varepsilon - p^0)\, \mathbf{e}_1)\, \varphi_1) \right|$$

$$\leqslant \left| \int_0^b \int_0^{h/2} ((u^\varepsilon \nabla)\, u_1^\varepsilon - (v^0 \nabla)\, v_1^0)\, \varphi_1 \right| + \left| \int_0^b \int_0^{h/2} \frac{\partial(u_1^\varepsilon - v_1^0)}{\partial x_2} \frac{\partial \varphi_1}{\partial x_2} \right|$$

$$\leqslant \left| \int_0^b \int_0^{h/2} ((u^\varepsilon - v^0) \nabla)\, u_1^\varepsilon \varphi_1 \right| + \left| \int_0^b \int_0^{h/2} v_1^0 \frac{\partial u_2^\varepsilon}{\partial x_2} \varphi_1 \right| + \left| \int_{\Sigma_1} (u_1^\varepsilon - v_1^0) \frac{\partial \varphi_1}{\partial x_2} \right|$$

$$+ \left| \int_0^b \int_0^{h/2} \frac{\partial^2 \varphi_1}{\partial x_2^2} (u_1^\varepsilon - v_1^0) \right| \leqslant C\varepsilon. \qquad (38)$$

This proves the proposition. ∎

Therefore, we have obtained the uniform a priori estimates for $\{u^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Poiseuille's flow in Ω is an $O(\varepsilon)$ L^2-approximation for u^ε. Following the formal asymptotic expansion from Section 1, the Navier friction law should correspond to the next order velocity correction.

3. THE NEXT ORDER VELOCITY CORRECTION AND NAVIER'S FRICTION LAW

The leading contribution for the estimate (20) was the interface integral terms $\int_{\Sigma_j} \varphi_1$. Following the approach from [11], we eliminate it by using the boundary-layer-type functions

$$\beta^{\mathrm{bl},\varepsilon}(x) = \varepsilon \beta^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \omega^{\mathrm{bl},\varepsilon}(x) = \omega^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right), \qquad x \in \Omega^\varepsilon. \qquad (39)$$

We have, for all $q \geqslant 1$,

$$\frac{1}{\varepsilon} \|\beta^{\mathrm{bl},\varepsilon} - \varepsilon(C_1^{\mathrm{bl}}, 0)\|_{L^q(\Omega)^2} + \|\omega^{\mathrm{bl},\varepsilon}\|_{L^q(\Omega)} + \|\nabla\beta^{\mathrm{bl},\varepsilon}\|_{L^q(\Omega)^4} = C\varepsilon^{1/q},$$
$$\|\beta^{\mathrm{bl},\varepsilon}(0,\cdot) - \varepsilon(C_1^{\mathrm{bl}}, 0)\|_{L^q(0,h)^2} = C\varepsilon^{1+1/q}, \tag{40}$$

and

$$\|\omega^{\mathrm{bl},\varepsilon}(0,\cdot)\|_{H^{-1/2}(\mathbb{R})} + \sqrt{\varepsilon}\,\|\omega^{\mathrm{bl},\varepsilon}(0,\cdot)\|_{L^2(\mathbb{R})} = C\varepsilon,$$
$$\varepsilon^{-1/2}\,\|\beta^{\mathrm{bl},\varepsilon}(0,\cdot) - \varepsilon(C_1^{\mathrm{bl}}, 0)\,H(\cdot)\|_{L^2(\mathbb{R})^2} + \left\|\frac{\partial\beta^{\mathrm{bl},\varepsilon}}{\partial x_2}(0,\cdot)\right\|_{H^{-1/2}(\mathbb{R})^2} = C\varepsilon. \tag{41}$$

Finally,

$$-\triangle\beta^{\mathrm{bl},\varepsilon} + \nabla\omega^{\mathrm{bl},\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon\backslash\Sigma_1 \tag{42}$$

$$\operatorname{div}\,\beta^{\mathrm{bl},\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon \tag{43}$$

$$[\beta^{\mathrm{bl},\varepsilon}]_\Sigma(\cdot, 0) = 0 \qquad \text{on } \Sigma_1 \tag{44}$$

$$[\{\nabla\beta^{\mathrm{bl},\varepsilon} - \omega^{\mathrm{bl},\varepsilon} I\}\, e_2]_{\Sigma_1}(\cdot, 0) = e_1 \qquad \text{on } \Sigma_1. \tag{45}$$

An analogous boundary layer corresponds to Σ_2. Without loss of the generality we can neglect it and restrict our considerations to Σ_1. As in [11], stabilization of $\beta^{\mathrm{bl},\varepsilon}$ toward a nonzero constant velocity $\varepsilon(C_1^{\mathrm{bl}}, 0)$ at the upper boundary generates a counterflow. It is given by the Oseen system in Ω

$$-\mu \triangle d + \nabla g + v_1^0 \frac{\partial d}{\partial x_1} + d_2 \frac{\partial v_1^0}{\partial x_2} e_1 = 0 \qquad \text{in } \Omega,$$
$$\operatorname{div} d = 0 \qquad \text{in } \Omega, \tag{46}$$
$$d = e_1 \quad \text{on } \Sigma_1, \qquad d = 0 \quad \text{on } (0, b) \times \{h\},$$
$$d_2 = 0 \quad \text{and} \quad g = 0 \qquad \text{on} \quad (\{0\} \cup \{b\}) \times (0, h).$$

Under the assumption $|p_b - p_0|/b\mu \leqslant C(b, h)\,\mu$ from Proposition 1, the problem (46) has a unique solution in the form of 2D Couette flow $d = (1 - x_2/h)\, e_1$ and $g = 0$.

There is an analogous counterflow at Σ_2, and, as with the boundary layers, we do not write it.

Now, we want to prove that the following quantities are $o(\varepsilon)$ for the velocity and $O(\varepsilon)$ for the pressure:

$$\mathscr{U}_0^\varepsilon(x) = u^\varepsilon - v^0 + (\beta^{\mathrm{bl},\varepsilon} - \varepsilon(C_1^{\mathrm{bl}}, 0)\, H(x_2)) \frac{\partial v_1^0}{\partial x_2}(0) + \varepsilon C_1^{\mathrm{bl}} \frac{\partial v_1^0}{\partial x_2}(0)\, H(x_2) \left(1 - \frac{x_2}{h}\right) e_1, \tag{47}$$

$$\mathscr{P}_0^\varepsilon = p^\varepsilon - p^0 + \omega^{\mathrm{bl},\varepsilon} \mu \frac{\partial v_1^0}{\partial x_2}(0). \tag{48}$$

We note that the boundary layers corresponding to Σ_2 are omitted for simplicity. Then we have the following result.

