Revisiting the foundations of Barbilian’s metrization procedure

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In the present work we prove that one of Barbilian’s theorems from 1960 regarding the metrization procedure in the plane admits a natural extension depending on a bilinear form and the relative position of two Apollonian hyperspheres. This result allows us to pursue two fundamental ideas. First, that all the distances with constant curvature can be described by Barbilian’s metrization principle. Secondly, that all the Riemannian metric corresponding to these distances can be obtained with the same unique procedure derived from the main theorem in the text (Theorem 2.5). We show how the hyperbolic metric of the disk, the hyperbolic metric on the exterior of the disk and the hyperbolic metric on the half-plane can be obtained in the same way using Theorem 2.5, which appears here for the first time and is an extension of a Barbilian classical result (Barbilian, 1960 [7]). Furthermore, we obtain metrics corresponding to quadratic forms with signature that includes minus. By considering the norms provided by either Lorentz or Minkowski (pseudo-)inner product as influence functions, two oscillant distances can be generated in some subsets of Lorentz or Minkowski plane. The extension of 1960 Barbilian’s theorem mentioned above allow us to obtain the metrics attached to these two Barbilian distances on corresponding subsets of Lorentz and Minkowski 2-dimensional spaces. The geometric study concludes that these metrics are generalized Lagrange metrics. A result concerning the distance induced by a Riemannian metric as a local Barbilian distance is also proved.

1. Introduction: the metrization procedure and several classes of metrics

The present paper represents the continuation of the study pursued in [23] and [22]. To better present our results stated below, we would like to remind here a few facts of historical nature, as well as several notations and definitions. Historically, it is known that Barbilian’s metrization procedure was introduced in the much cited paper [3] published originally in Časopis Mathematicky a Fysiky, and it was the subject of an inspiring correspondence between D. Barbilian and W. Blaschke [4] in 1934 and thereafter. (A number of 27 letters between them have been discovered recently [40].) Later contributions include P.J. Kelly’s work [37] and major developments are due to D. Barbilian himself who worked on this projects in his last years [5–8]. Actually, in 1959 Barbilian generalized the metrization procedure to domains of a more general form, withstanding not necessarily on planar sets, but in a more abstract setting. For a full historical account, see [19,21].

Over the years, the original paper [3] has been cited many times. Recent developments have been initiated by A.F. Beardon in his important work [9], F. Gehring and K. Hag [25], as well as P. Hästö, Z. Ibragimov and other authors [26–36]. The geometric viewpoint is discussed in the monograph [15] and a recent extension of Barbilian’s metrization procedure is in [41]. All of these works cite and have a common source in Barbilian’s paper [3]. The study of Barbilian’s metrization procedure from a geometric viewpoint is pursued in [11–14,16–18]. The examples explored in the present work aim to discuss...
Barbilian’s metrization procedure in the context of its relations with various classes of metrics, as for example Riemann, Finsler, Lagrange or Lagrange generalized metrics. (for this terminology, our main references are [1,2,8,39]).

To establish the grounds for our present study, we would like to remind here the following construction, given originally by Barbilian [5] as a development of the idea from [3]. Consider two arbitrary sets A and B and the function \( f: A \times B \rightarrow \mathbb{R}_+ \) denoted \( f(A,B) = \frac{f(P,A)}{f(P,B)} \), where P is a point in the set K. Note that \( g_{AB}: K \rightarrow \mathbb{R}_+ \) has a maximum when \( P \in K \). In [5] it is pointed out that if we assume the existence of \( \max g_{AB}(P) \), when \( P \in K \), then there also exists \( m_{AB} = \min_{P \in K} g_{AB}(P) = \frac{1}{\max_{P \in K} g_{AB}(P)} \).

It is known since [5] that \( d: J \times \rightarrow \mathbb{R}_+ \) given by

\[
d(A, B) = \ln \left( \frac{\max_{P \in K} g_{AB}(P)}{\min_{P \in K} g_{AB}(P)} \right)
\]

is a semidistance, i.e.: (1) if \( A = B \) then \( d(A, B) = 0 \); (2) \( d \) is symmetric; (3) \( d \) satisfies triangle inequality.

The influence \( f: K \times J \rightarrow \mathbb{R}_+ \) is called effective if there is no pair \( (A, B) \in J \times J \) such that the ratio \( g_{AB}(P) = \frac{f(P,A)}{f(P,B)} \) is constant for all \( P \in K \). In [5] it is shown that if \( f: K \times J \rightarrow \mathbb{R}_+ \) is an effective influence, then \( (1) \) is a distance. (For a recent exploration of a more general classes of distances than the ones produced by logarithmic oscillation, see [41].)

To better explain the setting of our study on Barbilian’s metrization procedure and to establish our notations, we briefly review a few basic well-known facts on Finsler, Lagrange and Lagrange generalized metrics. Our terminology and our context is consistent with e.g. [1,2,8,39] and our goal is to establish the same context as in [23,22]. Suppose that \( M \) is a real smooth finite-dimensional manifold and let \( \tau : TM \rightarrow M \) its tangent bundle. Let \( (U, (\xi')) \) be a local chart on \( M \); our convention is that indices \( i, j, k, \ldots \) run from 1 to \( n = \dim M \) and we are using Einstein’s convention on summation. Associate to any section \( \nu \in \tau^{-1}(U) \) the coordinates \((\xi'(\tau(\xi))) \) and \((\eta)\) and denote \( \hat{\xi}_i := \frac{\partial}{\partial \xi^i} \). A change of coordinates \((\xi', \eta') \rightarrow (\xi', \eta')\) on the smooth orientable manifold \( TM \) is

\[
x' = x'(x^1, \ldots, x^n), \quad y' = (\hat{\xi}_j x') y^j, \quad \text{rank}(\hat{\xi}_j x') = n.
\]

A Finsler structure or Finsler function of \( M \) is a function \( F : TM \rightarrow [0, \infty) \), \((x, y) \rightarrow F(x, y)\) with the properties

(i) \( F \) is smooth on the slit tangent bundle \( TM \setminus \{0\} \),
(ii) \( F \) is positively homogeneous of degree one in the \( y \)'s, that is \( F(x, \lambda y) = \lambda F(x, y) \), for all \( \lambda > 0 \), and
(iii) the matrix

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}
\]

is positive definite at every point of \( TM \setminus \{0\} \).

