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Partial cubes are distance graphs☆

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Abstract

Chatrand et al. [Graph similarity and distance in graphs, Aequationes Math. 55 (1998) 129–145] have recently conjectured that all bipartite graphs are distance graphs. Here we show that all graphs of a large subclass of bipartite graphs, i.e. partial cubes, are distance graphs.

MSC: 05C75; 05C12

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1. Introduction

If two graphs G_1 and G_2 are isomorphic, then there exists a one-to-one mapping $\phi : V(G_1) \to V(G_2)$ with the property that vertices u and v are adjacent in G_1 if and only if ϕu and ϕv are adjacent in G_2. Of course, ϕ is an isomorphism. In fact, if G_1 and G_2 are connected, then ϕ preserves the distance between every pair of vertices of G_1 (not only pairs of adjacent vertices), that is, if u and v are any two vertices of G_1, then $d_{G_1}(u, v) = d_{G_2}(\phi u, \phi v)$.

Let G_1 and G_2 be connected graphs of order n. There are $n!$ one-to-one mappings from $V(G_1)$ to $V(G_2)$. If G_1 and G_2 are isomorphic, then the number of isomorphisms among these $n!$ mappings is the order of the automorphism group Aut G_1 of G_1. For a one-to-one mapping ϕ and each pair u, v of vertices of G_1 it is of interest to compare $d_{G_1}(u, v)$ with $d_{G_2}(\phi u, \phi v)$. For this reason, we define the ϕ-distance between G_1 and G_2 as

$$d_\phi(G_1, G_2) = \sum |d_{G_1}(u, v) - d_{G_2}(\phi u, \phi v)|, \tag{1}$$

where the sum is taken over all $\binom{n}{2}$ unordered pairs u, v of vertices of G_1. Of course, if $d_\phi(G_1, G_2) = 0$ then ϕ is an isomorphism and $G_1 \cong G_2$, while if $G_1 \ncong G_2$, then $d_\phi(G_1, G_2) > 0$ for every one-to-one mapping ϕ. This suggests defining the distance $d(G_1, G_2)$ between G_1 and G_2 by

$$d(G_1, G_2) = \min\{d_\phi(G_1, G_2)\}, \tag{2}$$

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where the minimum is taken over all one-to-one mappings ϕ from $V(G_1)$ to $V(G_2)$. Thus, $d(G_1, G_2) = 0$ if and only if $G_1 \cong G_2$. Hence $d(G_1, G_2)$ can be interpreted as a measure of the similarity of G_1 and G_2, where then the smaller the value of $d(G_1, G_2)$, the more similar the structure of G_1 is to that of G_2.

This distance defined on the space of all connected graphs of a fixed order produces a metric space.

Let S be a set of connected graphs having the same order. Then the distance graph $D(S)$ of S has vertex set S and two vertices G_1 and G_2 of $D(S)$ are adjacent if and only if $d(G_1, G_2) = 1$. Further, we say that a graph G is a *distance graph* if there exists a set S of graphs having fixed order such that $D(S) \cong G$.

In [1] it has been conjectured:

Conjecture 1. A graph G is a distance graph if and only if G is bipartite.

The conjecture is based on the fact that every distance graph is bipartite, and that several classes of bipartite graphs are shown to be distance graphs, for example every even cycle is a distance graph, every tree is a distance graph, the graph $K_{2,n}$ is a distance graph for every positive integer n, and the graph $K_{3,3}$ is a distance graph [1].

The hypercube, Q_n, is defined recursively by $Q_1 = K_2$ and $Q_n = Q_{n-1} \Box K_2$, i.e. Q_n is a Cartesian product of n copies of K_2. The vertex set of the hypercube is $V(Q_n) = \{(x_1, x_2, \ldots, x_n) \mid x_i = 0 \text{ or } 1\}$, and two vertices are connected if they differ in exactly one coordinate. If $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$, then $d_{Q_n}(x, y) = \sum |x_i - y_i|$, where the sum is taken from 1 to n.

In this note we first show that hypercubes are distance graphs and consequently partial cubes are distance graphs. (Partial cubes are induced subgraphs of hypercubes [3].) We also show some other distance graphs.

2. Partial cubes are distance graphs

We will show that hypercubes are distance graphs. To this aim we first recall some results from [1].

