A Sharp Form of the Sobolev Trace Theorems*

RIDGWAY SCOTT

Applied Mathematics Department, Brookhaven National Laboratory,
Upton, New York 11973

Communicated by J. L. Lions

Received October 1975; revised November 1975

Trace theorems for Besov spaces are proved using piecewise polynomial approximation theory and the K-method of interpolating Banach spaces. These theorems are limiting cases of standard embedding results.

In this paper we give a new proof of some trace theorems for the Besov spaces, $B_p^{k,q}(\mathbb{R}^n)$ (cf. [3]). The first example of these is due to Peetre [10], who proved that $B_p^{k,p,1}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$, referring to the result as Bernstein's theorem. Recently, Agmon and Hörmander [1] showed that the restrictions of functions in $B_p^{k,p,1}(\mathbb{R}^n)$ to a submanifold $M \subset \mathbb{R}^n$ of codimension $k$ lie in $L_p(M)$. (Note that Peetre's result formally corresponds to the case $k = n$.) These theorems may be viewed as limiting cases of standard embedding results (cf. the survey by Nikol'skii [7]). Here we prove both results using approximation theory applied to interpolation theory, thus reversing the usual roles of these theories.

The plan of the paper is as follows: Section 1 reviews the interpolation theory [5] to be used, noting that the Besov spaces are interpolation spaces between Sobolev spaces. In Section 2, trace theorems are proved for Sobolev interpolation spaces, and, in Section 3, these theorems are proved to be sharp. Both sections rely on approximation theorems for piecewise polynomials [4]. Section 3 also uses a new approximation theorem (Theorem 3) first discovered by Babuška and Kellogg [2], which we present in a generalized form. In Section 4, the previous theorems are translated to the language of Besov spaces, and their relation to standard embedding theorems is discussed.

* Work performed under the auspices of ERDA. By acceptance of this article, the publisher acknowledges the U.S. Government's right to retain a nonexclusive, royalty-free license in and to any copyright covering this paper.
1. Review of Interpolation Theory

Recall the $K$-method (or real method) of interpolation [3, 6, 8]. Let $B_0$ and $B_1$ be Banach spaces continuously included in a topological vector space $\mathcal{B}$. For $u \in B_0 + B_1$ and $t > 0$, define

$$K(t, u) = \inf \{ \| v_0 \|_{B_0} + t \| v_1 \|_{B_1} : v_i \in B_i, \ u = v_0 + v_1 \}. \quad (1.1)$$

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, denote by $[B_0, B_1]_{\theta,q}$ the Banach space with norm

$$\| u \|_{[B_0, B_1]_{\theta,q}} = \left( \int_0^\infty K(t, u)^q t^{-\theta q - 1} \, dt \right)^{1/q} \quad (1.2)$$

(with the usual modification when $q = \infty$). Then

$$B_0 \cap B_1 \subset [B_0, B_1]_{\theta_0,q_0} \subset [B_0, B_1]_{\theta_1,q_1} \subset B_0 + B_1, \quad (1.3)$$

with the inclusions continuous, provided that $\theta_0 > \theta_1$ or that $\theta_0 = \theta_1$ and $q_0 \leq q_1$.

**Proposition 1 (Density).** Suppose that $1 < q < \infty$. Then $B_0 \cap B_1$ is dense in $[B_0, B_1]_{\theta,q}$.

**Proposition 2 (Duality).** Suppose that $1 \leq q < \infty$ and that $B_0 \cap B_1$ is dense in $B_0$ and $B_1$. Then

$$[B_0, B_1]^*_{\theta,q} \simeq [B_0^*, B_1^*]_{\theta,q'},$$

where $q' = q/q - 1$.

**Proof.** These are Théorème (2.1) and Théorème (3.1), respectively, in chapitre III of Lions and Peetre [6], except that the definition of interpolation space used here is different. The equivalence of the two definitions is proved in Peetre [8].

**Proposition 3 (Iteration).** Suppose that $A_0$ and $A_1$ are Banach spaces such that for some $\theta_0 < \theta_1$,

$$[B_0, B_1]_{\theta_i,1} \subset A_i \subset [B_0, B_1]_{\theta_i,\infty}, \quad i = 0, 1.$$

Then $[A_0, A_1]_{\lambda,q} \simeq [B_0, B_1]_{\theta,\lambda \theta}$ with $\theta = (1 - \lambda) \theta_0 + \lambda \theta_1$.