PROPOSITION 3. *Let $\mathscr{U}_0^\varepsilon$ be given by (47) and $\mathscr{P}_0^\varepsilon$ by (48). Then $\mathscr{U}_0^\varepsilon \in H^1(\Omega^\varepsilon)^2$, $\mathscr{U}_0^\varepsilon = 0$ on $\partial\Omega^\varepsilon \backslash \partial\Omega$, and* div $\mathscr{U}_0^\varepsilon = 0$ *in Ω^ε. Furthermore, we have the following estimate*

$$\left| \mu \int_{\Omega^\varepsilon} \nabla \mathscr{U}_0^\varepsilon \nabla \varphi - \int_{\Omega^\varepsilon} \mathscr{P}_0^\varepsilon \operatorname{div} \varphi + \int_{\Omega^\varepsilon} v_1^0 \frac{\partial \mathscr{U}_0^\varepsilon}{\partial x_1} \varphi + \int_{\Omega^\varepsilon} (\mathscr{U}_0^\varepsilon)_2 \frac{\partial v_1^0}{\partial x_2} \varphi_1 \right|$$
$$\leqslant \left| \mu \frac{\partial v_1^0}{\partial x_2}(0) \int_{-L}^h \omega^{\mathrm{bl},\varepsilon}(0, x_2)\, \varphi_1 \right| + \left| \mu \frac{\partial v_1^0}{\partial x_2}(0) \int_{-L}^h \omega^{\mathrm{bl},\varepsilon}(b, x_2)\, \varphi_1 \right| + \left| \int_{\Omega^\varepsilon \backslash \Omega} -\frac{p_b - p_0}{b} \varphi_1 \right| + C\varepsilon^{3/2} \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}, \qquad \forall \varphi \in \mathscr{Z}^\varepsilon. \tag{49}$$

Proof. First, we note that for $\varphi \in \mathscr{Z}^\varepsilon$ the equation (21) reads

$$\mu \int_{\Omega^\varepsilon} \nabla(u^\varepsilon - v^0) \nabla \varphi - \int_{\Omega^\varepsilon} (p^\varepsilon - p^0) \operatorname{div} \varphi + \int_\Omega v_1^0 \frac{\partial (u^\varepsilon - v^0)}{\partial x_1} \varphi + \int_\Omega u_2^\varepsilon \frac{\partial v_1^0}{\partial x_2} \varphi_1$$
$$= -\int_{\Omega^\varepsilon} ((u^\varepsilon - v^0) \nabla)(u^\varepsilon - v^0)\, \varphi + \mu \frac{\partial v_1^0}{\partial x_2}(0) \int_{\Sigma_1 \cup \Sigma_2} \varphi_1 - \int_{\Omega^\varepsilon \backslash \Omega} \nabla p^0 \varphi. \tag{50}$$

Next, the variational form of the problem (46) is

$$\int_\Omega (\mu \nabla d \nabla \varphi - g \operatorname{div} \varphi) + \int_\Omega v_1^0 \frac{\partial d}{\partial x_1} \varphi + \int_\Omega d_2 \frac{\partial v_1^0}{\partial x_2} \varphi_1$$
$$= \mu \int_{\Sigma_2} \frac{\partial d_1}{\partial x_2} \varphi_1 - \mu \int_{\Sigma_1} \frac{\partial d_1}{\partial x_2} \varphi_1 + \int_{\Sigma_1} g \varphi_2, \qquad \forall \varphi \in \mathscr{Z}^\varepsilon, \tag{51}$$

and, moreover, for $\{\beta^{\mathrm{bl},\varepsilon}, \omega^{\mathrm{bl},\varepsilon}\}$ we have

$$\int_{\Omega^\varepsilon} (\nabla(\beta^{\mathrm{bl},\varepsilon} - \varepsilon(C_1^{\mathrm{bl}}, 0)\, H(x_2))\, \nabla\varphi - \omega^{\mathrm{bl},\varepsilon} \operatorname{div} \varphi)$$

$$+\int_{\Omega} v_1^0 \frac{\partial \beta^{\mathrm{bl},\varepsilon}}{\partial x_1} \varphi + \int_{\Omega} \beta_2^{\mathrm{bl},\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1$$

$$= \int_{-L}^{h} (\omega^{\mathrm{bl},\varepsilon}(0, x_2)\, \varphi_1 - \omega^{\mathrm{bl},\varepsilon}(b, x_2)\, \varphi_1) - \int_{\Sigma} \varphi_1$$

$$+\int_{\Omega} \beta_2^{\mathrm{bl},\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1 - \int_0^h (\beta_1^{\mathrm{bl},\varepsilon} - \varepsilon C_1^{\mathrm{bl}})(0, x_2)\, v_1^0 \varphi_1$$

$$+\int_0^h (\beta_1^{\mathrm{bl},\varepsilon} - \varepsilon C_1^{\mathrm{bl}})(b, x_2)\, v_1^0 \varphi_1 - \int_{\Omega} v_1^0 (\beta^{\mathrm{bl},\varepsilon} - \varepsilon(C_1^{\mathrm{bl}}, 0)) \frac{\partial \varphi}{\partial x_1},$$

$$\forall \varphi \in \mathscr{Z}^\varepsilon. \tag{52}$$

Because of the estimates (40)–(41), we have for $\varphi \in \mathscr{Z}^\varepsilon$

$$\left| \int_{\Omega} \beta_2^{\mathrm{bl},\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1 - \int_0^h (\beta_1^{\mathrm{bl},\varepsilon} - \varepsilon C_1^{\mathrm{bl}})(0, x_2)\, v_1^0 \varphi_1 + \int_0^h (\beta_1^{\mathrm{bl},\varepsilon} - C_1^{\mathrm{bl}})(b, x_2)\, v_1^0 \varphi_1 \right|$$

$$+ \left| \int_{\Omega} v_1^0 (\beta^{\mathrm{bl},\varepsilon} - \varepsilon(C_1^{\mathrm{bl}}, 0)) \frac{\partial \varphi}{\partial x_1} \right|$$

$$\leqslant C\varepsilon^{3/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}, \tag{53}$$

$$\left| \varepsilon C_1^{\mathrm{bl}} \int_{\Sigma} \left(-\mu \frac{\partial d_1}{\partial x_2} \varphi_1 + g\varphi_2 \right) \right| \leqslant C\varepsilon^{3/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)^4}. \tag{54}$$