Theorem 2 (*Chartrand et al. [1]*). *Let G_1 and G_2 be two connected graphs of the same order having sizes p_1 and p_2, respectively, such that the size of a greatest common subgraph is s. Then*

$$d(G_1, G_2) \geqslant p_1 + p_2 - 2s. \tag{3}$$

Corollary 3 (*Chartrand et al. [1]*). *If G_1 and G_2 are connected graphs of the same order having sizes p_1 and p_2, respectively, such that $d(G_1, G_2) = 1$, then $|p_1 - p_2| = 1$ and one of G_1 and G_2 is a subgraph of the other.*

Theorem 4 (*Chartrand et al. [1]*). *Let G_1 and G_2 be connected graphs of the same order having sizes p_1 and p_2, respectively, with $p_1 \leqslant p_2$. Then $d(G_1, G_2) = 1$ if and only if $G_1 \subseteq G_2$, $p_2 = p_1 + 1$, and there exists a one-to-one mapping $\phi : V(G_1) \to V(G_2)$ such that for some 2-element subset $\{x, y\}$, it follows that $xy \notin E(G_1)$, $\phi x \phi y \in E(G_2)$, $d_{G_1}(x, y) = 2$, and if $\{u, v\} \neq \{x, y\}$, then $d_{G_1}(u, v) = d_{G_2}(\phi u, \phi v)$.*

Corollary 5 (*Chartrand et al. [1]*). *Let G_1 and G_2 be connected graphs of the same order having sizes p_1 and p_2, respectively, such that diam $G_1 = 2$ and $p_2 \geqslant p_1$. Then $d(G_1, G_2) = 1$ if and only if $G_1 \subseteq G_2$ and $p_2 = p_1 + 1$.*

Let $G_{m_1 m_2 \ldots m_n}$ ($m_i \in \{0, 1\}$) be a graph, depicted in Fig. 1, for $n = 6$.

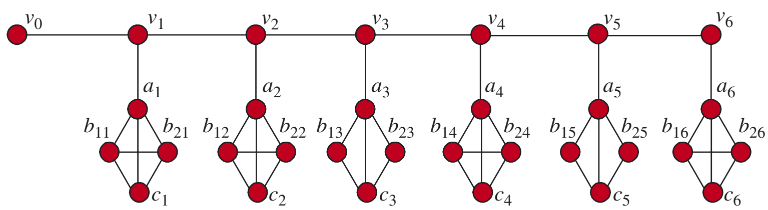

Fig. 1. The graph G_{110101}.

Theorem 6. *Let $m_1, m_2, \ldots, m_n \in \{0, 1\}$ and $m'_1, m'_2, \ldots, m'_n \in \{0, 1\}$. Graphs $G_{m_1 m_2 \cdots m_n}$ and $G_{m'_1 m'_2 \cdots m'_n}$ are isomorphic if and only if $m_i = m'_i$ for $i = 1, 2, \ldots, n$.*

Proof. Suppose that graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ are isomorphic. Any isomorphism φ from $V(G_{m_1 m_2 \ldots m_n})$ onto $V(G_{m'_1 m'_2 \ldots m'_n})$ maps v_0 into v'_0, because v_0 is the only vertex which has degree 1. Since φ preserves adjacency and nonadjacency, it follows that φ maps v_1 into v'_1 (only v_1 is adjacent to v_0), v_2 into v'_2 (from all vertices which are adjacent to v_1 only v_2 has degree 3), v_3 into v'_3 (from all vertices which are adjacent to v_2 only v_1 and v_3 have degree 3, so φ maps $\{v_1, v_3\}$ onto $\{v'_1, v'_3\}$; we have seen that φ maps v_1 into v'_1, therefore φ maps v_3 into v'_3), $\ldots$, v_n into v'_n (from all vertices which are adjacent to v_{n-1} only v_n has degree 2).

Let $i \in \{1, 2, \ldots, n\}$. Among all vertices which are adjacent to v_i only a_i has degree 4, so φ maps a_i into a'_i. The vertices b_{1i}, b_{2i} and c_i are adjacent to a_i, so φ maps $\{b_{1i}, b_{2i}, c_i\}$ onto $\{b'_{1i}, b'_{2i}, c'_i\}$.

1. If $m_i = 0$, then b_{1i} and b_{2i} have degree 2. Since c'_i has degree 3, it follows that φ maps $\{b_{1i}, b_{2i}\}$ onto $\{b'_{1i}, b'_{2i}\}$. Therefore b'_{1i} and b'_{2i} have degree 2, so $m'_i = 0$.
2. If $m_i = 1$, then b_{1i}, b_{2i} and c_i have degree 3. It follows that φ maps $\{b_{1i}, b_{2i}, c_i\}$ onto $\{b'_{1i}, b'_{2i}, c'_i\}$. Therefore b'_{1i} and b'_{2i} have degree 3, so $m'_i = 1$.

Hence $m_i = m'_i$ for $i = 1, 2, \ldots, n$.