**Proof.** This is Théorème 2 in Section 1 of Peetre [8]. See also Theorem 3.2.20 of Butzer and Berens [3].
Let \( W^k_p = W^k_p(\mathbb{R}^n) \) be the Sobolev space of functions with \( k \) derivatives in \( L_p(\mathbb{R}^n) \) (see, e.g. [11] for a precise definition). The norm in \( W^k_p \) is

\[
\| u \|_{W^k_p} = \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha u|^p \right)^{1/p}
\]

(only finite \( p \) will occur in this paper). The following will be used in conjunction with Proposition 3.

**Proposition 4.** Let \( j \) and \( k \) be integers with \( 0 < j < k \). Then

\[
[L_p, W^k_p]_{j/k, 1} \subset W^j_p \subset [L_p, W^k_p]_{j/k, \infty}.
\]

**Proof.** See Example 1.2 in Peetre [9].

We will be interested in a precise characterization of the spaces that arise from interpolating Sobolev spaces. For \( h \in \mathbb{R}^n \), let \( \Delta_h \) denote the difference operator

\[
\Delta_h u(x) = u(x + h) - u(x).
\]

The Besov space \( B^s_{p,q}(\mathbb{R}^n) \) is the Banach space with norm

\[
\| u \|_{B^s_{p,q}} + \left( \int_{\mathbb{R}^n} \| \Delta_h^{|s|} u \|_{L_p}^q \frac{dh}{|h|^n} \right)^{1/q},
\]

where \([s]\) is the smallest integer greater than \( s \) (and \( s \) may be any nonnegative real number).

**Proposition 5.** \( [L_p, W^k_p]_{\theta,q} \simeq B^s_{p,q} \).

**Proof.** See Theorem 4.3.6 of Butzer and Berens [3].

We collect here some standard notation that will be used later. Denote by \( C_0^0(\mathbb{R}^n) \) the Banach space of continuous functions that vanish at infinity, endowed with the supremum norm. Let \( C_c^\infty(\mathbb{R}^n) \) denote the space of infinitely differentiable functions with compact support. The dual space of \( C_c^\infty(\mathbb{R}^n) \) is the space of distributions, and this duality will be denoted by \( \langle , \rangle \).

### 2. Main Results

For \( x \in \mathbb{R}^n \), let \( \delta_x \) denote the Dirac distribution at \( x \):

\[
\langle \delta_x, \varphi \rangle = \varphi(x) \quad \text{for} \quad \varphi \in C_c^\infty(\mathbb{R}^n).
\]
LEMMA 1. There is a constant $C_n$ depending only on $n$ such that, for any $x \in \mathbb{R}^n$ and $t > 0$, there is a bounded function $\delta_x^t$ with compact support such that for $1 \leq p \leq \infty$,
\[
t^{-1/p'} \| \delta_x - \delta_x^t \|_{W_p^n(\mathbb{R}^n)} + t^{1/p} \| \delta_x^t \|_{L^p(\mathbb{R}^n)} \leq C_n,
\]
where $p' = p/p - 1$.

COROLLARY. If $1 \leq p \leq \infty$, then $\delta_x \in [L_p(\mathbb{R}^n), W_p^n(\mathbb{R}^n)^*]_{1/p, \infty}$ and
\[
\| \delta_x \|_{W_p^n(\mathbb{R}^n)^*} \leq C_n \quad \text{for all } x \in \mathbb{R}^n,
\]
where $C_n$ is the constant in Lemma 1.