Furthermore, a simple interpolation argument and the estimates (20) and (32)–(36) imply

$$\left| \int_{\Omega^\varepsilon} ((u^\varepsilon - v^0)\, \nabla)(u^\varepsilon - v^0)\, \varphi \right| \leqslant C\varepsilon^{3/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}. \tag{55}$$

Now the variational equations (50)–(52), the definition of $\mathscr{U}_0^\varepsilon$ and $\mathscr{P}_0^\varepsilon$, and the estimates (53)–(55) give the estimate (49). ∎

At this stage we want to follow the ideas from [11], take $\varphi = \mathscr{U}_0^\varepsilon$ as the test function, and get the required higher order a priori estimate. Nevertheless, here we are in the presence of the physical outer boundaries and $\mathscr{U}_0^\varepsilon \notin \mathscr{Z}^\varepsilon$. At $(0, b) \times \{h\}$ the velocity field $\mathscr{U}_0^\varepsilon$ is exponentially small with respect to x_2 and we can suppose it to be zero without losing generality.

At the inflow/outflow boundaries $(\{0\} \cup \{b\}) \times (0, h)$ the situation is different. We are going to correct the values of $\mathcal{U}_0^\varepsilon$ there. For this purpose we introduce the outer boundary layer in $(0, \ell) \times (0, +\infty)$, $\ell < 1$,

$$s_1^{\mathrm{bl}}(y) = -\frac{\ell}{3}\left(1 - \frac{y_1}{\ell}\right)^3 \frac{\partial \beta_1^{\mathrm{bl}}}{\partial y_1}(0, y_2), \tag{56}$$

$$s_2^{\mathrm{bl}}(y) = \left(1 - \frac{y_1}{\ell}\right)^2 \beta_2^{\mathrm{bl}}(0, y_2), \tag{57}$$

$$\vartheta^{\mathrm{bl}}(y) = -\frac{2\mu}{\ell}\frac{\partial \beta_1^{\mathrm{bl}}}{\partial y_1}(0, y_2)\left(y_1 - \frac{y_1^2}{2\ell}\right) + \frac{\ell^2 \mu}{12}\frac{\partial^3 \beta_1^{\mathrm{bl}}}{\partial y_1\, \partial^2 y_2^2}(0, y_2)\left(\left(1 - \frac{y_1}{\ell}\right)^4 - 1\right). \tag{58}$$

Obviously,

$$\operatorname{div}_y s^{\mathrm{bl}} = 0 \qquad \text{in} \quad (0, \ell) \times (0, +\infty)$$

and

$$\operatorname{div}_y(\mu \nabla_y s_1^{\mathrm{bl}} - \vartheta^{\mathrm{bl}} e_1) = 0 \qquad \text{in} \quad (0, \ell) \times (0, +\infty), \tag{59}$$
$$s_2^{\mathrm{bl}}(0, y_2) = \beta_2^{\mathrm{bl}}(0, y_2) \qquad \text{and} \qquad \vartheta^{\mathrm{bl}}(0, y_2) = 0.$$

We make an incompressible H^2-extension of s^{bl} to a function defined on $(Y - (0, 1)) \cap]0, \ell[\times]-1, 0[$ and having the zero trace on $\partial[(Y - (0, 1)) \cap [0, \ell] \times [-1, 0]]$. Analogously, we make an H^1-extension of the pressure field ϑ^{bl}.

Then we set

$$\begin{aligned} s^\varepsilon(x) &= \varepsilon s^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right), \quad x_1 \in [0, \varepsilon\ell); \qquad s^\varepsilon(x) = 0 \quad \text{for } x_1 \in [\varepsilon\ell, b - \varepsilon\ell] \\ \vartheta^\varepsilon(x) &= \vartheta^{\mathrm{bl}}\left(\frac{x}{\varepsilon}\right), \quad x_1 \in [0, \varepsilon\ell); \qquad \vartheta^\varepsilon(x) = 0 \quad \text{for } x_1 \in [\varepsilon\ell, b - \varepsilon\ell] \end{aligned} \tag{60}$$

and analogously for $x_1 \in (b - \varepsilon\ell, b]$. Because of the symmetry, we shall systematically neglect the right lateral boundary $\{b\} \times (0, h)$ and present the calculations only for the left one, $\{0\} \times (0, h)$.

Then, for every $q \in [1, +\infty]$, we have

$$\frac{1}{\varepsilon} \|s^\varepsilon\|_{L^q(\Omega^\varepsilon)^2} + \|\nabla s^\varepsilon\|_{L^q(\Omega^\varepsilon)^4} + \|\vartheta^\varepsilon\|_{L^q(\Omega^\varepsilon)} \leqslant C\varepsilon^{2/q}$$

$$\|\Delta s^\varepsilon\|_{L^q(\Omega^\varepsilon)^2} + \|\nabla \vartheta^\varepsilon\|_{L^q(\Omega^\varepsilon)^2} \leqslant C\varepsilon^{2/q-1} \tag{61}$$

$$\left\| \frac{\partial s_1^\varepsilon}{\partial x_2} \right\|_{L^q(\Sigma_j)} \leqslant C\varepsilon^{1/q}.$$

After introducing all auxiliary functions we are in a position to prove our main result,

