On \( L \)-Fuzzy Topological Spaces*

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\( L \)-fuzzy sets and, subsequently, \( L \)-fuzzy topological spaces are considered in this paper, where \( L \) is a complete and completely distributive non-atomic Boolean algebra. A few separation properties as well as subspace \( L \)-fuzzy topology are defined here, in the light of an earlier paper. These are more similar to ordinary topological spaces than to fuzzy topological spaces.

1. INTRODUCTION

Following Zadeh's concept of fuzzy sets in [10], Chang developed the theory of fuzzy topological spaces in [3]. Since then, many others have developed various aspects of fuzzy topology, a brief but comprehensive account of which is given in [9]. In most of these cases, however, fuzzy sets in a set \( X \) of points are characterized by functions from \( X \) to the closed unit interval \([0, 1]\). A further development of fuzzy topology is made in [2] and [6] (and possibly others), in which the ranges of these functions are other than \([0, 1]\).

In the present paper, the range has been taken as a complete and completely distributive non-atomic Boolean algebra \( L \). The resulting fuzzy sets are called \( L \)-fuzzy sets (following [6]), while those with range \([0, 1]\) are called ordinary fuzzy sets.

The Hausdorff separation axiom as well as some of the other separation axioms have been defined in \( L \)-fuzzy spaces. Although these had been dealt with in [7], the process here is different in some respects. Quite a number of results have been proved here which could not be shown to hold in [7], though most of those established in [7] are observed to hold here.

\( L \)-fuzzy subspace topology has been defined along the same lines as in [4], as well as [7]. But contrary to [7], we find here that most of the separation properties defined by us are hereditary.

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2. Preliminaries

We first give a few definitions; some others are included in the relevant sections. For those not given here, we refer the reader to [9].

Let \( X \) be a set of points \( \{x_p, p \in A\} \) (where \( A \) is an arbitrary index set) and \( L \) a complete and completely distributive non-atomic Boolean algebra. The \( L \)-fuzzy sets are all characterized by functions from \( X \) to \( L \), called membership functions.

If 0 and 1 are the bounds of \( L \), then, as usual, the empty set \( \emptyset \) and the full set \( X \) have membership functions \( \mu_{\emptyset} \) and \( \mu_X \), defined by

\[
\mu_{\emptyset}(x_p) = 0, \quad \text{for all } p \in A,
\]

and

\[
\mu_X(x_p) = 1, \quad \text{for all } p \in A.
\]

\( A \subset X \) is a crisp subset iff

\[
\mu_A(x_p) = 1, \quad \text{for } p \in A, \quad \mu_A(x_p) = 0, \quad \text{for } p \notin A.
\]

If \( A \) and \( B \) are two \( L \)-fuzzy sets, then \( A \subset B, A = B, A \cap B, A \cup B \) and \( A' \) are all defined according to [2].

So an \( L \)-fuzzy set and its complement have positions in the set \( X \) analogous to a set and its complement in ordinary set theory, which is a marked deviation from ordinary fuzzy sets.

\( L \)-fuzzy (and crisp) points as well as the idea "\( p \) belongs to \( A \)" (written as \( p \in A \)), where \( p \) is an \( L \)-fuzzy point and \( A \) an \( L \)-fuzzy set, are defined as in [9]. If, however, \( p \) is crisp, then \( p \in A \Rightarrow \mu_p(x_p) = \mu_A(x_p) = 1 \), where \( x_p \) is the support of \( p \).

By points (subsets) of \( X \), we mean both crisp and fuzzy points (subsets).

Remark. It is easy to see that \( p \in A \Rightarrow p \notin A' \), but not conversely.

Therefore it is quite possible that both \( p \in A \) and \( p \notin A' \) hold, but still \( p \in A \cup A' \), which again is a deviation from ordinary fuzzy sets.

Before going into further details of the present work, we first recall Stone’s Representation Theorem [8], which, for completeness, we include below.

Theorem [Stone] 2.1. If \( L \) is a Boolean algebra, then there exists a totally disconnected compact Hausdorff space \( H \) such that \( L \) is isomorphic to the Boolean algebra of all open-closed subsets of \( H \).