The pair \((M, F)\) is called a Finsler manifold and one says that \( g_{ij}(x, y) \) is its Finsler metric. Notice that by the positive homogeneity it follows that \( F^2(x, y) = g_{ij}(x, y) y^i y^j \). If \( g_{ij}(x, y) \) do not depend on \( y \)'s, the Finsler manifold \((M, F)\) becomes a Riemannian manifold with the metric \( ds^2 = g_{ij}(x) dx^i dx^j \).

There are several generalizations of Finsler geometry (see [1,8,39]). To us, for the present paper, of interest are Lagrange and generalized Lagrange geometries.

It is said that a set of matrices \( g_{ij}(x, y) \) define a generalized Lagrange metric if they satisfy the following three requirements:

(i) A change of coordinates \( (2) \) implies

\[
g_{ij}(x, y) = (\hat{\xi}_i x'(\xi')) (\hat{\xi}_j x'(\xi')) g_{ij'}(x', y') .
\]

(ii) Symmetry: \( g_{ij}(x, y) = g_{ji}(x, y) \).

(iii) Non-degeneracy: \( \det(g_{ij}(x, y)) \neq 0 \).

A generalized Lagrange metric is said to be a Lagrange metric if there exists a smooth function \( L : TM \rightarrow \mathbb{R} \) (called a Lagrangian) such that

\[
g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} .
\]

The pair \((M, L)\) is called a Lagrange manifold. Every Finsler manifold is a particular Lagrange manifold with \( L = F^2 \). A necessary and sufficient condition for a generalized Lagrange metric to be a Lagrange metric is that \( C_{ijk} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \frac{\partial^2 L}{\partial y^k} \) be totally symmetric.

In the next section, in Theorem 2.5, we generalize both a result from [7, part 2, paragraph 7], and a version of the argument used by P.A. Hästö in [29], in the proof of his Lemma 3.5.

In [22] it is proved the following classification result.

\[
A \text{ necessary and sufficient condition for a generalized Lagrange metric to be a Lagrange metric is that } C_{ijk} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \frac{\partial^2 L}{\partial y^k} \text{ be totally symmetric.}
\]
Theorem 1.1. Barbilian’s metrization procedure for \( K \) and \( J \) two subsets of the Euclidean plane \( \mathbb{R}^2 \), and \( f(M, A) = \|MA\| \), yields either a Riemannian metric or a Lagrange generalized metric irreducible to a Finslerian or a Lagrangian metric.

By using Theorem 2.5, we will revisit the problem studied in [22] and the above mentioned Theorem 1.1.

2. A generalization of Barbilian’s metrization procedure in \( \mathbb{R}^n \)

The results we present in this section have an intuitive geometric motivation. We have seen that Barbilian has determined the infinitesimal expression of the distance generated by logarithmic oscillation. However, the original description of the phenomenon depends upon the relative position of the points representing the extrema \( \sigma_1 \) and \( \sigma_2 \). Let us consider on the real vector space \( \mathbb{R}^n \) an arbitrary symmetric bilinear form \( \varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \). The vectors of \( \mathbb{R}^n \) can be described as follows.

Definition 2.1. The vector \( x \in \mathbb{R}^n \) having:

- \( \varphi(x, x) = 0 \) is called **null vector**;
- \( \varphi(x, x) > 0 \) is called **positive vector**;
- \( \varphi(x, x) < 0 \) is called **negative vector**.

In the affine space \( \mathbb{R}^n \) we define the following types of hyperspheres.

Definition 2.2. Let \( p \in \mathbb{R}^n \) be an arbitrary point and \( r > 0 \) a real number. The set of all points \( x \in \mathbb{R}^n \) which satisfy

\[
\varphi(x - p, x - p) = r^2
\]

is called **positive hypersphere** with center \( p \) and radius \( r \).

Definition 2.3. Let \( p \in \mathbb{R}^n \) be an arbitrary point and \( r > 0 \) a real number. The set of all points \( x \in \mathbb{R}^n \) which satisfy

\[
\varphi(x - p, x - p) = -r^2
\]

is called **negative hypersphere** with center \( p \) and radius \( r \).

A description of a pair of sets is now required for a better understanding of Theorem 2.5.

Definition 2.4. The pair of sets \( (K, J) \subseteq \mathbb{R}^n \times \mathbb{R}^n \) is called \( \varphi \)-pair of Barbilian sets if satisfy the conditions:

1. \( K \cap J = \emptyset \);
2. only one of the following
   a. \( \varphi(x - y, x - y) > 0 \), for all \( \forall x \in K, \forall y \in J \), or
   b. \( \varphi(x - y, x - y) < 0 \), for all \( \forall x \in K, \forall y \in J \).

