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Witt and Virasoro algebras as Lie bialgebras

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Abstract

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We give a countably infinite number of Lie coalgebra structures on the Witt algebra $W = \mathrm{Der}\, k[x]$ over a field k, and on the Virasoro algebras $W_1 = \mathrm{Der}\, k[x, x^{-1}]$ and $V = W_1 \oplus kc$ with central charge c. These come from certain solutions of the classical Yang–Baxter equation, and yield Lie bialgebra structures in each case. For k of characteristic 0, we show that these Lie coalgebra structures on W are mutually non-isomorphic, using an analysis of the locally finite part of W. We also discuss the Lie bialgebra duals of each of these constructions, which can be identified with linearly recursive sequences (one-sided or two-sided).

1. Introduction

We discuss various Lie bialgebra structures on the Witt and Virasoro algebras, and also on their continuous duals of linearly recursive sequences. The basic idea of a (triangular coboundary) Lie bialgebra has been suggested by Drinfeld [1].

2. Triangular coboundary Lie bialgebras

We work over a field F of characteristic $\neq 2$. A *Lie algebra* L over F has a skew-symmetric multiplication $[\ ,\ ]$ satisfying the Jacobi identity. Reversing the arrows, a *Lie coalgebra* M over F has a comultiplication δ from M into $M \wedge M$, the skew-symmetric tensors in $M \otimes M$, which satisfies the co-Jacobi identity

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$(1+\sigma+\sigma^2)(1\otimes\delta)\delta=0$, where σ is the permutation (123) in S_3 acting in the usual way on $M\otimes M\otimes M$. In contrast to the associative coalgebra theory (see [13]), Lie coalgebras need not be locally finite. See [4] for a general discussion of Lie coalgebras and local finiteness in particular. Set $\mathrm{Loc}(M)$ equal to the sum of all finite dimensional Lie subcoalgebras of M. Examples will appear later where $\mathrm{Loc}(M)\neq M$, as will others where $\mathrm{Loc}(M)=M$ (see also [7]).

A Lie algebra L which is simultaneously a Lie coalgebra is called a *Lie bialgebra* if $\delta\in Z^1(L, L\wedge L)$, where L acts on $L\wedge L$ by the adjoint action $[a\wedge b, x]=[a,x]\wedge b+a\wedge[b,x]$. Thus the compatibility (invariance) condition is that $\delta[x,y]=[\delta x, y]-[\delta y, x]$. (Note: we sometimes omit the comma in [,] if no confusion is possible.) If $\delta=\delta_r\in B^1(L, L\wedge L)$ for some $r\in L\wedge L$, L is called a *coboundary Lie bialgebra* (see [1]). The condition is that $\delta_r(x)=[r,x]$ for all $x\in L$.

For a Lie algebra L, we recall the classical Yang–Baxter equation (CYBE) for an element r in $L\otimes L$:

$$\text{(CYBE)}\quad [r^{12}, r^{13}]+[r^{12}, r^{23}]+[r^{13}, r^{23}]=0\,.$$

(CYBE) is an equation in $U(L)\otimes U(L)\otimes U(L)$, where $U(L)$ is the universal enveloping algebra of L. The notation is that if $r=\sum a_i\otimes b_i$, then $r^{12}=\sum a_i\otimes b_i\otimes 1$, $r^{13}=\sum a_i\otimes 1\otimes b_i$ and $r^{23}=\sum 1\otimes a_i\otimes b_i$. The following proposition was stated in [1]. A proof is outlined in [3]. The calculation for the case $r=a\wedge b=a\otimes b-b\otimes a$ and $[ab]=b$ is given in [9]. We give a complete proof here.

Proposition 1. *Let L be a Lie algebra. Let $r\in L\wedge L$ satisfy* (CYBE). *Then $\delta=\delta_r$ defined by $\delta(x)=[r,x]$ gives L the structure of a Lie coalgebra, and hence the structure of a coboundary Lie bialgebra.*

Proof. Let $c(r)=[r^{12}, r^{13}]+[r^{12}, r^{23}]+[r^{13}, r^{23}]$, so that (CYBE) is the condition $c(r)=0$.