Proof of Lemma 1. Denote by $\mathcal{P}_n$ the space of polynomials of degree $n - 1$ in $n$ variables, and let $Q \in \mathcal{P}_n$ be such that $\int_{|y| \leq 1} Q(y) P(y) dy = P(0)$ for all $P \in \mathcal{P}_n$. Let $x \in \mathbb{R}^n$ and $t > 0$. Define
\[
\delta_x^t(y) = t^{-1} Q \left( \frac{y - x}{t^{1/n}} \right) \quad \text{for } |y - x| < t^{1/n},
\]
and
\[
= 0 \quad \text{for } |y - x| > t^{1/n}.
\]
Note that $\langle \delta_x - \delta_x^t, P \rangle = 0$ for $P \in \mathcal{P}_n$ and that, with $c_0 = 2^n \sup_{|y| \leq 1} |Q(y)|$,
\[
\| \delta_x^t \|_{L^p(\mathbb{R}^n)} \leq c_0 t^{-1/p}.
\]
(2.1)
It is well known (cf. [4]) that there exists a linear map $\mathcal{I}: W_p^n(\mathbb{R}^n) \rightarrow \mathcal{P}_n$ such that, for all $\varphi \in W_p^n(\mathbb{R}^n)$, $\mathcal{I}\varphi(x) = \varphi(x)$ and
\[
\| \varphi - \mathcal{I}\varphi \|_{L_p(\{y-x\leq t^{1/n}\})} \leq c t \| \varphi \|_{W_p^n(\{y-x\leq t^{1/n}\})},
\]
where $c$ is a constant depending only on $n$ ($\mathcal{I}\varphi$ interpolates $\varphi$ at various points in $\{|y - x| \leq t^{1/n}\}$). Thus for $\varphi \in W_p^n(\mathbb{R}^n)$,
\[
\langle \delta_x - \delta_x^t, \varphi \rangle = \langle \delta_x - \delta_x^t, \varphi - \mathcal{I}\varphi \rangle
\]
\[
= -\langle \delta_x^t, \varphi - \mathcal{I}\varphi \rangle
\]
\[
\leq \| \delta_x^t \|_{L^p(\{y-x\leq t^{1/n}\})} \cdot \| \varphi - \mathcal{I}\varphi \|_{L_p(\{y-x\leq t^{1/n}\})}
\]
\[
\leq (c_0 c) t^{1-1/p} \| \varphi \|_{W_p^n(\mathbb{R}^n)}.
\]
Since $\varphi$ is arbitrary, this implies that
\[
\| \delta_x - \delta_x^t \|_{W_p^n(\mathbb{R}^n)^*} \leq (c_0 c) t^{1/p'}.
\]
Combined with (2.1), this proves the lemma, with $C_n = c_0 (1/c)$.  ☐
Theorem 1. Let $1 < p < \infty$. Then

$$[L_p(\mathbb{R}^n), W_p^n(\mathbb{R}^n)]_{1/p, 1} \subset C_0^0(\mathbb{R}^n)$$

with the inclusion map continuous.

Proof. Let $x \in \mathbb{R}^n$ and $\varphi \in W_p^n(\mathbb{R}^n)$. Then

$$I_{\varphi(x)} = \langle \delta_x, \varphi \rangle \leq \| \delta_x \|_{L_p, W_p^n(\mathbb{R}^n)} \| \varphi \|_{L_p, W_p^n(\mathbb{R}^n)}.$$  \hspace{1cm} (2.2)

By Proposition 2, we have $[L_p, W_p^n]_{1/p, 1} \simeq [L_p, (W_p^n)^*]_{1/p, \infty}$ so (2.2) and the corollary imply that, for some constant $c$,

$$\| \varphi \|_{L_p} \leq c \| \varphi \|_{L_p, W_p^n(\mathbb{R}^n)} \quad \text{for all} \quad \varphi \in W_p^n(\mathbb{R}^n).$$  \hspace{1cm} (2.3)

Now $W_p^n(\mathbb{R}^n)$ is contained in $C_0^0(\mathbb{R}^n)$ and, by Proposition 1, is dense in $[L_p, W_p^n]_{1/p, 1}$, so the theorem follows from (2.3) by continuity. \hspace{1cm} \Box

We now consider restrictions to linear submanifolds of $\mathbb{R}^n$. Let us denote the coordinates of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$ by $(x, y) = ((x_1, ..., x_{n-k}), (y_1, ..., y_k))$. Denote by $M$ the linear manifold $\{(x, y): y = 0\}$ of codimension $k$. For $\psi \in C_0^\infty(\mathbb{R}^n)$, define a distribution $\tilde{\psi}$ by

$$\langle \tilde{\psi}, f \rangle = \int_M \psi f \quad \text{for all} \quad f \in C_0^\infty(M).$$

Lemma 2. For $1 \leq p \leq \infty$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, $\tilde{\psi} \in [L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)^*]_{1/p, \infty}$ and

$$\| \tilde{\psi} \|_{L_p^*, (W_p^k)^*} \leq 2 C_k \| \psi \|_{L_p^*(M)},$$

where $p' = p/p - 1$ and $C_k$ is the constant in Lemma 1.