\( H \) is called the “Stone space” [1] of \( L \).
We further state the following well known result on Boolean algebras (see [1] and [5]), for a ready reference.

**Lemma 2.1.** If \((X, \tau)\) is an extremally disconnected compact Hausdorff space, then no singleton is open-closed in it iff \((X, \tau)\) is the Stone space of a complete non-atomic Boolean algebra.

From this lemma, we get the following corollaries, in which we assume \(x \in L\) to correspond to the open-closed subset \(H_x\) of \(H\).

**Corollary 2.1.** Let \(L\) be a complete non-atomic Boolean algebra and \(a \in L\). If \(M = \{\beta; \beta \in L, \beta > a\}\), then \(a = \inf_{\beta \in M} \beta\). Dually, if \(N = \{\gamma; \gamma \in L, \gamma < a\}\), then \(a = \sup_{\gamma \in N} \gamma\).

**Proof.** In the first case, \(M\) is isomorphic to \(\mathcal{H} = \{H_\beta; H_\alpha \supseteq H_\beta\}\), in our notation.

If \(H_\alpha \neq \bigcap_{\beta \in M} H_\beta\), let \(x \in H\) be such that \(x \in \bigcap_{\beta \in M} H_\beta\), but \(\notin H_\alpha\). Then there exists an open-closed set \(U_x \subseteq H\) such that \(x \in U_x \supseteq H_\alpha\) [8], as \(H_\alpha\) is open and \(L\) is non-atomic. Hence \(H_\beta \supseteq U_x\) is open and closed and \(\mathcal{H}\), but \(x \notin H_\beta - U_x\), which implies a contradiction.

The dual can be proved similarly.

**Corollary 2.2.** If \(L\) is as in Corollary 2.1, then \(x, y \in L\) and \(x < y \Rightarrow\) there exists \(t \in L\) such that \(x < t < y\).

**Proof.** Let \(z \in H_y - H_x\), with is non-empty. Then there exists an open-closed set \(U_z \subset H\) such that \(z \in U_z \supseteq H_y - H_x\), as in Corollary 2.1. If \(H_t = H_x \cup U_z\), then there exists \(t \in L\) such that \(t \rightarrow H_t\) and hence \(x < t < y\).

**Corollary 2.3.** If \(L\) is as in Corollary 2.1 and \(x, y \in L\) be incomparable, then there exists \(p \in L\) such that \(p > x\) but \(p\) and \(y\) are incomparable.

3. **Hausdorff Separation Axiom**

In the following, \(L\) is assumed to be completely distributive, in addition and \(x_p\), the support of the \(L\)-fuzzy point \(p \in X\). The \(L\)-fuzzy topological space \((X, \tau)\) is defined exactly as in [6] and is written as \(L-f\)ts, for short.

The following theorem is obvious.
THEOREM 3.1. Closed subsets of a compact (Lindelof) L-fts are compact (Lindelof).

We now define the Hausdorff separation axiom in L-fts's as follows:

DEFINITION 3.1. \((X, \tau)\) is defined to be Hausdorff \((L\cdot F\cdot T_2)\) iff the following conditions are satisfied:

Let \(p, q\) be two points belonging to \(X\).

(I) If \(x_p \neq x_q\), then there exist two open sets \(U_p\) and \(U_q\), such that
\[
    p \in U_p, \quad \mu_q(x_q) > \mu_{U_p}(x_q)
\]
and
\[
    q \in U_q, \quad \mu_p(x_p) > \mu_{U_q}(x_p).
\]

(II) Let \(x_p = x_q\).

(i) If \(\mu_p(x_p) < \mu_q(x_p)\), then there exists \(U_p \in \tau\) such that \(p \in U_p\) but \(\mu_{U_p}(x_p) < \mu_q(x_p)\).