For the converse we assume that $m_i = m'_i$ for $i = 1, 2, \ldots, n$. Then it is easy to see that graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ are isomorphic. □

Theorem 7. *Let $m_1, m_2, \ldots, m_n \in \{0, 1\}$ and $m'_1, m'_2, \ldots, m'_n \in \{0, 1\}$. Then $d(G_{m_1 m_2 \ldots m_n}, G_{m'_1 m'_2 \ldots m'_n}) = 1$ if and only if the corresponding tuples differ in precisely one position.*

Proof. Let $p_1 = |E(G_{m_1 m_2 \ldots m_n})|$ and $p_2 = |E(G_{m'_1 m'_2 \ldots m'_n})|$.

Suppose that $d(G_{m_1 m_2 \ldots m_n}, G_{m'_1 m'_2 \ldots m'_n}) = 1$. By Corollary 3, $|p_1 - p_2| = 1$ and one of the graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ is a subgraph of the other. We may, without loss of generality, assume that $p_2 = p_1 + 1$. In this case the graph $G_{m_1 m_2 \ldots m_n}$ is a subgraph of $G_{m'_1 m'_2 \ldots m'_n}$. In other words, we can get $G_{m'_1 m'_2 \ldots m'_n}$ from $G_{m_1 m_2 \ldots m_n}$ by adding one edge. By definition of these graphs, the corresponding tuples differ in precisely one place.

For the converse we suppose that corresponding tuples of graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ differ in precisely one place. Without loss of generality, we may assume that there exists such $j \in \{1, 2, \ldots, n\}$ that $m_i = m'_i$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$ and $m_j = 0$, $m'_j = 1$. Then graph $G_{m_1 m_2 \ldots m_n}$ is a subgraph of $G_{m'_1 m'_2 \ldots m'_n}$, $p_2 = p_1 + 1$ and the identity mapping ϕ, which maps vertices from $V(G_{m_1 m_2 \ldots m_n})$ onto corresponding vertices in $V(G_{m'_1 m'_2 \ldots m'_n})$ has the following properties: (1) $b_{1j} b_{2j} \notin E(G_{m_1 m_2 \ldots m_n})$ and $\phi b_{1j} \phi b_{2j} \in E(G_{m'_1 m'_2 \ldots m'_n})$, (2) $d_{G_{m_1 m_2 \ldots m_n}}(b_{1j}, b_{2j}) = 2$, (3) if $\{u, v\} \neq \{b_{1j}, b_{2j}\}$ then $d_{G_{m_1 m_2 \ldots m_n}}(u, v) = d_{G_{m'_1 m'_2 \ldots m'_n}}(\phi u, \phi v)$. By Theorem 4, $d(G_{m_1 m_2 \ldots m_n}, G_{m'_1 m'_2 \ldots m'_n}) = 1$. □

Theorem 8. *Every hypercube is a distance graph.*

Proof. By Theorem 7 and the definition of hypercubes, Q_n is the distance graph of the collection $\{G_{m_1 m_2 \ldots m_n} \mid m_i \in \{0, 1\}\}$. □

Theorem 9. *Every induced subgraph of a hypercube is a distance graph.*

Proof. Let G be an induced subgraph of a hypercube Q_n ($n \in \mathbb{N}$) and let u and v be arbitrary vertices of the graph G.

If vertices u and v are adjacent in G then u and v are also adjacent in the hypercube Q_n, because G is a subgraph of Q_n. Therefore the distance between the corresponding graphs from the proof of Theorem 8 is 1.

If the vertices u and v are not adjacent in G then u and v are also not adjacent in the hypercube Q_n, because G is an induced subgraph of Q_n. Therefore the distance between the corresponding graphs from the proof of Theorem 8 is more than 1.

Thus the subset of corresponding graphs from the proof of Theorem 8 has distance graph isomorphic to G. □

An isometric embedding of G in H is a map $f : V(G) \to V(H)$ which preserves distances:

$$d_H(f(u), f(v)) = d_G(u, v) \tag{4}$$

for any $u, v \in V(G)$. We say that a graph G can be isometrically embedded into a graph H, if there exists an isometric embedding of G in H.

Theorem 10. *Every graph which can be isometrically embedded into a hypercube is a distance graph.*

Proof. Let G be a graph which can be isometrically embedded in a hypercube Q_n ($n \in \mathbb{N}$). Then G is isomorphic to an isometric subgraph of Q_n. Every isometric subgraph is also an induced subgraph, therefore G is isomorphic to an induced subgraph of Q_n. By Theorem 9, G is a distance graph. □

We say e is in relation Θ to f, if

$$d(x, u) + d(y, v) \neq d(x, v) + d(y, u), \tag{5}$$

where $e = xy$ and $f = uv$ are two edges of connected graph G.