Proof. We must construct, for each $t > 0$, a function $\psi^t$ such that

$$t^{-1/p'} \| \tilde{\psi} - \psi^t \|_{L_p^*, (W_p^k)^*} = t^{1/p} \| \psi^t \|_{L_p^*(\mathbb{R}^n)} \leq 2 C_k \| \psi \|_{L_p^*(M)}. \hspace{1cm} (2.4)$$

Note that $\tilde{\psi} = \psi \mid M \otimes \delta_0$, so we take

$$\psi^t(x, y) = \psi(x, 0) \delta_0^t(y),$$

where $\delta_0^t$ is the approximation to $\delta_0$ from Lemma 1 in $k$ variables (i.e., in the $y$ variables). By Fubini's theorem

$$\| \psi^t \|_{L_p^*(\mathbb{R}^n)} = \int_M \psi(x, 0) \int_{\mathbb{R}^m} \delta_0^t(y)^{p'} dy dx \int_{\mathbb{R}^k} \psi(x, 0)^{p} dx,$$

so that by Lemma 1,

$$\| \psi^t \|_{L_p^*(\mathbb{R}^n)} \leq C_k t^{-1/p'} \| \psi \|_{L_p^*(M)}. \hspace{1cm} (2.5)$$
Let $\varphi \in W_p^k(\mathbb{R}^n)$. Then

$$\langle \tilde{\psi} - \psi', \varphi \rangle = \int_M \varphi(x, 0) \langle \delta_0 - \delta_0', \varphi(x, \cdot) \rangle \, dx$$

$$\leq \int_M \varphi(x, 0) C_k \delta_0^{1/p'} \| \varphi(x, \cdot) \|_{W_p^k(\mathbb{R}^n)} \, dx$$

$$\leq C_k \delta_0^{1/p'} \| \varphi \|_{L_p'(M)} \left( \int_M \| \varphi(x, \cdot) \|_{W_p^k(\mathbb{R}^n)}^p \, dx \right)^{1/p}$$

$$\leq C_k \delta_0^{1/p'} \| \varphi \|_{L_p'(M)} \| \varphi \|_{W_p^k(\mathbb{R}^n)}.$$

Since $\varphi$ was arbitrary, this proves that

$$\| \tilde{\psi} - \psi' \|_{W_p^k(\mathbb{R}^n)} \leq C_k \delta_0^{1/p} \| \varphi \|_{L_p'(M)}.$$

Combined with (2.5), this proves (2.4) and completes the lemma. \[ \]

**Theorem 2.** Let $1 < p < \infty$. The mapping $\varphi \mapsto \varphi | M$ extends by continuity to a continuous map of $[L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)]_{1/p, 1}$ to $L_p(M)$.

**Proof.** Because $W_p^k$ is dense in $[L_p, W_p^k]_{1/p, 1}$ (Proposition 1), we need only show that for all $\varphi \in W_p^k(\mathbb{R}^n)$,

$$\| \varphi \|_{L_p(M)} \leq c \| \varphi \|_{[L_p, W_p^k]_{1/p, 1}}. \tag{2.6}$$

To do so, note that

$$\| \varphi \|_{L_p(M)} = \sup_{\psi \in C_c^\infty(\mathbb{R}^n)} \int_M \varphi \psi \| \psi \|_{L_p'(M)}. \tag{2.7}$$

By Proposition 2 and Lemma 2, there is a constant $c < \infty$ such that for $\psi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_M \varphi \psi = \langle \tilde{\psi}, \varphi \rangle$$

$$\leq \| \tilde{\psi} \|_{L_p, W_p^k_{1/p, 1}} \| \varphi \|_{[L_p, W_p^k]_{1/p, 1}}$$

$$\leq c \| \psi \|_{L_p'(M)} \| \varphi \|_{[L_p, W_p^k]_{1/p, 1}}.$$

This combines with (2.7) to prove (2.6) and completes the theorem.