THEOREM 1. *Let*

$$\mathscr{U}^\varepsilon(x) = u^\varepsilon - v^0 + (\beta^{\mathrm{bl},\varepsilon} - s^\varepsilon) \frac{\partial v_1^0}{\partial x_2}(0) - \varepsilon C_1^{\mathrm{bl}} \frac{\partial v_1^0}{\partial x_2}(0)\, H(x_2) \frac{x_2}{h} e_1, \tag{62}$$

$$\mathscr{P}^\varepsilon = p^\varepsilon - p^0 + (\omega^{\mathrm{bl},\varepsilon} - \vartheta^\varepsilon)\, \mu \frac{\partial v_1^0}{\partial x_2}(0), \tag{63}$$

where $\{v^0, p^0\}$ *is defined by* (16), $\{\beta^{\mathrm{bl},\varepsilon}, \omega^{\mathrm{bl},\varepsilon}\}$ *by* (39), *and* $\{s^\varepsilon, \vartheta^\varepsilon\}$ *by* (60). *Then we have the estimates*

$$\|\nabla \mathscr{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \leqslant C\varepsilon \tag{64}$$

$$\|\mathscr{U}^\varepsilon\|_{L^2(\Omega^\varepsilon \setminus \Omega)^2} \leqslant C\varepsilon^2 \tag{65}$$

$$\frac{1}{\sqrt{\varepsilon}} \|\mathscr{U}_2^\varepsilon\|_{H^{-1/2}(\Sigma_1 \cup \Sigma_2)} + \|\mathscr{U}^\varepsilon\|_{L^2(\Sigma_1 \cup \Sigma_2)^2} \leqslant C\varepsilon^{3/2} \tag{66}$$

$$\|\mathscr{U}^\varepsilon\|_{L^2(\Omega)^2} \leqslant C\varepsilon^{3/2} \tag{67}$$

$$\|\mathscr{P}^\varepsilon\|_{L^2(\Omega)} \leqslant C\varepsilon. \tag{68}$$

Proof. In analogy with Proposition 3 and after using (40) and (41), we have

$$\left| \mu \int_{\Omega^\varepsilon} \nabla \mathscr{U}^\varepsilon \nabla \varphi - \int_{\Omega^\varepsilon} \mathscr{P}^\varepsilon \operatorname{div} \varphi + \int_\Omega v_1^0 \frac{\partial \mathscr{U}^\varepsilon}{\partial x_1} \varphi + \int_\Omega \mathscr{U}_2^\varepsilon \frac{\partial v_1^0}{\partial x_2} \varphi_1 \right|$$

$$\leqslant C\varepsilon \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4} + \mu \left| \frac{\partial v_1^0}{\partial x_2}(0) \right| \left| \int_{\Omega^\varepsilon} (\nabla s^\varepsilon \nabla \varphi - \vartheta^\varepsilon \operatorname{div} \varphi) \right|, \qquad \forall \varphi \in \mathscr{Z}^\varepsilon.$$

By the definition of $\{s^\varepsilon, \vartheta^\varepsilon\}$ and (61),

$$\left|\int_{\Omega^\varepsilon} (\nabla s^\varepsilon \nabla \varphi - \vartheta^\varepsilon \operatorname{div} \varphi)\right| \leqslant C\varepsilon \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}, \qquad \forall \varphi \in \mathscr{L}^\varepsilon.$$

Therefore, we obtain the estimate

$$\left|\mu \int_{\Omega^\varepsilon} \nabla \mathscr{U}^\varepsilon \nabla \varphi - \int_{\Omega^\varepsilon} \mathscr{P}^\varepsilon \operatorname{div} \varphi + \int_\Omega v_1^0 \frac{\partial \mathscr{U}^\varepsilon}{\partial x_1} \varphi + \int_\Omega \mathscr{U}_2^\varepsilon \frac{\partial v_1^0}{\partial x_2} \varphi_1\right| \\ \leqslant C\varepsilon \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}, \qquad \forall \varphi \in \mathscr{L}^\varepsilon. \tag{69}$$

Now let us note that $\mathscr{U}^\varepsilon \in \mathscr{L}^\varepsilon$ and $\operatorname{div} \mathscr{U}^\varepsilon = 0$. Hence it is possible to take $\varphi = \mathscr{U}^\varepsilon$. With this choice we get the estimate (64). Poincaré's inequality (17) applied to (64) gives (65). Equation (66) is a consequence of (18) and (68) follows from (64).

It remains to prove (67). We note that $\{\mathscr{U}^\varepsilon, \mathscr{P}^\varepsilon\}$ satisfies the following Oseen system in Ω,

$$\begin{aligned} -\mu \triangle \mathscr{U}^\varepsilon + \nabla \mathscr{P}^\varepsilon + v_1^0 \frac{\partial \mathscr{U}^\varepsilon}{\partial x_1} + (\mathscr{U}^\varepsilon)_2 \frac{\partial v_1^0}{\partial x_2} e_1 = G^\varepsilon &\qquad \text{in } \Omega, \\ \operatorname{div} \mathscr{U}^\varepsilon = 0 &\qquad \text{in } \Omega, \\ \mathscr{U}^\varepsilon = \xi^\varepsilon \quad \text{on } \Sigma_1 \cup \Sigma_2, \qquad \|\xi^\varepsilon\|_{L^2(\Sigma_1 \cup \Sigma_2)^2} \leqslant C\varepsilon^{3/2}&, \\ \mathscr{U}_2^\varepsilon = 0 \quad \text{and} \quad \mathscr{P}^\varepsilon = \omega^{\mathrm{bl},\varepsilon} &\qquad \text{on } (\{0\} \cup \{b\}) \times (0, h), \end{aligned} \tag{70}$$

where

$$G^\varepsilon = v_1^0 \frac{\partial}{\partial x_1} (\beta^{\mathrm{bl},\varepsilon} - s^\varepsilon) + (\beta_2^{\mathrm{bl},\varepsilon} - s_2^\varepsilon) \frac{\partial v_1^0}{\partial x_2} e_1 + e_2 (\operatorname{div}(\mu \nabla s_2^\varepsilon - \vartheta^\varepsilon e_2)). \tag{71}$$

The adjoint problem for (70) reads

$$\begin{aligned} -\mu \triangle \Phi + \nabla \eta - v_1^0 \frac{\partial \Phi}{\partial x_1} + \Phi_1 \frac{\partial v_1^0}{\partial x_2} e_2 = \bar{g} &\qquad \text{in } \Omega, \\ \operatorname{div} \Phi = z &\qquad \text{in } \Omega, \\ \Phi = 0 &\qquad \text{on } \Sigma_1 \cup \Sigma_2, \\ \Phi_2 = 0 \quad \text{and} \quad \eta - v_1^0 \Phi_1 = \mu z &\qquad \text{on } (\{0\} \cup \{b\}) \times (0, h). \end{aligned} \tag{72}$$

It is easily seen that for $\bar{g} \in L^2(\Omega)^2$ and $z \in H^1(\Omega)$ the problem (72) has a unique solution $\{\Phi, \eta\} \in H^2(\Omega)^2 \times H^1(\Omega)$, which depends continuously on the data.