(ii) If \(\mu_p(x_p)\) and \(\mu_q(x_p)\) are incomparable, and if either

\[
    \text{a) } \mu_p(x_p) \land \mu_q(x_p) \neq 0,
\]

or

\[
    \text{b, a) } \mu_p(x_p) = \mu_q(x_p),
\]

then there exist \(U_p \in \tau\) and \(U_q \in \tau\) such that \(p \in U_p\), \(q \notin U_p\) and \(q \in U_q\), \(p \notin U_q\). But if in (ii) we have

\[
    \text{b, b) } \mu_p(x_p) < \mu_q(x_p),
\]

then a condition identical to Case II(i) holds.

DEFINITION 3.2. \((X, \tau)\) is \(L\cdot F\cdot T_1\), if singletons are closed.

From this we get the following theorem straightaway.

THEOREM 3.2. \((X, \tau)\) is \(L\cdot F\cdot T_2 \Rightarrow (X, \tau)\) is \(L\cdot F\cdot T_1\).

Proof. First let \(p\) be a crisp point and \(q \in A - \{p\}\) arbitrary. If \(q\) is any fuzzy point with support \(x_q\), then there exists \(U_{pq} \subset \tau\) such that
\[
    \mu_{U_{pq}}(x_p) = 1 \quad \text{and} \quad \mu_q(x_q) > \mu_{U_{pq}}(x_q),
\]

as \((X, \tau)\) is \(L\cdot F\cdot T_2\).
Considering all points with support \( x_q \), we get a collection \( \{ \bar{U}_{pq} \} \) of closed sets, such that

\[
\bigwedge \mu_{\bar{U}_{pq}}(x_p) = 1 \quad \text{and} \quad \bigwedge \mu_{\bar{U}_{pq}}(x_q) = 0,
\]

by Corollary 2.1.

So if \( P_q = \bigcap \bar{U}_{pq} \), then \( P_q \) is open and \( \mu_{P_q}(x_p) = 0 \) and \( \mu_{P_q}(x_q) = 1 \).

Hence

\[
\bigvee _{p \in L - \{ p \}} \mu_{P_q}(x_r) = 0, \quad \text{for} \quad r = p,
\]

\[
= 1, \quad \text{otherwise}.
\]

If now \( P = \bigcup P_q \), then \( P \) is a closed set identical with \( p \).

If \( p \) is fuzzy, let \( \{ p_i, i \in \mu \} \) be the collection of all fuzzy points with support \( x_p \) and value \( \mu_p(x_p) \). Then, since \((X, \tau)\) is Hausdorff, there exists a collection \( \{ U_{p_i}, i \in \mu \} \) of open sets such that

\[
\mu_{p_i}(x_p) < \mu_{U_{p_i}}(x_p) \leq \mu_{\bar{U}_{p_i}}(x_p) < \mu_p(x_p).
\]

Therefore, \( \mu_p(x_p) = \sup_{i \in \mu} \mu_{U_{p_i}}(x_p) \), by Corollary 2.1. If \( B = \bigcup_{i \in \mu} U_{p_i} \cup P \), then

\[
\mu_B(x_r) = \mu_p(x_p), \quad \text{for} \quad r = p,
\]

\[
= 1, \quad \text{otherwise},
\]

\( \Rightarrow P' (= B) \) is open \( \Rightarrow p \) is closed.

The following theorem shows that the definition of compactness given in [9] does not seem very natural in \( L \)-\( F \)-\( T_2 \)-spaces, as was the case in [7] also.

**Theorem 3.3.** No subset of an \( L \)-\( F \)-\( T_2 \)-space can be compact.

**Proof.** Let \( A \subset X \) be such that \( \mu_A(x_p) > 0 \) for some \( p \in A \). Let \( \{ p_i, i \in \mu \} \) be the collection of all fuzzy points with support \( x_p \) and value \( \mu_p(x_p) \). Then as in Theorem 3.2, we can form a collection \( \{ U_{p_i}, i \in \mu \} \) of open sets such that

\[
\mu_A(x_p) = \sup_{i \in \mu} \mu_{U_{p_i}}(x_p).
\]

Now \( P \cup \{ U_{p_i}, i \in \mu \} \) is clearly an open cover of \( A \) which does not have any finite subcover, where \( P \) is as in Theorem 3.2.