Theorem 2.5. Let \( (K, J) \subseteq \mathbb{R}^n \times \mathbb{R}^n \) be a \( \varphi \)-pair of Barbilian sets. Consider the influence \( f : K \times J \rightarrow \mathbb{R}^+ \) defined by \( f(M, A) = \sqrt{\varphi(MA, MA)} \) and the function \( g_{AB}(M) = \frac{f(M, A)}{f(M, B)} \) such that on \( J \) is defined a Barbilian distance \( d^B(A, B) \). Suppose furthermore that each extrema \( \max_{M \in K} g_{AB}(M) \) and \( \min_{M \in K} g_{AB}(M) \) is attained in a unique point \( P \) and \( P' \) from \( K \), respectively. Then:

(a) For any \( A \in J \) and any hyperplane \( \pi \) passing through \( A \) there exist exactly two hyperspheres tangent to \( \pi \) in \( A \) and passing through the points \( P \) and \( P' \) from \( K \), respectively.
(b) The metric induced by the Barbilian distance has the form

\[
ds^2 = \frac{1}{4} \left( \frac{1}{r_1} \pm \frac{1}{r_2} \right)^2 d\sigma^2_{\varphi},
\]

where \( r_1 \) and \( r_2 \) are the radii of the hyperspheres described in (a) and \( d\sigma^2_{\varphi} \) is the metric induced by the bilinear form \( \varphi \).

Proof. Let \( A(a) \) and \( B(b) \) be fixed in \( J \) such that the vector \( \vec{AB} \) is not zero and let \( M(x) \) be arbitrary in \( J \). Let us consider the Apollonius hypersphere described by the equation
\[ \varphi(MA, MA) = \varphi(MB, MB) = 1, \]

where \( \lambda \neq 0 \) and \( \lambda \neq 1 \).

By a straightforward computation, we obtain that

\[ \varphi \left( x - \frac{a - \lambda b}{1 - \lambda}, x - \frac{a - \lambda b}{1 - \lambda} \right) = \frac{\lambda \varphi(a - b, a - b)}{(1 - \lambda)^2}. \]

The above relation reveals a hypersphere of which the radius \( R \) is given by

\[ R^2 = \text{sgn}(\varphi(a - b, a - b)) \frac{\lambda \varphi(a - b, a - b)}{(1 - \lambda)^2}. \]

Extremas \( M_{AB} \) and \( m_{AB} \) of ratio \( \frac{\varphi(MA)}{\varphi(MB)} \) yield to

\[ r_1^2 = \frac{M_{AB}}{(1 - M_{AB})^2} |\varphi(\overrightarrow{AB}, \overrightarrow{AB})|, \]

\[ r_2^2 = \frac{m_{AB}}{(1 - m_{AB})^2} |\varphi(\overrightarrow{AB}, \overrightarrow{AB})|. \]

Therefore,

\[ \left( \frac{1 + M_{AB}}{1 - M_{AB}} \right)^2 = \frac{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})| + 4r_1^2}{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|}, \]

which gives

\[ M_{AB} = 1 + \frac{2 \sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|}}{-\sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|} + \sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|} + 4r_1^2}. \]

In the same way we obtain that

\[ m_{AB} = 1 - \frac{2 \sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|}}{\sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|} + \sqrt{|\varphi(\overrightarrow{AB}, \overrightarrow{AB})|} + 4r_2^2}. \]

Assuming that the points \( A \) and \( B \) are close enough, we denote \( B = A + dA \). In this case, \( \varphi(\overrightarrow{AB}, \overrightarrow{AB}) \) becomes

\[ d\sigma^2 = d\sigma_\varphi^2. \]

The Barbilian distance between the points \( A \) and \( dA \) gives the arc element \( d\beta(A, A + dA) \), denoted by \( ds \), where \( ds = d\beta(A, A + dA) = \frac{1}{2} m_{AB} - m_{AB} \). To complete our proof, the following geometric argument is essential. At least for the case when the bilinear form is the Euclidean inner product, the geometric locus of the points \( P \) in \( \mathbb{R}^n \) with the property that \( \|PA\| = 1 \) is the perpendicular bisector hyperplane of the line segment \( AB \). The geometric locus of the points \( P \) in \( \mathbb{R}^n \) with the property \( \|PA\| = m < 1 \) is an Apollonian hypersphere that encloses \( A \) and leaves \( B \) in its exterior. The geometric locus of the points \( P \) in \( \mathbb{R}^n \) with the property \( \|PA\| = M > 1 \) is an Apollonian hypersphere that encloses \( B \) and this time leaves \( A \) in its exterior. When we let \( B = A + dA \rightarrow A \), the two hyperspheres are separated by the hyperplane perpendicular to the direction \( dA \) in \( A \). Depending upon the relative position of the points representing the extrema \( M_{AB} \) and \( m_{AB} \), it may be possible that the Apollonian hyperspheres are or are not separated by the hyperplane perpendicular to the direction \( dA \) in \( A \). However, if the two hyperspheres are not separated by this hyperplane, we must assign a signature to the radii, due to the fact that we actually have the symmetrical of the usual Apollonian hypersphere, which would hold the appropriate positive sign. Based on this argument, we can see that the metric depends essentially on the relative position of the Apollonian hyperspheres corresponding to the extrema \( M_{AB} \) and \( m_{AB} \). Taking into account that

\[ \frac{2 d\sigma_\varphi}{-d\sigma_\varphi + \sqrt{d\varphi^2 + 4r_1^2}} = \frac{d\sigma_\varphi}{r_1}, \]

and

\[ \frac{2 d\sigma_\varphi}{d\sigma_\varphi + \sqrt{d\sigma_\varphi^2 + 4r_2^2}} = \frac{d\sigma_\varphi}{r_2}. \]
by a straightforward computation and taking into account our previous observation, we can easily see that

\[
ds = \frac{1}{2} \left( \frac{r_1}{d\sigma_\psi} \pm \frac{r_2}{d\sigma_\psi} \right) \cdot \frac{d\sigma_\psi}{r_1} \cdot \frac{d\sigma_\psi}{r_2} = \frac{1}{2} \left( \frac{1}{r_1} \pm \frac{1}{r_2} \right) \ d\sigma_\psi,
\]

so the metric has the form

\[
ds^2 = \frac{1}{4} \left( \frac{1}{r_1} \pm \frac{1}{r_2} \right)^2 d\sigma_\psi^2.
\]