Using results [2,3] on sufficient conditions for the existence of isometric embeddings into hypercubes Theorem 10 has the following three corollaries:

Corollary 11. *Every bipartite graph for which relation Θ is transitive ($\Theta^* = \Theta$) is a distance graph.*

Corollary 12. *Let G be a bipartite graph such that for any edge vw of G, the set of vertices that are closer to v than w is closed under taking shortest paths. Then G is a distance graph.*

Let G and H be the bipartite graphs, depicted in Fig. 2.

Fig. 3 shows that for any edge vw of G, the set of vertices that are closer to v than w is closed under taking shortest paths. Therefore G is a distance graph.

Fig. 4 shows that there exists such edge vw of H, that the set of vertices that are closer to v than w is not closed under taking shortest paths. Namely, one of the two shortest paths between x and y meets both sets. Therefore the Corollary 12 does not show whether H is a distance graph or not.

Corollary 13. *Let G be a bipartite graph and for any edge xy of G, if $a, b, c \in V(G)$ such that $d(a, x) < d(a, y)$, $d(b, x) < d(b, y)$ and $d(a, b) = d(a, c) + d(b, c)$ then $d(c, x) < d(c, y)$. It follows that G is a distance graph.*

3. Some more results

At first sight one might guess that a subgraph of a hypercube is a distance graph if and only if it is an isometric subgraph of a hypercube. The next example shows that it is not true.

Let G be a graph, depicted in Fig. 5.

For $i \in \{1, 2, \ldots, 8\}$, consider the graphs $F_i = K_6 - E(H_i)$, where each H_i is shown in Fig. 6. Since the diameter of each graph in $S = \{F_1, F_2, \ldots, F_8\}$ is 2, Corollary 5 implies that $D(S) \cong G$.

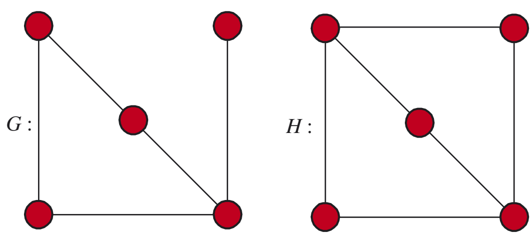

Fig. 2. The graphs G and H.

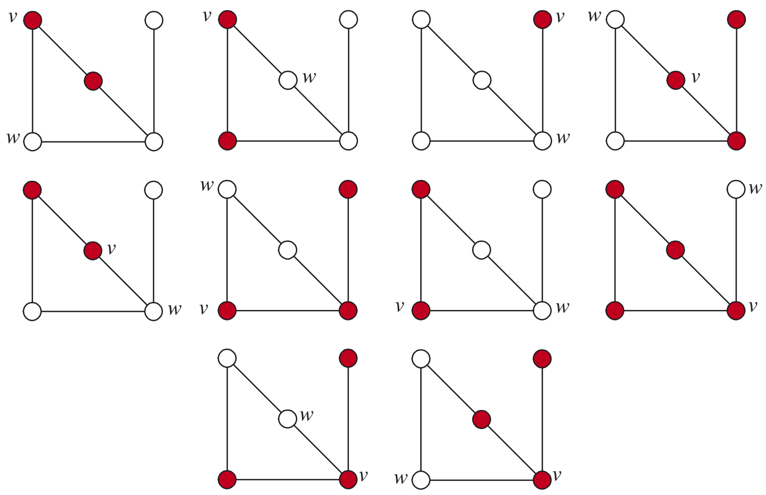

Fig. 3. The graph G.

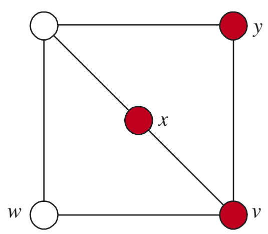

Fig. 4. The graph H.

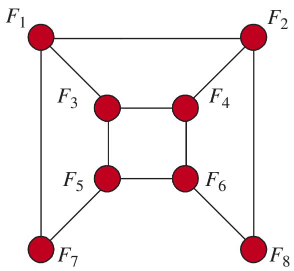

Fig. 5. The graph G.