Following [9] we write the very weak formulation corresponding to (70) as

$$\int_\Omega \mathscr{U}^\varepsilon \bar{g} - \langle \mathscr{P}^\varepsilon, z \rangle_\Omega$$

$$= \int_{\Sigma_1} (\mu \nabla \Phi - \eta I)\, e_2 \xi^\varepsilon - \int_{\Sigma_2} (\mu \nabla \Phi - \eta I)\, e_2 \xi^\varepsilon$$

$$+ \int_\Omega (\beta_2^{\mathrm{bl},\varepsilon} - s_2^\varepsilon) \frac{\partial v_1^0}{\partial x_2} \Phi_1 - \int_\Omega v_1^0 (\beta^{\mathrm{bl},\varepsilon} - s^\varepsilon - \varepsilon (C_1^{\mathrm{bl}}, 0)) \frac{\partial \Phi}{\partial x_1}$$

$$- \int_{\{0\} \times (0,h)} (\beta_1^{\mathrm{bl},\varepsilon} - s_1^\varepsilon - \varepsilon C_1^{\mathrm{bl}})\, v_1^0 \Phi_1 - \int_\Omega \left(\mu \nabla s_2^\varepsilon \nabla \Phi_2 - \vartheta^\varepsilon e_2 \nabla \Phi_2 \right)$$

$$\forall \bar{g} \in L^2(\Omega)^2, \quad \forall z \in H^1(\Omega). \tag{73}$$

Thus, for every $\kappa > 0$, we have obtained the estimate

$$\|\mathscr{U}^\varepsilon\|_{L^2(\Omega)^2} \leqslant C \{ \|\xi^\varepsilon\|_{L^2(\Sigma_1 \cup \Sigma_2)^2} + \|\beta^{\mathrm{bl},\varepsilon} - \varepsilon (C_1^{\mathrm{bl}}, 0)\|_{L^{1+\kappa}(\Omega)} + \|s^\varepsilon\|_{W^{1,1+\kappa}(\Omega)^2}$$

$$+ \|\vartheta^\varepsilon\|_{L^{1+\kappa}(\Omega)^2} + \|\beta_1^{\mathrm{bl},\varepsilon} - s_1^\varepsilon - \varepsilon C_1^{\mathrm{bl}}\|_{L^1(\{0\} \times (0,h))} \}$$

$$\leqslant C (\|\xi^\varepsilon\|_{L^2(\Sigma_1 \cup \Sigma_2)^2} + \varepsilon^{2-\kappa}). \tag{74}$$

Now (74) and (66) imply (67). ∎

The estimates (64)–(68) allow us to justify Navier's slip condition.

We start with a result related to the behavior of the velocity field u^ε at the interface Σ_1. Let $H_{00}^{1/2}(\Sigma_1)$ be a subspace of $L^2(\Sigma_1)$ consisting of the functions w for which there exists an element of $H^1(\Omega)$ which is zero on $\partial\Omega \backslash \Sigma_1$ and equal to w on Σ_1. Then we have the following result.

THEOREM 2. *Let u^ε be the velocity field determined in Proposition* 1 *and let the boundary layer tangential velocity at infinity C_1^{bl} be given by* (12). *Then we have*

$$\left\| u_1^\varepsilon + \varepsilon C_1^{\mathrm{bl}} \frac{\partial u_1^\varepsilon}{\partial x_2} \right\|_{(H_{00}^{1/2}(\Sigma_1))'} \leqslant C \varepsilon^{3/2}. \tag{75}$$

Proof. Using the definition of the correction $\mathscr{U}_0^\varepsilon$, we get

$$\left\| u_1^\varepsilon + \varepsilon C_1^{\mathrm{bl}} \frac{\partial u_1^\varepsilon}{\partial x_2} \right\|_{(H_{00}^{1/2}(\Sigma_1))'} \leqslant C\varepsilon^2 + \left\| \mathscr{U}_{01}^\varepsilon + \varepsilon C_1^{\mathrm{bl}} \frac{\partial \mathscr{U}_{01}^\varepsilon}{\partial x_2} \right\|_{(H_{00}^{1/2}(\Sigma_1))'}$$

$$+ C \|\beta_1^{\mathrm{bl},\varepsilon}(0,\cdot) - \varepsilon C_1^{\mathrm{bl}}\|_{(H_{00}^{1/2}(\Sigma_1))'}$$

$$+ C\varepsilon \left\| \frac{\partial \beta_1^{\mathrm{bl},\varepsilon}}{\partial x_2}(0,\cdot) \right\|_{(H_{00}^{1/2}(\Sigma_1))'}. \tag{76}$$

It should be noted that $(H_{00}^{1/2}(\Sigma_1))' = [L^2(\Sigma_1), H^{-1}(\Sigma_1)]_{1/2}$ and that $\beta_1^{\mathrm{bl},\varepsilon}(0,\cdot) - \varepsilon C_1^{\mathrm{bl}}$ and $(\partial \beta_1^{\mathrm{bl},\varepsilon}/\partial x_2)(0,\cdot)$ are ε-periodic functions with zero mean.

Consequently, by the simple duality argument we obtain

$$\| \beta_1^{\mathrm{bl},\varepsilon}(0,\cdot) - \varepsilon C_1^{\mathrm{bl}} \|_{(H_{00}^{1/2}(\Sigma_1))'} + \varepsilon \left\| \frac{\partial \beta_1^{\mathrm{bl},\varepsilon}}{\partial x_2}(0,\cdot) \right\|_{(H_{00}^{1/2}(\Sigma_1))'} \leqslant C \varepsilon^{3/2}. \tag{77}$$

It remains to estimate the first term on the right-hand side of the inequality (76). The difficulty comes from the derivative of $\mathscr{U}_{01}^{\varepsilon}$. Since we have no information on the H^2-norm of $\mathscr{U}_0^{\varepsilon}$, the only possibility is to use the generalized Green formula.