Let us call natural position for the Apollonian hyperspheres the position which accomplishes the following task. The Apollonian hyperspheres are tangent in \(A\) and each one intersects \(K\) in a unique point. Hence, for a given point and a given direction \(dA\) we have only one possible sign assigned in formula (3) to this natural position.

To clarify the ideas and be precise about the meaning of the theorem presented in this section, we should point out that \(\pm\) in equation (3) does not necessarily infer automatically that we get two disjoint metrics on set \(J\). The relevant argument is Example 3.2 below, which shows that on the exterior of the unit disk the two possible signs yield one single metric. Furthermore, the metric we obtain in Example 3.2 is important, more specifically is the Riemannian metric that corresponds to non-Euclidean Lobatchevsky's geometry outside the disk.

In the next section we are pursuing this idea on some important particular cases.

3. Applications: the metrics on some domains of interest in \(\mathbb{R}^2\)

3.1. The case when \(J\) is the disk

Let \(O\) be the origin of \(\mathbb{R}^2\). Let \(K\) be the circle having \(O\) as a center and \(R\) as its radius. Denote its interior by \(J\). It is known that a Barbilian distance can be induced in \(J\) by considering as influence the Euclidean scalar product (see [5]). In order to obtain the natural induced metric associated to this distance, consider a fixed point \(A\) and a direction \(dA\). Denote by \((x, y)\) the coordinates of the point \(A\) and let \(d\) be the line passing through \(A\) and perpendicular to \(dA\). Let \(m = \frac{y}{x}\) be the slope of this line. We compute the radii of the Apollonian circles, which are

\[
r_1 = \frac{\sqrt{m^2 + 1}}{2} \cdot \frac{R^2 - OA^2}{\sqrt{m^2 + 1} - mx + y},
\]

\[
r_2 = \frac{\sqrt{m^2 + 1}}{2} \cdot \frac{R^2 - OA^2}{\sqrt{m^2 + 1} + mx - y}.
\]

Therefore, using Theorem 2.5, we obtain the metric

\[
ds^2 = \frac{4R^2}{(R^2 - (x^2 + y^2))^2} (dx^2 + dy^2),
\]

which is Riemannian. This constant Gaussian curvature metric is studied in [20,23].

3.2. The metric on the exterior of the disk

Denote by \(K\) the circle centered in the origin, with radius \(R\) and denote by \(J\) its exterior. Consider the Euclidean scalar product as influence. In order to simplify the notations let us denote by \(MA\) the Euclidean norm \(||MA||\).

We aim to study the metric generated by Barbilian's metrization procedure outside this circle. To this goal, we need to describe geometrically the points where the maximum and the minimum are reached. Consider the inversion with pole \(B\) and power \(\mu\), where \(\mu\) is the power of \(B\) with respect to the given circle.

Then, by this inversion the circle \(K\) remains invariant; we also have \(A \rightarrow A'\) such that \(BA \cdot BA' = \mu\), where \(A'\) is a fixed point lying on the line \(AB\).

Since any point \(M \in K\) is mapped into \(M' \in K\), we have that

\[
A'M' = \mu \cdot \frac{AM}{BA \cdot BM},
\]

denote the constant ratio \(k = \frac{AM}{BM}\), therefore \(A'M' = k \cdot \frac{AM}{BM}\).

Remark that the ratio \(\frac{AM}{BM}\) reaches its maximum or its minimum, respectively, whenever \(A'M'\) is maximum or minimum, respectively.

The endpoints \(S_0\) and \(S'_0\) of the diameter through \(A'\), are the points where \(A'M'\) is maximum or minimum, respectively. Their inverses, the points \(M'_0\) and \(M_0\), are the points where the minimum and the maximum of the ratio \(\frac{AM}{BM}\) are reached, respectively. Since \(S_0S'_0\) is orthogonal to the circle \(K\), we get that \(M_0\) and \(M'_0\) are the endpoints of the arc twice orthogonal to \(K\) passing through \(A\) and \(B\). Thus, it makes sense to consider the Barbilian distance for \(J\).
First, we claim that the metric is Riemannian. To get it, we use both forms of the metrics emphasized by Theorem 2.5. Consider the arbitrary point \((x_0, y_0)\) in \(\mathbb{R}^2 - \{(x, y) \in \mathbb{R}^2; \; x^2 + y^2 < R^2\}\) and the arbitrary line \(d\) passing through \((x_0, y_0)\), \((d)\); \(y - y_0 = m(x - x_0)\).

We distinguish three cases.