We have checked all non-isometric subgraphs of the 3-cube and we found that they are all distance graphs. It seems reasonable to work on the following:

Problem 14. *Prove* (*or disprove*) *that every subgraph of a hypercube is a distance graph.*

We have also considered some complete bipartite graphs. For example, it is shown in [1] that:

1. the graph $K_{1,n}$ is a distance graph for every positive integer n (because every tree is a distance graph),
2. the graph $K_{2,n}$ is a distance graph for every positive integer n,
3. $K_{3,3}$ is a distance graph.

The following examples show that $K_{3,4}$ and $K_{3,5}$ are distance graphs.

For $i = 1, 2, \ldots, 7$, let $U_i = K_6 - E(L_i)$, where graphs L_i are given in Fig. 7. Since the diameter of each graph in $S = \{U_1, U_2, \ldots, U_7\}$ is 2, Corollary 5 implies that $D(S) \cong K_{3,4}$, where the bipartite sets of $D(S)$ are $\{U_1, U_2, U_3\}$ and $\{U_4, U_5, U_6, U_7\}$.

For $i = 1, 2, \ldots, 8$, let $R_i = K_9 - E(Z_i)$, where graphs Z_i are given in Fig. 8. Since the diameter of each graph in $S = \{R_1, R_2, \ldots, R_8\}$ is 2, Corollary 5 implies that $D(S) \cong K_{3,5}$, where the bipartite sets of $D(S)$ are $\{R_1, R_2, R_3\}$ and $\{R_4, R_5, R_6, R_7, R_8\}$.

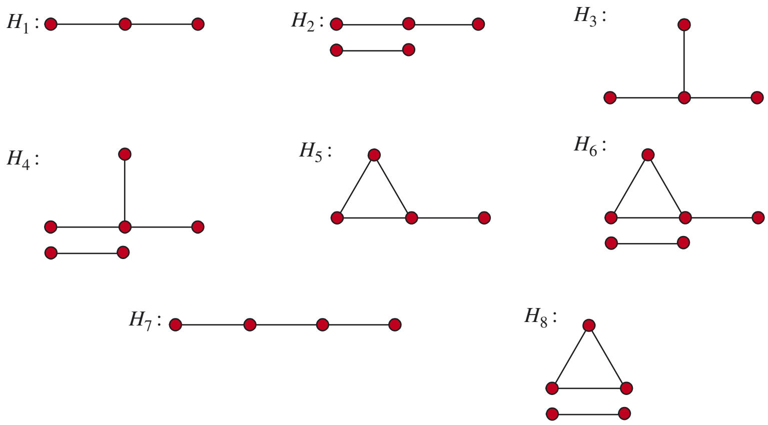

Fig. 6. Graphs H_i $(i = 1, 2, \ldots, 8)$.

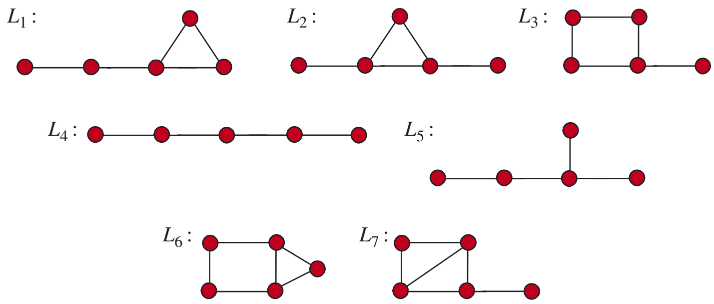

Fig. 7. Graphs L_i $(i = 1, 2, \ldots, 7)$.

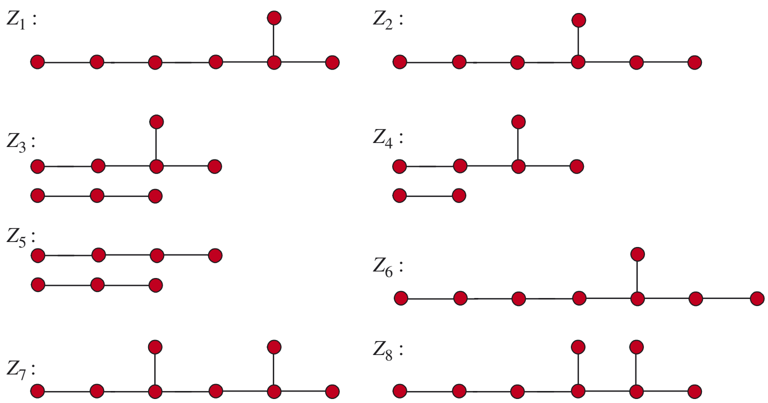

Fig. 8. Graphs Z_i $(i = 1, 2, \ldots, 8)$.

These examples motivate our next working problem:

Problem 15. *Prove that the graph $K_{3,n}$ is a distance graph for every positive integer n.*

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