First, we have

$$\operatorname{div}(\mu \nabla \mathscr{U}_{01}^{\varepsilon} - \mathscr{P}_0^{\varepsilon} e_1) = v_1^0 \frac{\partial \mathscr{U}_{01}^{\varepsilon}}{\partial x_1} + \mathscr{U}_{02}^{\varepsilon} \frac{\partial v_1^0}{\partial x_2} + v_1^0 \frac{\partial \beta_1^{\mathrm{bl},\varepsilon}}{\partial x_1} + \beta_2^{\mathrm{bl},\varepsilon} \frac{\partial v_1^0}{\partial x_2} \quad \text{in } \Omega. \tag{78}$$

Hence

$$\|\operatorname{div}(\mu \nabla \mathscr{U}_{01}^{\varepsilon} - \mathscr{P}_0^{\varepsilon} e_1)\|_{L^2(\Omega)} \leqslant C\sqrt{\varepsilon}. \tag{79}$$

Now, by the generalized Green formula,

$$\begin{aligned} \left\| \frac{\partial \mathscr{U}_{01}^{\varepsilon}}{\partial x_2} \right\|_{(H_{00}^{1/2}(\Sigma_1))'} &\leqslant C\{ \|\mu \nabla \mathscr{U}_{01}^{\varepsilon} - \mathscr{P}_0^{\varepsilon} e_1\|_{L^2(\Omega)^2} + \|\operatorname{div}(\mu \nabla \mathscr{U}_{01}^{\varepsilon} - \mathscr{P}_0^{\varepsilon} e_1)\|_{L^2(\Omega)} \} \\ &\leqslant C\sqrt{\varepsilon}. \end{aligned} \tag{80}$$

Since (66) implies $\|\mathscr{U}_{01}^{\varepsilon}\|_{(H_{00}^{1/2}(\Sigma_1))'} \leqslant C\varepsilon^{3/2}$, after inserting (77) and (80) into (76) we obtain the estimate (75). ▮

Now we denote by $\bar{C}_1^{\mathrm{bl}}$ Navier's constant corresponding to Σ_2, and we introduce the effective flow equations in Ω through the following boundary value problem:

Find a velocity field u^{eff} and a pressure field p^{eff} such that

$$-\mu \triangle u^{\mathrm{eff}} + (u^{\mathrm{eff}} \nabla) u^{\mathrm{eff}} + \nabla p^{\mathrm{eff}} = 0 \quad \text{in } \Omega, \tag{81}$$

$$\operatorname{div} u^{\mathrm{eff}} = 0 \quad \text{in } \Omega, \tag{82}$$

$$u_2^{\mathrm{eff}} = 0 \quad \text{on } (\{0\} \cup \{b\}) \times (0,h), \tag{83}$$

$$p^{\varepsilon} = p_0 \quad \text{on } \{0\} \times (0,h) \qquad \text{and} \qquad p^{\varepsilon} = p_b \quad \text{on } \{b\} \times (0,h), \tag{84}$$

$$
\begin{aligned}
&u_2^{\text{eff}}=0 \quad \text{and} \quad u_1^{\text{eff}}+\varepsilon C_1^{\text{bl}}\frac{\partial u_1^{\text{eff}}}{\partial x_2}=0 \qquad \text{on} \quad \Sigma_1 \\
&u_2^{\text{eff}}=0 \quad \text{and} \quad u_1^{\text{eff}}-\varepsilon \bar{C}_1^{\text{bl}}\frac{\partial u_1^{\text{eff}}}{\partial x_2}=0 \qquad \text{on} \quad \Sigma_2.
\end{aligned}
\tag{85}
$$

Under the assumptions of Proposition 1, the problem (81)–(85) has a unique solution,

$$
\begin{aligned}
&u^{\text{eff}}=\left(\frac{p_b-p_0}{2b\mu}\,(x_2^2-h(x_2-\varepsilon C_1^{\text{bl}})\,\frac{h-2\varepsilon\bar{C}_1^{\text{bl}}}{h-\varepsilon(C_1^{\text{bl}}+\bar{C}_1^{\text{bl}})}\right),0\right) && \text{for} \quad 0\leqslant x_2\leqslant h; \\
&p^{\text{eff}}=p^0=\frac{p_b-p_0}{b}\,x_1+p_0 && \text{for} \quad 0\leqslant x_1\leqslant b.
\end{aligned}
\tag{86}
$$

The effective mass flow rate through the channel is then

$$
M^{\text{eff}}=b\int_0^h u_1^{\text{eff}}(x_2)\,dx_2=-\frac{p_b-p_0}{12\mu}\,h^3\,\frac{h-4\varepsilon(C_1^{\text{bl}}+\bar{C}_1^{\text{bl}})+12\varepsilon^2C_1^{\text{bl}}\bar{C}_1^{\text{bl}}/h}{h-2\varepsilon(C_1^{\text{bl}}+\bar{C}_1^{\text{bl}})}\,,
\tag{87}
$$

where C_1^{bl} and $\bar{C}_1^{\text{bl}}$ are strictly negative.

Let us estimate the error made when replacing $\{u^\varepsilon, p^\varepsilon, M^\varepsilon\}$ by $\{u^{\text{eff}}, p^{\text{eff}}, M^{\text{eff}}\}$. We have

PROPOSITION 4. *Under the assumptions of Proposition* 1 *we have*

$$
\|\nabla(u^\varepsilon-u^{\text{eff}})\|_{L^1(\Omega)^4}\leqslant C\varepsilon,
\tag{88}
$$

$$
\|u^\varepsilon-u^{\text{eff}}\|_{L^2(\Omega)^2}\leqslant C\varepsilon^{3/2-\gamma}, \qquad \forall\gamma>0,
\tag{89}
$$

$$
|M^\varepsilon-M^{\text{eff}}|\leqslant C\varepsilon^{3/2}.
\tag{90}
$$

Proof. After also taking into account Σ_2, we have

$$
\begin{aligned}
u^\varepsilon-u^{\text{eff}}&=\mathscr{U}^\varepsilon+v^0-u^{\text{eff}}-\varepsilon C_1^{\text{bl}}\frac{\partial v_1^0}{\partial x_2}(0)\left(1-\frac{x_2}{h}\right)e_1+\varepsilon\bar{C}_1^{\text{bl}}\frac{\partial v_1^0}{\partial x_2}(h)\,\frac{x_2}{h}\,e_1 \\
&\quad-(\beta^{\text{bl},\varepsilon}-s^\varepsilon-\varepsilon C_1^{\text{bl}}e_1)\frac{\partial v_1^0}{\partial x_2}(0)+(\bar{\beta}^{\text{bl},\varepsilon}-\bar{s}^\varepsilon-\varepsilon\bar{C}_1^{\text{bl}}e_1)\frac{\partial v_1^0}{\partial x_2}(h) \\
&\quad\text{in } \Omega.
\end{aligned}
\tag{91}
$$