(i) Suppose that the straight line \(d\) does not intersect the circle \(K\). The two Apollonian circles have radii \(r_1\) and \(r_2\), and are centered in \((x_1, y_1)\) and \((x_2, y_2)\), respectively and let us suppose that the circle centered in \((x_2, y_2)\) having \(r_2\) as radius contains in its interior both \(K\) and the other Apollonian circle. The common tangent to the Apollonian circles through \((x_0, y_0)\) is the straightline \(d\). The line of the centers is perpendicular to the radical axis and has the equation \(y - y_0 = \frac{1}{R}(x - x_0)\). Besides this equation, we have also that \((x_2 - x_0)^2 + (y_2 - y_0)^2 = r_2^2\). Additionally, the distance from the origin to \((x_2, y_2)\) is \(r_2 - R\), thus \(x_2^2 + y_2^2 = (r_2 - R)^2\). We obtain

\[
r_2 = \frac{1}{2} \cdot \sqrt{1 + m^2} (x_0^2 + y_0^2 - R^2) - R \sqrt{1 + m^2} + y_0 - mx_0. \tag{4}
\]

Similarly, \((x_1 - x_0)^2 + (y_1 - y_0)^2 = r_1^2\). The distance from the origin to \((x_1, y_1)\) is \(r_1 + R\), thus \(x_1^2 + y_1^2 = (r_1 + R)^2\). As before we have

\[
r_1 = \frac{1}{2} \cdot \sqrt{1 + m^2} (x_0^2 + y_0^2 - R^2) R \sqrt{1 + m^2} + y_0 - mx_0 \tag{5}
\]

and we compute directly the expression of the arclength using that both Apollonian circles lie on the same part of \(d\):

\[
\frac{1}{r_1} - \frac{1}{r_2} = \frac{4R}{x_0^2 + y_0^2 - R^2}. \tag{6}
\]

(ii) Suppose that the straight line \(d\) intersects the circle \(K\). In this case, the value of the radii are computed by using the same ideas as before. By direct computations, we have:

\[
r_1 = \frac{1}{2} \cdot \sqrt{1 + m^2} (x_0^2 + y_0^2 - R^2) \cdot R \sqrt{1 + m^2} + y_0 - mx_0, \tag{7}
\]

\[
r_2 = \frac{1}{2} \cdot \sqrt{1 + m^2} (x_0^2 + y_0^2 - R^2) \cdot R \sqrt{1 + m^2} - y_0 + mx_0. \tag{8}
\]

Bearing in mind that the Apollonian circles are separated by \(d\), we have

\[
\frac{1}{r_1} + \frac{1}{r_2} = \frac{4R}{x_0^2 + y_0^2 - R^2}. \tag{9}
\]

(iii) For the last case, suppose that the line \(d\) is tangent to the circle \(K\). This yields the tangency condition

\[
(R \sqrt{1 + m^2} + y_0 - mx_0)(- R \sqrt{1 + m^2} - y_0 + mx_0) = 0 \tag{10}
\]

which implies

\[
\frac{1}{r_1} = 0 \quad \text{and} \quad \frac{1}{r_2} = \frac{4R}{x_0^2 + y_0^2 - R^2}. \tag{11}
\]

Therefore in all the cases the natural induced metric in the exterior of the disk is Riemannian and has the expression

\[
ds^2 = \left(\frac{2R}{x^2 + y^2 - R^2}\right)^2 (dx^2 + dy^2). \tag{12}
\]

3.3. The half-plane

Consider the case when \(K = \{(x, y) \in \mathbb{R}^2; \; y = 0\}\) and \(J = \{(x, y) \in \mathbb{R}^2; \; y > 0\}\). Let \(M \in K\) and \(A(x_0, y_0) \in J\) and \(B(x_1, y_1) \in J\). Consider the influence function \(\|MA\|\) and the associated ratio

\[
g_{AB}(M) = \frac{\|MA\|}{\|MB\|}. \tag{13}
\]

As in our previous example, let us denote by \(MA\) the Euclidean norm \(\|MA\|\).

We describe geometrically the points where the maximum and the minimum are reached. Let \(B_1\) be the foot of the perpendicular drawn from \(B\) to \(x\)-axis. Consider the inversion with pole \(B\) and power \(BB_1^2\).
Then, by this inversion we have $B_1 \rightarrow B_1$ and $K \rightarrow C(BB_1)$, where we denote by $C(BB_1)$ the circle of diameter $BB_1$; we also have $A \rightarrow A'$ such that $BA \cdot BA' = BB_1^2$, where $A'$ is a fixed point lying on the line $AB$.

Since any point $M \in K$ is mapped into $M' \in C(BB_1)$, we have that

$$A'M' = BB_1^2 \cdot \frac{AM}{BA \cdot BM};$$

denote the constant ratio $k = \frac{AA_1^2}{BM}$, therefore $A'M' = k \cdot \frac{AM}{BM}$.

Remark that the ratio $\frac{AM}{BM}$ reaches its maximum or its minimum whenever $A'M'$ is maximum or minimum, respectively.

Therefore, the antipodal points $S_0$ and $S'_0$, bound the diameter through $A'$ in $C(AA_1)$. Their inverse images, the points $\{M_0\} = AS_0 \cap (y = 0)$, and $\{M_0'\} = AS_0' \cap (y = 0)$, are the points where the minimum and the maximum of the ratio $\frac{AM}{BM}$ are reached, respectively.

Since $S_0S'_0$ is orthogonal to the circle $C(AA_1)$ and since $B' \in SS_0$, we get that $M_0$ and $M'_0$ are the endpoints of the arc twice orthogonal onto $K$ passing through $A$ and $B$. Thus, it makes sense to consider the Barbilian distance for the half-plane $J$, since the two extrema needed in the logarithmic oscillation formula exist.