We therefore define proper open cover as in [7], as follows:

**Definition 3.3.** A collection \( \mathcal{U} = \{ U_\alpha \in \tau, \alpha \in \mu \} \) is defined to be a proper open cover of the set \( A \subset X \) if for each \( p \in A \), there exists \( U_{\alpha_p} \in \mathcal{U} \), such that \( \mu_{U_{\alpha_p}}(x_p) \geq \mu_A(x_p) \).
DEFINITION 3.4. \( A \subseteq X \) is defined to be *properly compact* if each proper open cover of \( A \) has a proper open finite subcover.

We now get the following theorem:

**Theorem 3.4.** *Properly compact sets in an L-F-\( T_2 \)-space are closed.*

**Proof.** Although this is somewhat similar to [7], we give a complete version of it as applied to L-F-\( T_2 \)-spaces.

Let \( A \subseteq X \) be properly compact and \((X, \tau)\) an L-F-\( T_2 \)-space. If \( p \in X \) be such that

\[
\mu_A(x_p) > \mu_A(x_p),
\]
then there exists \( U_{x_p} \in \tau \), such that

\[
\mu_A(x_p) < \mu_{U_{x_p}}(x_p) \quad \text{and} \quad \mu_{U_{x_p}}(x_p) > \mu_{U_{x_p}}(x_p). \tag{2}
\]

Hence for each \( p \) satisfying (1), there exists a collection \( \{U_{pq}, q \in A\} \) of open sets such that

\[
\mu_A(x_q) < \mu_{U_{pq}}(x_q) \quad \text{and} \quad \mu_{U_{pq}}(x_q) > \mu_{U_{pq}}(x_q). \tag{3}
\]

If, however, \( \mu_A(x_q) = 1 \), then we must have

\[
\mu_A(x_q) = \mu_{U_{pq}}(x_q). \tag{4}
\]

But this implies that

\[
\mu_A(x_q) < \sup_{q \in A} \mu_{U_{pq}}(x_q) \quad \text{and} \quad \mu_{U_{pq}}(x_q) > \mu_{U_{pq}}(x_q). \tag{5}
\]

as \( A \) is properly compact,

\[
\Rightarrow \mu_A(x_q) \leq \sup_{q \in A} \mu_{U_{pq}}(x_q). \tag{6}
\]

If \( \bigcup_{k=1}^{n} U_{pq_k} = F_p \), then (6) reduces to,

\[
\mu_A(x_p) \leq \mu_{F_p}(x_p). \tag{7}
\]
This if of course accompanied by

\[ \mu_p(x_p) \geq \mu_{F_p}(x_p). \]  \hspace{1cm} (8)

Considering now all points \( p \in X \) satisfying (1), we get the collection \( \{F_p\} \) of closed sets, satisfying both (7) and (8). So \( \mu_A(x_p) = \inf \mu_{F_p}(x_p) \Rightarrow A \) is closed.

Remark. It should be noted here that the non-atomicity of \( L \) is essential in arriving at this result.

Definition 3.5. An \( L \)-fts is called an \( L \)-fuzzy \( P \) space if countable union of closed sets is closed.

We therefore obtain,

Theorem 3.5. Lindelof sets of a Hausdorff \( L \)-fuzzy \( P \) space are closed.

As in \([7]\), we easily get:

Theorem 3.6. Compact sets in an \( L \)-\( F\)-\( T_1 \), \( P \)-space have finite supports.

Corollary 3.1. Properly compact sets in an \( L \)-\( F\)-\( T_2 \), \( P \)-space have finite supports.

4. Other Separation Properties

In this section we consider some of the other separation properties of \( L \)-fts's. These are found to have stronger similarities to the corresponding ideas in ordinary topological spaces than what the ordinary fuzzy spaces have.