After a simple calculation we find the identity

$$v^0 - u^{\mathrm{eff}} - \varepsilon C_1^{\mathrm{bl}} \frac{\partial v_1^0}{\partial x_2}(0)\left(1 - \frac{x_2}{h}\right) e_1 + \varepsilon \bar{C}_1^{\mathrm{bl}} \frac{\partial v_1^0}{\partial x_2}(h) \frac{x_2}{h} e_1$$

$$= \frac{p_b - p_0}{2b\mu} (\bar{C}_1^{\mathrm{bl}} - C_1^{\mathrm{bl}}) \frac{\varepsilon C_1^{\mathrm{bl}} h - x_2(C_1^{\mathrm{bl}} + \bar{C}_1^{\mathrm{bl}})}{h - \varepsilon(C_1^{\mathrm{bl}} + \bar{C}_1^{\mathrm{bl}})} \varepsilon^2. \tag{92}$$

Now (88) follows from (61) and (92). Equation (89) follows from the theory of very weak solutions for the Oseen system (73) and from the estimates on $\mathcal{U}^\varepsilon$, $\beta^{\mathrm{bl},\varepsilon}$, and s^ε.

It remains to prove the estimate (90). Using (91), (92), (40), (41), and (61), we get the simple estimate

$$|M^\varepsilon - M^{\mathrm{eff}}| \leqslant \left| \int_0^h \mathcal{U}_1^\varepsilon(0, x_2)\, dx_2 \right| + C\varepsilon^2. \tag{93}$$

Now let $0 < m < b$ and $\varphi = \varphi(x_1) \in C^1[0, m]$, $\varphi(0) = 1$, and $\varphi(m) = 0$. Then we have

$$-\mathcal{U}_1^\varepsilon(0, x_2) = \int_0^m \frac{\partial}{\partial \eta} (\mathcal{U}_1^\varepsilon(\eta, x_2)\, \varphi(\eta))\, d\eta = \int_0^m \left(\frac{\partial \mathcal{U}_1^\varepsilon}{\partial \eta} \varphi + \mathcal{U}_1^\varepsilon \frac{d\varphi}{d\eta} \right) d\eta$$

$$= \int_0^m \left(-\frac{\partial}{\partial x_2} (\mathcal{U}_2^\varepsilon \varphi) + \mathcal{U}_1^\varepsilon \frac{d\varphi}{d\eta} \right) d\eta, \tag{94}$$

and (90) follows from (66), (67) and (93), (94). ∎

Our next step is to calculate the *tangential drag force* or the *skin friction*

$$\mathcal{F}_{\mathrm{t}}^\varepsilon = \int_0^b \mu \frac{\partial u_1^\varepsilon}{\partial x_2}(x_1, 0)\, dx_1. \tag{95}$$

THEOREM 3. *Let the skin friction $\mathcal{F}_t^\varepsilon$ be defined by (95). Then we have*

$$\left| \mathcal{F}_t^\varepsilon + \frac{(p_b - p_0)\, h}{2} \left(1 + \varepsilon \frac{C_1^{\mathrm{bl}} - \bar{C}_1^{\mathrm{bl}}}{h} \right) \right| \leqslant C\varepsilon^{3/2}. \tag{96}$$

Proof. Let $0<m<h$ and let $\varphi=\varphi(x_2)\in C^2[0,m]$, $\varphi(0)=1$, and $\varphi(m)=\varphi'(m)=0$. Let $\mathcal{Q}^\varepsilon=\mathcal{P}^\varepsilon-\mu(\partial v_1^0/\partial x_2)(0)\,\omega^{\mathrm{bl},\varepsilon}(0,x_2)$. Then we have

$$\left|\mu\int_0^b\frac{\partial\mathcal{U}_1^\varepsilon}{\partial x_2}(x_1,0)\,dx_1\right|=\left|\int_\Omega \operatorname{div}((\mu\,\nabla\mathcal{U}_1^\varepsilon-\mathcal{Q}^\varepsilon e_1)\,\varphi)\right|$$
$$\leqslant\left|\int_\Omega\mu\frac{\partial\mathcal{U}_1^\varepsilon}{\partial x_2}\frac{d\varphi}{dx_2}\right|+\left|\int_\Omega \operatorname{div}(\mu\,\nabla\mathcal{U}_1^\varepsilon-\mathcal{Q}^\varepsilon e_1)\,\varphi\right|. \tag{97}$$

First, we estimate the second term on the left-hand side of (97). Let $w^\varepsilon=u^\varepsilon-v^0$. Then we have

$$\left|\int_\Omega \operatorname{div}(\mu\,\nabla\mathcal{U}_1^\varepsilon-\mathcal{Q}^\varepsilon e_1)\,\varphi\right|$$
$$=\left|\int_\Omega (u^\varepsilon\nabla)\,u_1^\varepsilon\varphi\right|$$
$$=\left|\int_\Omega\left(w_1^\varepsilon\frac{\partial w_1^\varepsilon}{\partial x_1}+v_1^0\frac{\partial w_1^\varepsilon}{\partial x_1}+w_2^\varepsilon\frac{\partial w_1^\varepsilon}{\partial x_2}+w_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\right)\varphi\right|$$
$$\leqslant\left|\int_\Omega\left(w_1^\varepsilon\frac{\partial w_1^\varepsilon}{\partial x_1}+w_2^\varepsilon\frac{\partial w_1^\varepsilon}{\partial x_2}\right)\varphi\right|+\left|\int_\Omega\left(\frac{\partial}{\partial x_2}(v_1^0w_2^\varepsilon)\right)\varphi\right|$$
$$+2\left|\int_\Omega\mathcal{U}_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\varphi\right|+2\mu\left|\frac{\partial v_1^0}{\partial x_2}(0)\right|\left|\int_\Omega\frac{\partial v_1^0}{\partial x_2}\varphi(s_2^\varepsilon-\beta_2^\varepsilon)\right|$$
$$\leqslant C\varepsilon^{3/2}+\left|\int_\Omega\left(\frac{\partial}{\partial x_2}(v_1^0w_2^\varepsilon)\right)\varphi\right|. \tag{98}$$