Our assertion is that the natural induced metric is an important Riemannian metric, the Poincaré metric on the half-plane. Let $A(x_0, y_0)$ and the arbitrary line through $A$ given by $y - y_0 = m(x - x_0)$. Consider the circles of centers $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ tangents in $A$ to the line and also tangents to $K$. Then we have:

$$y_1 - y_0 = -\frac{1}{m}(x_1 - x_0),$$

$$y_2 - y_0 = -\frac{1}{m}(x_2 - x_0),$$

$$y_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2,$$

$$y_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2,$$

$$y_1 < y_0,$$

$$y_2 > y_0.$$  

From (13), (15), and (17), we get:

$$r_1 = y_1 = \frac{y_0\sqrt{m^2 + 1}}{1 + \sqrt{m^2 + 1}}.$$  

Similarly, from (14), (16), and (18), we obtain

$$r_2 = y_2 = \frac{y_0\sqrt{m^2 + 1}}{-1 + \sqrt{m^2 + 1}}.$$  

By using Theorem 2.5, we have:

$$ds^2 = \frac{1}{y^4}(dx^2 + dy^2),$$

don that is the Poincaré metric on the half-plane, which is a Riemannian one. The Gaussian curvature is $-1$.

4. The 2-dimensional Lorentz case

The following concepts are outlined and studied in [10]; for all the definitions in Lorentzian geometry, we will keep the terminology from [10].

Consider $L^2$ the vector space $\mathbb{R}^2$ endowed with the Lorentzian inner product

$$\langle x, y \rangle_L := x_1y_1 - x_2y_2, \quad \forall x = (x_1, x_2), \ y = (y_1, y_2).$$

The group of matrices

$$G_L = \left\{ B(u) \left| B(u) = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \ u \in \mathbb{R} \right. \right\}$$

preserves Lorentzian inner product and yields the Lorentz geometry in $L^2$. 
Definition 4.1. Let $O(x_0, y_0)$ and $r > 0$. We call **positive circle** having the center $O$ and radius $r$ in the plane $\mathcal{P}$ the set of all points $X(x, y)$ satisfying

$$(OX, OX)_L = r^2.$$ 

In [10] such a circle is called **time-like circle**.

Obviously, the equation of a Lorentz time-like circle is

$$(x - x_0)^2 - (y - y_0)^2 = r^2,$$

which is the equation of a hyperbola in Euclidean geometry.

Applying a rotation centered at the origin, with the angle $\alpha$, e.g.

$x \mapsto x \cosh \alpha + y \sinh \alpha,$

$y \mapsto x \sinh \alpha + y \cosh \alpha,$

we obtain that

$$x^2 - y^2 - 2(x_0 \cosh \alpha - y_0 \sinh \alpha)x + 2(y_0 \cosh \alpha - x_0 \sinh \alpha)y +$$

$$+ (x_0 \cosh \alpha - y_0 \sinh \alpha)^2 - (y_0 \cosh \alpha - x_0 \sinh \alpha)^2 - r^2 = 0.$$

Therefore, the equation of a Lorentz time-like circle is of the form

$$x^2 - y^2 - 2ax + 2by + a^2 - b^2 - r^2 = 0.$$ 

Denote by $|MA|_L = \sqrt{\langle MA, MA \rangle}_L$.

Next, we will show that in the above Lorentz geometry can be defined a Barbilian distance in a set $J$ using Barbilian procedure.

Let us consider the sets

$$K := \{(0,y) \mid y \in [0,1]\},$$

$$J := \{(x,y) \mid x \in (1,2), y \in (3,4)\},$$

and the influence function $f : K \times J \to \mathbb{R}^+_+$ defined by $f(M, A) := |MA|_L$, for all $M \in K$ and $A \in J$.

Let $(a_1, a_2), (b_1, b_2) \in J$ and consider the function $g_{AB} : K \to \mathbb{R}^+_+$ defined by

$$g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = \frac{|MA|_L}{|MB|_L} = \sqrt{\frac{(a_2 + a_1 - y)(a_2 - a_1 - y)}{(b_2 + b_1 - y)(b_2 - b_1 - y)}}$$

The extrema for the function $g_{AB}$ can be found by studying the monotonicity of the continuous function $h : [0,1] \to \mathbb{R}^+_+$ defined for $a, b, c, d \in (1, \infty)$ by

$$h(y) = \frac{(y - a)(y - b)}{(y - c)(y - d)}, \quad \forall y \in [0,1].$$

The extrema of $h$ exist because of the Weierstrass theorem since $h$ is continuous.

It is not very difficult to see that either

$$h(0) = \max_{y \in [0,1]} h(y),$$

$$h(1) = \min_{y \in [0,1]} h(y),$$

or

$$h(1) = \max_{y \in [0,1]} h(y),$$

$$h(0) = \min_{y \in [0,1]} h(y).$$

Therefore, a Barbilian distance exists in $J$. Next, we adapt Theorem 2.5 to the context of Lorentzian geometry.

Theorem 4.2. Let $K \subseteq \mathbb{R}^2$, $J \subseteq \mathbb{R}^2$, $K \cap J = \emptyset$. Consider the influence $f(M, A) = |MA|_L$ and $g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = |MA|_L$ such that on $J$ is defined a Barbilian distance $d^B(A, B)$. Suppose furthermore that each of extrema $\max_{M \in K} g_{AB}(M)$ and $\min_{M \in K} g_{AB}(M)$ is attained in a unique point in $K$. Then:

(a) For any $A \in J$ and any straight line $d$ passing through $A$ there exist exactly two Lorentz circles tangent to each other on $A$ to $d$ and passing through the points from $K$ described above.
(b) The metric induced by the Barbilian distance has the form

$$ds^2 = \frac{1}{4} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^2 (dx^2 - dy^2),$$

where $r_1$ and $r_2$ are the radii of the Lorentz circles described in (a).