3. **Complement to Theorems 1 and 2**

We begin with some general preliminaries concerning the application of interpolation theory to approximation theory. Let $B_0, B_1$ be two Banach spaces, with $B_1 \subset B_0$ continuously. A family of approximating subspaces $\{S_t: 0 < t \leq 1\}$
for $B_0$, $B_1$ is a family of subspaces $S^t \subset B_0$ with the property that for all $u \in B_1$,

$$\inf_{v \in S^t} \| u - v \|_{B_0} \leq t \| u \|_{B_1} \quad \text{for} \quad 0 < t \leq 1. \quad (3.1)$$

**Example.** Divide $\mathbb{R}^n$ uniformly into cubes of side $h$, and let $S_k^h$ be the linear subspace of $L_p(\mathbb{R}^n)$ consisting of functions whose restriction to each cube is a polynomial of degree $k - 1$. It is well known (cf. [4]) that for $1 \leq p \leq \infty$,

$$\inf_{v \in S_k^h} \| u - v \|_{L_p(\mathbb{R}^n)} \leq c h^k \| u \|_{W_p^k(\mathbb{R}^n)} \quad (3.2)$$

for all $u \in W_p^k(\mathbb{R}^n)$. Defining $S^t = S_k^h$ with $h = (t/c)^{1/k}$, we thus have a family of approximating subspaces for the pair $L_p$, $W_p^k$.

Estimate (3.1) says that each $u \in B_1$ may be approximated to order 1 (with respect to the parameter $t$) from the subspaces $\{S^t\}$. With only the assumption that $u \in B_0$, we may still assert that $u$ is approximated to order 0, and hence for $u \in [B_0, B_1]_{\theta, q}$ one may well expect that $u$ is approximated to some order between 0 and 1 depending on $\theta$ and $q$. In case $B_0$ is a Hilbert space,

$$\inf_{v \in S^t} \| u - v \|_{B_0} = \| u - P_t u \|_{B_0},$$

where $P_t$ is the orthogonal projection of $B_0$ onto $S^t$. The norm of $I - P_t$, as a map of $B_1$ to $B_0$, is at most $t$, and its norm as a map of $B_0$ to $B_0$ is at most 1; hence, by the operator interpolation property [3, 6], the norm of $I - P_t$ as a map of $[B_0, B_1]_{\theta, q}$ to $B_0$ is at most $c t^\theta$. That is,

$$\inf_{v \in S^t} \| u - v \|_{B_0} \leq c t^\theta \| u - v \|_{[B_0, B_1]_{\theta, q}} \quad \text{for} \quad 0 < t \leq 1. \quad (3.3)$$

In fact, this is true for $B_0$ a Banach space, and for $1 \leq q \leq \infty$. However, when $q$ is finite, a more striking estimate is valid which indicates that, for $u \in [B_0, B_1]_{\theta, q}$, the approximation error is at most $o(t^\theta)$, rather that $O(t^\theta)$. This was discovered by Babuska and Kellogg [2] in the context of Sobolev spaces. We give below a statement and proof of the result in the context of general Banach spaces.

**Theorem 3.** Let $B_0 \subset B_1$ continuously and let $\{S^t\}$ be a family of approximating subspaces for $B_0$, $B_1$. For $u \in B_0$, define

$$E(t, u) = \inf_{v \in S^t} \| u - v \|_{B_0}. \quad (3.4)$$

If $1 \leq q < \infty$ and $u \in [B_0, B_1]_{\theta, q}$, then

$$\lim_{t \to 0} E(t, u)/t^\theta = 0,$$
and in fact

\[(\int_0^1 (E(t, u)/t^\theta)^q \, dt/t)^{1/q} \leq \| u \|_{[B_0, B_1]_{q, q}}.\]  \hspace{1cm} (3.5)

**Proof.** Estimate (3.5) follows from the observation that

\[E(t, u) \leq K(t, u) \quad \text{for all } u \in B_0 \text{ and } 0 < t \leq 1,\]  \hspace{1cm} (3.6)

where \(K\) is the function used to define the interpolation norms (see Section 1). To prove (3.6), let \(w \in B_1\), then

\[\inf_{v \in S_t} \| u - v \|_{B_0} \leq \| u - w \|_{B_0} + \inf_{v \in S_t} \| w - v \|_{B_0}\]

\[\leq \| u - w \|_{B_0} + t \| w \|_{B_1}\]

by (3.1). Taking the infimum over \(w \in B_1\) proves (3.6).