Using that $\int_{\Sigma_j}\beta_2^{\mathrm{bl},\varepsilon}(x_1,0)\,dx_1=0$, (98) implies

$$\left|\int_\Omega \operatorname{div}(\mu\,\nabla\mathcal{U}_1^\varepsilon-\mathcal{Q}^\varepsilon e_1)\,\varphi\right|$$
$$\leqslant C\varepsilon^{3/2}+\left|\int_{\Sigma_1}v_1^0\mathcal{U}_2^\varepsilon\right|+\left|\int_\Omega v_1^0\frac{d\varphi}{dx_2}\,\mathcal{U}_2^\varepsilon\right|$$
$$+\mu\left|\frac{\partial v_1^0}{\partial x_2}(0)\right|\left|\left(-\int_{\Sigma_1}v_1^0s_2^\varepsilon+\int_\Omega v_1^0\frac{d\varphi}{dx_2}(\beta_2^{\mathrm{bl},\varepsilon}-s_2^\varepsilon)\right)\right|\leqslant C\varepsilon^{3/2}. \tag{99}$$

Estimating for the first term on the left-hand side of (97) is much simpler. We have

$$\left|\int_{\Omega} \frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_2} \frac{d\varphi}{dx_2}\right| \leqslant \left|\int_{\Sigma_1} \mathcal{U}_1^\varepsilon \frac{d\varphi}{dx_2}(0)\right| + \left|\int_{\Omega} \mathcal{U}_1^\varepsilon \frac{d^2\varphi}{dx_2^2}\right| \leqslant C\varepsilon^{3/2}. \tag{100}$$

After inserting (99) and (100) into (97), we obtain

$$\left|\mu \int_0^b \frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_2}(x_1, 0)\, dx_1\right| \leqslant C\varepsilon^{3/2}. \tag{101}$$

Now we note that $\int_\Sigma (\partial \beta_1^{\mathrm{bl}, \varepsilon}/\partial x_2)(x_1, 0)\, dx_1 = 0$, and at $x_2 = b$ the first component of the boundary layer function is $-s_1^\varepsilon(b - x_1, x_2)$. Consequently, the corresponding terms do not contribute to the tangential drag and (101) implies (96). ∎

Corollary 1. *Let*

$$\mathcal{F}_{\mathrm{t}}^{\mathrm{eff}} = -\frac{h(p_b - p_0)}{2} \frac{h - 2\varepsilon \bar{C}_1^{\mathrm{bl}}}{h - \varepsilon(C_1^{\mathrm{bl}} + \bar{C}_1^{\mathrm{bl}})}$$

be the tangential drag force corresponding to the effective velocity u^{eff}. *Then we have*

$$|\mathcal{F}_{\mathrm{t}}^{\mathrm{eff}} - \mathcal{F}_{\mathrm{t}}^{\varepsilon}| \leqslant C\varepsilon^{3/2}. \tag{102}$$

Remark 1. Let $\Omega_{a\varepsilon} = (0, b) \times (a\varepsilon, h - a\varepsilon)$ for $a > 0$ and let $\{u^{a,\mathrm{eff}}, p^{a,\mathrm{eff}}\}$ be a solution for (81)–(85) in $\Omega_{a\varepsilon}$, with (85) replaced by

$$\begin{aligned} u_2^{a,\mathrm{eff}} = 0 \quad &\text{and} \quad u_1^{a,\mathrm{eff}} + \varepsilon C_1^{a,\mathrm{bl}} \frac{\partial u_1^{a,\mathrm{eff}}}{\partial x_2} = 0 \qquad \text{on } \Sigma_{a,1} \\ u_2^{a,\mathrm{eff}} = 0 \quad &\text{and} \quad u_1^{a,\mathrm{eff}} - \varepsilon \bar{C}_1^{a,\mathrm{bl}} \frac{\partial u_1^{a,\mathrm{eff}}}{\partial x_2} = 0 \qquad \text{on } \Sigma_{a,2}. \end{aligned} \tag{103}$$

Under the assumptions of Proposition 1, the unique solution $\{u^{a,\mathrm{eff}}, p^{a,\mathrm{eff}}\}$ is given by

$$\begin{aligned} u^{a,\mathrm{eff}} = \Bigg(\frac{p_b - p_0}{2b\mu}\Bigg(&(x_2 - a\varepsilon)^2 - (h - 2a\varepsilon)(x_2 - a\varepsilon - \varepsilon C_1^{a,\mathrm{bl}}) \\ &\times \frac{h - 2a\varepsilon - 2\varepsilon \bar{C}_1^{a,\mathrm{bl}}}{h - 2a\varepsilon - \varepsilon(C_1^{a,\mathrm{bl}} + \bar{C}_1^{a,\mathrm{bl}})}\Bigg), 0\Bigg) \end{aligned}$$

for $a\varepsilon \leqslant x_2 \leqslant h - a\varepsilon$, and

$$p^{a,\,\mathrm{eff}} = p^0 = \frac{p_b - p_0}{b} x_1 + p_0 \qquad \text{for} \quad 0 \leqslant x_1 \leqslant b.$$

By Lemma 2, $C_1^{a,\,\mathrm{bl}} = C_1^{\mathrm{bl}} - a$ and

$$u^{a,\,\mathrm{eff}}(x) = u^{\mathrm{eff}}(x) - \left(2\,\frac{x_2(C_1^{\mathrm{bl}} - \bar{C}_1^{\mathrm{bl}}) + C_1^{\mathrm{bl}}(h - 2\varepsilon \bar{C}_1^{\mathrm{bl}})}{h - \varepsilon(C_1^{\mathrm{bl}} + \bar{C}_1^{\mathrm{bl}})} - a \right) \frac{a\varepsilon^2 (p_b - p_0)}{2b\mu}\, e_1.$$

Therefore, a perturbation of the interface position of order $O(\varepsilon)$ implies a perturbation in the solution of $O(\varepsilon^2)$. Consequently, there is freedom in fixing the position of Σ_j. This influences the result only at the next order of the asymptotic expansion.

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