For the Barbilian distance $dB_L(A, B)$ we compute the radii of the Lorentz time-like circles. (See Fig. 1.) Suppose that the first Lorentz time-like circle ($H_1$) passes through $M_1(0, 1) \in K$ and is tangent in $A(x_0, y_0) \in J$ to the straight line

$$y - y_0 = m(x - x_0).$$

Its radius is

$$r_1^2 = \frac{(m^2 - 1)(y_0^2 - 1)^2}{4(1 + mx_0 - y_0)^2}. \quad (21)$$

The second Lorentz time-like circle ($H_2$), which passes through $M_2(0, 0) \in K$ and is tangent in $A(x_0, y_0) \in J$ to the straight line

$$y - y_0 = m(x - x_0),$$

has the radius

$$r_2^2 = \frac{(m^2 - 1)y_0^2 - x_0^2}{4(mx_0 - y_0)^2}. \quad (22)$$

By Theorem 4.2, the induced metric is

$$ds^2 = \frac{1}{4} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^2 (dx^2 - dy^2).$$

Replacing $m = \frac{\dot{y}}{\dot{x}} > 4$, after a few computations, we obtain

$$ds^2 = \frac{(x\dot{y} + (x^2 + y - y^2)(2\dot{y} - 2x\dot{y} - \dot{x}))^2}{(y^2 - x^2)(y^2 - x^2)(y - 1)^2 - x^2} (dx^2 - dy^2).$$

Since

$$\frac{\partial g_{11}}{\partial \dot{y}} \neq \frac{\partial g_{12}}{\partial \dot{x}} = 0,$$

the tensor $C_{ijk} = \frac{\partial g_{ij}}{\partial x^k}$ is not totally symmetric, that is the metric is Langrange generalized.
5. The Minkowski 2-dimensional case

Similarly, we observe that Minkowski plane \( M^2 \) is produced by the vector space \( \mathbb{R}^2 \) endowed with the pseudo-inner product

\[
\langle x, y \rangle_N := x_1 y_2 + x_2 y_1,
\]

\( \forall x = (x_1, x_2), y = (y_1, y_2) \).

It follows easily that the group of matrices

\[
G_N = \left\{ C(a) \mid C(a) = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in \mathbb{R}^* \right\}
\]

preserves the pseudo-inner product defined above in the Minkowski geometry.

**Definition 5.1.** Let \( O(x_0, y_0) \) and \( r > 0 \). We call **Minkowski positive circle** with center \( O \) and radius \( r \) in the plane \( P \) the set of all points \( X(x, y) \) satisfying

\[
\langle OX, OX \rangle_N = r^2.
\]

The equation of a Minkowski positive circle is

\[
2(x - x_0)(y - y_0) = r^2,
\]

which is also the equation of a hyperbola in Euclidean geometry.

Applying a rotation centered at the origin, with the angle \( \alpha \neq 0 \), e.g.

\[
x \mapsto \alpha x,
\]

\[
y \mapsto \frac{1}{\alpha} y,
\]

we obtain the general equation of a Minkowski negative circle

\[
2xy - 2\alpha y_0 x - 2\frac{x_0}{\alpha} y + 2x_0 y_0 - r^2 = 0.
\]

Denote by \( \|MA\|_N = \sqrt{\langle MA, MA \rangle_N} \).

Next, we prove that in the above Minkowski geometry can be defined a Barbilian distance in a set \( J \) using Barbilian procedure.

Let us consider the sets

\[ K := \{(x, x) \mid x \in [0, 1]\}, \]

\[ J := \{(x, y) \mid x \in (3, 4), y \in (1, 2)\}, \]

and the influence function \( f : K \times J \to \mathbb{R}_+ \) defined by \( f(M, A) := \|MA\|_N \), for all \( M \in K \) and \( A \in J \).

Let \( A(a_1, a_2), B(b_1, b_2) \in J \) and consider the function \( g_{AB} : K \to \mathbb{R}_+ \) defined by

\[
g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = \frac{\|MA\|_N}{\|MB\|_N}.
\]

For arbitrary \( M(x, x) \in K \), we have that \( x \in [0, 1] \) and

\[
\|MA\|_N = \sqrt{2(a_1 - x)(a_2 - x)},
\]

\[
\|MB\|_N = \sqrt{2(b_1 - x)(b_2 - x)}.
\]

Then

\[
g_{AB}(M) = \frac{\sqrt{(a_1 - x)(a_2 - x)}}{\sqrt{(b_1 - x)(b_2 - x)}}, \quad \forall x \in [0, 1].
\]

The function \( g_{AB} \) is obviously continuous and strictly monotone, so then the extrema (maximum and minimum) are attained for \( x = 0 \) and, respectively, \( x = 1 \) (not necessarily in this order).

Therefore, a Barbilian distance exists in \( J \). It makes sense now to state the form of **Theorem 2.5** in the Minkowski geometry.
Theorem 5.2. Let $K \subseteq \mathbb{R}^2$, $J \subseteq \mathbb{R}^2$, $K \cap J = \emptyset$. Consider the influence $f(M, A) = \|MA\|_N$ and $g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = \frac{\|MA\|_N}{\|MB\|_N}$ such that on $J$ is defined a Barbilian distance $d_N^B(A, B)$. Suppose furthermore that each extrema $\max_{M \in K} g_{AB}(M)$ and $\min_{M \in K} g_{AB}(M)$ is attained in a unique point in $K$. Then:

(a) For any $A \in J$ and any straight line $d$ passing through $A$ there exist exactly two Minkowski positive circles tangent to each other on $A$ to $d$ and passing through the points from $K$ described above.
(b) The metric induced by the Barbilian distance has the form

$$ds^2 = \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)^2 (dx dy),$$

where $r_1$ and $r_2$ are the radii of the Minkowski positive circles described in (a).