Definition 4.1. \((X, \tau)\) is regular if for each point \( p \in X \) and for each \( U \in \tau \) such that \( p \in U \), there exists \( G_p \in \tau \), where

\[ p \in G_p \subset \bar{G}_p \subset U. \]

\((X, \tau)\) is \( L \)-\( F\)-\( T_3 \) if it is \( L \)-\( F\)-\( T_1 \) and regular.

Definition 4.2. \((X, \tau)\) is normal if for each \( A \subset X \) and for each \( U \in \tau \) where \( A \subset U \), there exists \( G_A \in \tau \) such that

\[ A \subset G_A \subset \bar{G}_A \subset U. \]

\((X, \tau)\) is \( L \)-\( F\)-\( T_4 \) if it is \( L \)-\( F\)-\( T_1 \) and normal.

So we obtain:
THEOREM 4.1. $(X, \tau)$ is L-F-T $\Rightarrow (X, \tau)$ is L-F-T$_3$.

THEOREM 4.2. $(X, \tau)$ is L-F-T $\Rightarrow (X, \tau)$ is L-F-T$_2$.

Proof. Let $p, q$ be two points belonging to $X$.

(I) Let $x_p \neq x_q$.

Then $q \in P \in \tau$, where $P$ is as in Theorem 3.2.

So there exists $U_q \in \tau$ such that $q \in U_q \subset \bar{U}_q \subset P$, where $\mu_p(x_p) > \mu_{\bar{t}_q}(x_p)$ certainly, as

$$\mu_{\bar{t}_q}(x_p) = \mu_p(x_p) = 0.$$

(II) Let now $x_p = x_q$.

If $\mu_p(x_p) < \mu_q(x_p)$, then consider the point $p_1$ with support $x_p$ and value $t'$, where $\mu_p(x_p) < t < \mu_q(x_p)$ (Corollary 2.2).

Since $p_1'$ is open, there exists $U_p \in \tau$ such that

$$p \in U_p \subset \bar{U}_p \subset p_1'.$$

That is,

$$\mu_{\bar{t}_p}(x_p) \leq \mu_{p_1}(x_p) < \mu_q(x_p).$$

For both the cases II(ii)(a) and II(ii)(b, $\alpha$) of Definition 3.1, let $p_1$ be the point with support $x_p$ such that $\mu_p(x_p) < \mu_{p_1}(x_p)$ and $\mu_q(x_p), \mu_{p_1}(x_p)$ are incomparable (Corollary 2.3).

Hence there exists $U_p \in \tau$ such that

$$p \in U_p \subset \bar{U}_p \subset p_1' \Rightarrow q \notin \bar{U}_p.$$

If, however, $\mu_p(x_p) < \mu_q(x_p)$ (i.e., Case II(ii)(b, $\beta$) of Definition 3.1), then $p_1$ is to be so chosen that

$$\mu_p(x_p) < \mu_{p_1}(x_p) < \mu_q(x_p).$$

This completes the proof.

THEOREM 4.3. Let $(X, \tau)$ be an L-fits. Then the following are equivalent:

(i) $(X, \tau)$ is regular.

(ii) If $A \subset X$ be closed and $p \in A'$ then there exist open sets $G_p$ and $G_A$ such that $p \in G_p$ and $A \subset G_A$, where $G_p \cap G_A = \phi$.

THEOREM 4.4. Let $(X, \tau)$ be an L-fits. Then the following are equivalent.

(i) $(X, \tau)$ is normal.
(ii) If $P$ and $Q$ are closed sets satisfying $P \cap Q = \phi$, then there exist open sets $U_P$ and $U_Q$ such that $P \subseteq U_P$ and $Q \subseteq U_Q$, where $U_P \cap U_Q = \phi$.

**Theorem 4.5.** Let $(X, \tau)$ be an L-fts. Then the following are equivalent:

(i) Conditions (I) or II(ii, $\alpha, \beta$) of Definition 3.1 hold.

(ii) For points $p, q \in X$ and satisfying $p \in q'$, there exist open sets $U_p$ and $U_q$ such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \phi$.