To prove (3.4), we would like to appeal to the fact that

\[E(t, u)t^{-\theta} \in L_0(0, \infty; dt/t).\]

However, not every such function tends to zero as \(t \to 0\). Thus we need

\[K(t, u) t^{-\theta} \leq \left((1 - \theta)q \int_0^t (K(s, u)s^\theta) t^{s}/s \, ds/s\right)^{1/q}.\]  \hspace{1cm} (3.7)

With this and (3.6), we have \(E(t, u)t^{-\theta}\) dominated by the tail of a convergent integral, proving (3.4). To prove (3.7), we observe that, for any \(u \in B_0\), \(K(t, u)t^{-1}\) is a decreasing function of \(t\). Thus

\[(K(t, u)t^\theta)^q = (K(t, u)t^\theta)(1 - \theta)q \int_0^t s^{(1-\theta)\alpha-1} \, ds\]

\[\leq (1 - \theta)q \int_0^t (K(s, u)s^\theta) t^{s}/s \, ds/s,

proving (3.7). This completes the proof of Theorem 3.

We now use this refined approximation theorem to prove the following complement to Theorem 1.

**Theorem 4.** Suppose \(1 < p < \infty\). If \(q > 1\), then \([L_p, W^{\alpha, q}_{1/p, q}]\) is not continuously contained in \(C_0\), i.e.,

\[
\sup_{\varphi \in C_0 \cap [L_p, W^{\alpha, q}_{1/p, q}]} \frac{\| \varphi \|_{C_0}}{\| \varphi \|_{[L_p, W^{\alpha, q}_{1/p, q}]} - \infty.}
\]
Proof. Suppose not. We may assume that $q$ is finite in view of (1.3). Since $W^n_p \subset C^0$ is dense in $[L_p, W^n_p]_{1/p,q}$, we have $\delta_0 \in [L_p, (W^n_p)_1/p,q']$, where $q' = q/q - 1$, etc. Suppose that $n$ is even, and let $g$ solve $(1 - \Delta)^{n/2} g = \delta_0$. By elliptic regularity (cf. [5]) and the operator interpolation property [3, 6], we have $g \in [W^n_p, L_p]_{1/p,q'} = [L_p, W^n_p]_{1/p,q'}$. From Theorem 3 and the example, we have

$$\inf_{v \in S^n} \| g - v \|_{L_p} = o(h^{n/p'}) \quad \text{as} \quad h \to 0.$$ 

In the example, choose the mesh so that the origin is in the center of one of the cubes. We have $g \sim \log |x|$ near the origin, so the following lemma yields the desired contradiction.

**Lemma 3.** Let $\mathcal{P}_n$ denote the space of polynomials of degree $n - 1$ in $n$ variables. Let $1 \leq p < \infty$. Then there is a positive constant $c = c(n, p)$ such that

$$\inf_{P \in \mathcal{P}_n} \int_{|x| \leq h} | log |x| - P(x)|^p dx = ch^n \quad \text{for all} \quad h.$$ 

**Proof of Lemma 3.** Let $P_h$ minimize $\int_{|x| \leq h} | log |x| - P(x)|^p dx$ over $\mathcal{P}_n$. Then $P_h$ is the unique solution to

$$\int_{|x| \leq h} (log |x| - P_h(x)) | log |x| - P_h(x)|^{p-2} Q(x) dx = 0 \quad \text{for all} \quad Q \in \mathcal{P}_n.$$ 

The change of variables $x \to x/h$ in this equation shows that $P_h(x) = P_1(x/h) + \log h$. Changing variables back again yields

$$\int_{|x| \leq h} | log |x| - P_h(x)|^p dx = h^n \int_{|x| \leq 1} | log |x| - P_1(x)|^p dx = ch^n.$$ 

Because $log |x| \notin \mathcal{P}_n$, we have $c > 0$. 

Now suppose $n$ is odd. Solve $(1 - \Delta)^{(n+1)/2} g = \delta_0$. Then $g \in [W^1_p, W^{n+1}_{1/p',q'}]$. In view of Propositions 3 and 4, we have

$$[W^1_p, W^{n+1}_{1/p',q'}] \sim [L_p, W^{n+1}_{1/p'+1/n, q'}] \sim [L_p, W^n_p]_{1/p'+1/n, q'}.$$ 

Thus, we would have $\inf_{v \in S^n} \| g - v \|_{L_p} = o(h^{(n/p'+1)}. However, $g \sim |x|$ near $x = 0$, and, analogous to Lemma 3, we have

$$\inf_{P \in \mathcal{P}_n} \int_{|x| \leq h} | x | - P(x)^{p'} dx = ch^{n+p'},$$ 

again a contradiction. 