Computing as in the Lorentz case the radii of the Minkowski positive circles tangent in $A(x_0, y_0)$ to the straightline (see Fig. 2)

$$y - y_0 = m(x - x_0),$$

we obtain

$$r_1^2 = \frac{2mx_0^2y_0^2}{(mx_0 - y_0)^2}$$

and

$$r_2^2 = \frac{2m(x_0 - 1)^2(y_0 - 1)^2}{(1 + m(x_0 - 1) - y_0)^2}. \quad (23)$$

Using Theorem 5.2 and replacing $m = \frac{\dot{y}}{\dot{x}} > \frac{2}{3}$, the metric induced by Barbilian distance is

$$ds^2 = \frac{(\dot{x}\dot{y} - \dot{y}\dot{x})(2\dot{x}\dot{y} - \dot{x} - \dot{y} + 1) + \dot{x}\dot{y}(1 - \dot{y})^2}{4\dot{x}\dot{y}(x - 1)^2(y - 1)^2x^2y^2} dx dy. \quad (24)$$

Since

$$\frac{\partial g_{11}}{\partial \dot{y}} \neq \frac{\partial g_{12}}{\partial \dot{x}} = 0,$$

the tensor $C_{ijk} = \frac{\partial g_{ij}}{\partial x^k}$ is not totally symmetric, that is the metric above is Langrange generalized.
6. The distance induced by a Riemannian metric as a local Barbilian distance

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. For any two points $x, y \in M$, we consider the following sets:

$$C(x, y) := \{ \sigma \mid \sigma : [0, 1] \to M, \ \sigma(0) = x, \ \sigma(1) = y \ \text{and} \ \sigma \ \text{piecewise smooth} \},$$

(the set of piecewise smooth curves which connects $x$ and $y$),

$$L(\sigma) := \int_0^1 \| \dot{\sigma}(t) \| \, dt = \int_0^1 \sqrt{g(\dot{\sigma}(t), \dot{\sigma}(t))} \, dt, \quad \forall \sigma \in C(x, y)$$

(the length of a piecewise smooth curve).

It is known that $d^R(x, y) = \inf_{\sigma \in C(x, y)} L(\sigma)$ is a distance on $M$ induced by the Riemannian metric. Denote by $S(x_0, R) := \{ T \in M \mid d^R(x_0, T) = R \}$ the sphere centered in $x_0$ with radius $R$ and by $B(x_0, R) := \text{int} \ S(x_0, R)$ the ball centered in $x_0$ with radius $R$.

Next we show the local coincidence between a well defined Barbilian distance and the distance $d^R$ induced by the Riemannian metric.

**Theorem 6.1.** For any ball included in a normal neighborhood of a complete Riemannian manifold, a convenient Barbilian metrization procedure induces the distance $d^R$.

**Proof.** Let $x_0 \in M$, $W_{x_0}$ normal neighborhood of $x_0$ in $M$ and let $S(x_0, \epsilon) \subset W_{x_0}$. Denote by $K := S(x_0, \epsilon)$ and by $J := B(x_0, \epsilon)$. Consider $p$ and $q$ two distinct points in $J$.

Let $v \in T_pM$, $\|v\| = 1$, be the vector such that $c(t) = \exp_p tv$, $c(0) = p$ and $c(1) = q$ yields the geodesic which connects $p$ and $q$. Note that $c_1 : [0, \infty) \to \mathbb{R}$, $c_1(t) = \exp_p (-tv)$, has as image a continuous arc which intersects for the first time $S(x_0, \epsilon)$ in a point $k$.

Let $v' = \dot{c}(1)$ and consider $c_2 : [0, \infty) \to \mathbb{R}$, $c_2(t) = \exp_p tv'$. Then $c_2$ has as image a continuous arc which intersects for the first time $S(x_0, \epsilon)$ in $k' \neq k$. Since $\text{Im} c = \text{Im} c_1 \cup \text{Im} c_1[0,1] \cup \text{Im} c_2$, it results that $\text{Im} c$ intersects $S(x_0, \epsilon)$ in two points $k$ and $k'$ such that the order is $k, q, k'$.

Consider the influence function $I(u, p) = e^{\frac{1}{2}d^R(u, p)}$, $u \in K$ and $p \in J$. We have

$$g_{pq}(u) = e^{\frac{1}{2}d^R(u, p)}.$$  

The triangle inequality can be expressed in the form

$$|d^R(u, p) - d^R(u, q)| \leq d^R(p, q).$$

Taking into account the existence of the points $k$ and $k'$, we have

$$\max_{u \in K} g_{pq}(u) = e^{\frac{1}{2}d^R(p, q)} \quad \text{and} \quad \min_{u \in K} g_{pq}(u) = e^{-\frac{1}{2}d^R(p, q)}.$$

It results that

$$d^B(p, q) = \ln \frac{\max_{u \in K} g_{pq}(u)}{\min_{u \in K} g_{pq}(u)} = d^R(p, q).$$

As a consequence, all the distances induced by the metrics of complete Riemannian manifolds seen in [24] can be expressed locally as Barbilian distances. In particular, the $n$-dimensional elliptic spaces (see [24, p. 118]) and the $n$-dimensional hyperbolic spaces (see [24, p. 119]) allow distances which can be expressed using the Barbilian procedure. We also proved that the Poincaré distance for the disk, the Cayley–Klein distance for the disk (see [24, pp. 120–122]) and Poincaré distance in halfplane (see [24, p. 121]) are local Barbilian distances. Furthermore, in Theorem 2.5 we have obtained the well-known Riemannian metric corresponding to these distances.

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References