**Proof:** (i) $\Rightarrow$ (ii).

Let $x_p \neq x_q$. Consider the point $t$ with support $x_q$ and value $\mu_q(x_q)$. By (I) of Definition 3.1, there exists $U_p \in \tau$ such that

$$\mu_p(x_p) < \mu_{U_p}(x_p)$$

and

$$\mu_q(x_q) > \mu_{U_q}(x_q) \Rightarrow \mu_q(x_q) < \mu_{U_q}(x_p).$$

Since $U_p \cap U_q = \phi$, the result follows.

If $x_p = x_q$ and $\mu_p(x_p) < \mu_q(x_p)$, then there exists $U_p \in \tau$, such that

$$\mu_p(x_p) < \mu_{U_p}(x_p) \leq \mu_{T_p}(x_p) < \mu_q(x_p),$$

by II(i) of Definition 3.1.

(ii) $\Rightarrow$ (i).

Let $x_p \neq x_q$. If $t$ is as above, then $p \in t'$. So there exist $U_p, U_t \in \tau$ such that $p \in U_p$, $t \in U_t$ and $U_p \cap U_t = \phi$. Therefore, $\mu_q(x_q) > \mu_{U_q}(x_q) \geq \mu_{U_p}(x_q) \Rightarrow \mu_q(x_q) > \mu_{U_t}(x_q)$, as $U_t$ is closed. If $x_p = x_q$ and $\mu_p(x_p) < \mu_q(x_p)$, then $\mu_p(x_p) < \mu_{U_t}(x_p) < \mu_q(x_p)$.

**Remark.** It is interesting to note that Theorems 4.3–4.5 establish a sort of similarity between L-fts’s and ordinary topological spaces, so far as these separation properties are concerned.

We now modify the definition of $A \subset U$ as follows:

**Definition 4.3.** $A \subset U$ iff $\mu_A(x_p) < \mu_U(x_p)$, for each $p \in A$, for which $\mu_A(x_p) < 1$, where $A$ and $U$ are subsets of $X$.

Using this definition, let us enunciate the following:

**Definition 4.4.** $(X, \tau)$ is properly regular iff for each $p \in X$, the condition $p \in U \Rightarrow p \in G_p \subset G_p \subset U$, for some $G_p \in \tau$, where $U \in \tau$ is arbitrary.

**Definition 4.5.** $(X, \tau)$ is properly normal iff for each closed set $A \subset X$,
the condition \( A \subset U \Rightarrow A \subset G_A \subset \bar{G}_A \subset U \), for some \( G_A \in \tau \), where \( U \in \tau \) is arbitrary.

We are now in a position to prove the following theorems.

**Theorem 4.6.** A compact, properly regular \( L \)-fts is properly normal.

**Proof.** Let \( A \subset X \) be closed and \((X, \tau)\) compact. Let \( A \subset U \in \tau \). If \( \mu_A(x_p) < 1 \), then by Definition 4.4, there exists \( U_{x_p} \in \tau \), such that

\[
\mu_A(x_p) < \mu_{U_{x_p}}(x_p) < \mu_{U_{x_p}}(x_p) < \mu_U(x_p).
\]

The rest of the proof is straightforward, as \( A \) is compact (Theorem 3.1).

**Theorem 4.7.** A properly regular Lindelof \( P \)-space is properly normal.

5. **L-Fuzzy Subspaces**

We now define a topology on \( A \subset X \) (as inherited from the topology \( \tau \) of \( X \)), in the line of [4].

**Definition 5.1.** Let \((X, \tau)\) be an \( L \)-fts and \( A \subset X \). Then the collection \( \tau_A = \{U \cap A : U \in \tau\} \) forms an \( L \)-fuzzy topology on \( A \), called the \( L \)-fuzzy subspace topology on \( A \). Members of \( \tau_A \) are the open subsets of \( A \).

**Definition 5.2.** \( B \subset A \) is the complement of \( C \subset A \) in \( A \), iff \( B \cap C = \emptyset \) and \( B \cup C = A \). We write this as \( C' = B \).