We now present a complement to Theorem 2. Let $M$ be a linear submanifold of $\mathbb{R}^n$ of codimension $k$.

**Theorem 5.** Suppose $1 < p < \infty$. If $q > 1$, then the mapping $\varphi \mapsto \varphi | M$ does not extend to a continuous map of $[L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)]_{1/p,q}$ to $L_p(M)$, i.e.,

$$\sup_{\varphi \in C^0 \cap [L_p, W_p^k]} \frac{\|\varphi\|_{L_p(M)}}{\|\varphi\|_{[L_p, W_p^k]_{1/p,q}}} = +\infty.$$  

*Proof.* Write $M = \{(x, y) \in \mathbb{R}^n : y = 0\}$. Using Theorem 4, choose functions $\varphi_j \in C^0(\mathbb{R}^n) \cap [L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)]_{1/p,q}$ such that

$$\|\varphi_j\|_{[L_p, W_p^k]_{1/p,q}} = 1 \quad \text{and} \quad \|\varphi_j\|_{C^0} > j.$$ 

By the translation invariance of the Sobolev norms, we may assume that $|\varphi_j(0)| \geq j$. Define

$$\bar{\varphi}_j(x, y) = e^{-|x|^2} \varphi_j(y).$$

A simple calculation shows that

$$\|\bar{\varphi}_j\|_{[L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)]_{1/p,q}} \leq C$$

for some constant $C$ independent of $j$. Thus

$$\|\bar{\varphi}_j\|_{L_p(M)}\|\bar{\varphi}_j\|_{[L_p, W_p^k]_{1/p,q}} \geq (j/C) \left( \int e^{-p|x|^2} \, dx \right)^{1/p} \to \infty.$$ 


4. **Summary of Results**

Let us define an ordering of the Besov smoothness indices. Given two non-negative real numbers $s_1$ and $s_2$, and two real numbers $q_1$ and $q_2$ such that $1 \leq q_i < \infty$, define

$$(s_1, q_1) < (s_2, q_2) \quad \text{iff} \quad \begin{cases} s_1 < s_2 \\ \text{or} \\ s_1 = s_2 \text{ and } q_1 > q_2 \end{cases}.$$ 

Applying Proposition 5 to the results in the previous sections and using (1.3) yields the following.

**Theorem 6.** Let $1 < p < \infty$. Then $B_p^{s,q}(\mathbb{R}^n) \subset C_0^0(\mathbb{R}^n)$ with the inclusion map continuous if and only if $(s, q) \geq (n/p, 1)$. 

$\S 80.25/1-6$
THEOREM 7. Let $1 < p < \infty$ and let $M$ be a linear submanifold of $\mathbb{R}^n$ of codimension $k$. Then "restriction to $M" extends as a continuous linear map of $B^p_\infty(\mathbb{R}^n)$ to $L_q(M)$ if and only if $(s, q) \geq (k/p, 1)$.

The cases $p = 1$ and $\infty$ are excluded in the above because of our use of interpolation theory. However, the case $p = 1$ formally corresponds to the known results (cf. [11, pp. 128–130])

$$W^s_1(\mathbb{R}^n) \subset C^0_\infty(\mathbb{R}^n) \quad \text{and} \quad W^{k}_1(\mathbb{R}^n) \subset L_1(\mathbb{R}^{n-k}).$$

The case $p = \infty$ formally corresponds to false results, namely, that $L_\infty(\mathbb{R}^n) \subset C^0_\infty(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^{n-k})$.

The above theorems may be viewed as limiting cases of standard embedding theorems (cf. [7], Theorem 2), namely,

$$B^{p,q}_r(\mathbb{R}^n) \subset L_r(\mathbb{R}^m)$$

provided that $1 < p < r < \infty, q \leq p$, and $s = n/p - m/r$ ($m = n$ being allowed).

ACKNOWLEDGMENTS

The author would like to thank I. Babuška, H. Brezis, B. Kellogg, and L. Wahlbin for their helpful discussions and suggestions concerning these results.

REFERENCES