**Definition 5.3.** \( F \subset A \) is closed in \( A \) iff \( F = U A \), where \( U \in \tau_A \).

Therefore, we obtain

**Proposition 5.1.** \( F \subset A \) is closed in \( A \) iff \( F = A \cap K \) where \( K' \in \tau \).

**Proposition 5.2.** If \( A \subset X \) is closed, then \( F \subset A \) is closed in \( A \) \( \Rightarrow F \) is closed in \( X \) also.

**Definition 5.4.** A topological property \( P \) is said to be hereditary iff \((X, \tau)\) possesses \( P \Rightarrow (A, \tau_A) \) also possesses \( P \), where \( A \subset X \) is arbitrary.

The question that naturally comes after this is whether the separation properties are hereditary. The following theorems give the answer:

**Theorem 5.1.** \( L \)-\( F \)-\( T_1 \) is a hereditary property.
THEOREM 5.2. \( L\text{-}F\text{-}T_2 \) is a hereditary property.

\textit{Proof}: Let \((X, \tau)\) be an \(L\text{-}F\text{-}T_2\)-space and \(A \subseteq X\) arbitrary.
Let \(p, q \in A\).

(I) If \(x_p \neq x_q\), then let \(t \in A\) be such that \(\mu_t(x_p) < \mu_t(x_p)\) (Corollary 2.2), where \(x_p\) is the support of \(t\) also.
Since \((X, \tau)\) is \(L\text{-}F\text{-}T_2\), there exist \(U_p \in \tau\) and \(U_q \in \tau\) such that
\[
\mu_t(x_p) < \mu_{U_p}(x_p) \quad \text{and} \quad \mu_t(x_q) > \mu_{U_q}(x_q),
\]
\((\alpha)\)
together with
\[
\mu_q(x_q) < \mu_{U_q}(x_q) \quad \text{and} \quad \mu_t(x_p) > \mu_{U_q}(x_p).
\]
\((\beta)\)
Relations \((\alpha)\) imply
\[
\mu_t(x_p) \leq (\mu_{U_p} \land \mu_A)(x_p)
\]
So if
\[
V_p = U_p \cap A,
\]
then
\[
\mu_p(x_p) < \mu_{V_p}(x_p)
\]
and
\[
\mu_q(x_q) > \mu_{\overline{V_p \cap A}}(x_q) \geq \mu_{\overline{U_p \land A}}(x_p) = \mu_{\overline{V_p}}(x_p),
\]
where \(\overline{V_p}\) denotes the closure of \(V_p\) in \((A, \tau_A)\).

(II) Now let \(x_p = x_q\), but let II(ii)(a) or II(ii)(b, \(a\)) hold.
Then there exists a point \(t \in A\) with support \(x_p\) such that \(\mu_t(x_p) < \mu_t(x_p)\) and \(\mu_t(x_p), \mu_q(x_p)\) are not comparable.
Therefore, there exists \(U_t \in \tau\) such that \(\mu_t(x_p) < \mu_{U_t}(x_p)\) and \(\mu_q(x_p), \mu_{\overline{U_t}}(x_p)\) are not comparable, as \((X, \tau)\) is \(L\text{-}F\text{-}T_2\).
Taking intersection with \(A\), the result follows.
The other cases are straightforward.

THEOREM 5.3. \((X, \tau)\) is regular \(\Rightarrow\) \((A, \tau_A)\) is also regular, where \(A \subseteq X\) is arbitrary.

COROLLARY 5.1. \((X, \tau)\) is \(L\text{-}F\text{-}T_3 \Rightarrow \) \((A, \tau_A)\) is also \(L\text{-}F\text{-}T_3\).

The observation regarding normality is not as general as the others, as in ordinary topology. The corresponding theorem is as follows.
Theorem 5.4. \((X, \tau)\) is normal \(\Rightarrow (A, \tau_A)\) is also normal, where \(A \subset X\) is an arbitrary closed subset.

Corollary 5.2. \((X, \tau)\) is L-F-T \(\Rightarrow (A, \tau_A)\) is also L-F-T, where \(A \subset X\) is closed.

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References