We write $r=\sum a_i\wedge b_i=\sum a_i\otimes b_i-\sum b_i\otimes a_i$. We organize (CYBE) as follows. We write

$$r^{12}=\sum a_i\otimes b_i\otimes 1-\sum b_j\otimes a_j\otimes 1\,,$$
$$r^{23}=\sum 1\otimes a_k\otimes b_k-\sum 1\otimes b_l\otimes a_l\,.$$

We write $r^{13}=\sum a_k\otimes 1\otimes b_k-\sum b_l\otimes 1\otimes a_l$ in the product $[r^{12}, r^{13}]$, but write $r^{13}=\sum a_i\otimes 1\otimes b_i-\sum b_j\otimes 1\otimes a_j$ in the product $[r^{13}, r^{23}]$. The result is the following 12 terms, where we drop the summation signs over i, j, k and l:

$$\begin{aligned}
\text{(CYBE)}\quad 0 &= [a_i a_k]\otimes b_i\otimes b_k - [a_i b_l]\otimes b_i\otimes a_l\\
&- [b_j a_k]\otimes a_j\otimes b_k + [b_j b_l]\otimes a_j\otimes a_l\\
&+ a_i\otimes [b_i a_k]\otimes b_k - a_i\otimes [b_i b_l]\otimes a_l\\
&- b_j\otimes [a_j a_k]\otimes b_k + b_j\otimes [a_j b_l]\otimes a_l\\
&+ a_i\otimes a_k\otimes [b_i b_k] - a_i\otimes b_l\otimes [b_i a_l]\\
&- b_j\otimes a_k\otimes [a_j b_k] + b_j\otimes b_l\otimes [a_j a_l]\,.
\end{aligned}$$

Since $\delta\in B^1(L, L\wedge L)$, we have only to verify the co-Jacobi identify for δ. For $x\in L$,

$$\begin{aligned}
\delta(x) &= \left[\sum a_i\wedge b_i, x\right] = \sum [a_i x]\wedge b_i + \sum a_i\wedge [b_i x]\\
&= \sum a_i\otimes [b_i x] + \sum [a_j x]\otimes b_j - \sum b_k\otimes [a_k x] - \sum [b_l x]\otimes a_l\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&(\delta\otimes 1)\delta(x)\\
&= \sum_{i,k} (a_k\otimes [b_k a_i] + [a_k a_i]\otimes b_k - b_k\otimes [a_k a_i] - [b_k a_i]\otimes a_k)\otimes [b_i x]\\
&+ \sum_{j,l} (a_l\otimes [b_l[a_j x]] + [a_l[a_j x]]\otimes b_l\\
&\qquad - b_l\otimes [a_l[a_j x]] - [b_l[a_j x]]\otimes a_l)\otimes b_j\\
&- \sum_{k,l} (a_l\otimes [b_l b_k] + [a_l b_k]\otimes b_l\\
&\qquad - b_l\otimes [a_l b_k] - [b_l b_k]\otimes a_l)\otimes a_l\otimes [a_k x]\\
&- \sum_{l,k} (a_k\otimes [b_k[b_l x]] + [a_k[b_l x]]\otimes b_k\\
&\qquad - b_k\otimes [a_k[b_l x]] - [b_k[b_l x]]\otimes a_k)\otimes a_l\,.
\end{aligned}$$

Notice that there are 16 terms here (dropping the summation signs). Hence $(1+\sigma+\sigma^2)(\delta\otimes 1)\delta(x)$ has 48 terms. We break this into 3 groups of 16 terms each.

First consider the 16 terms whose third factor is a product of 2 or 3 elements, one of which is x. We wish to write this group as $1\otimes 1\otimes R_x$ acting on an element of $L\otimes L\otimes L$, where R_x is the adjoint action $R_x[y]=[y,x]$ of L on L.

8 of the 16 terms above already are in this form (the first and third sums). The 8 remaining terms (after recycling) are

$$b_j \otimes a_l \otimes [b_l[a_j x]] - b_k \otimes a_k \otimes [a_k[b_l x]]$$
$$+ b_l \otimes b_j \otimes [a_l[a_j x]] - b_j \otimes b_l \otimes [a_l[a_j x]]$$
$$- a_l \otimes b_j \otimes [b_l[a_j x]] + a_l \otimes b_k \otimes [a_k[b_l x]]$$
$$- a_l \otimes a_k \otimes [b_k[b_l x]] + a_k \otimes a_l \otimes [b_k[b_l x]] .$$

Using the Jacobi identity on each line, these 8 terms reduce to the 4 terms

$$b_j \otimes a_j \otimes [[b_l a_j]x]$$
$$+ b_l \otimes b_j \otimes [[a_l a_j]x]$$
$$+ a_l \otimes b_k \otimes [[a_k b_l]x]$$
$$+ a_k \otimes a_l \otimes [[b_k b_l]x] .$$

Thus the 16 terms we are now considering reduce to 12 terms, whose sum is the image under $1 \otimes 1 \otimes R_x$ of

$$a_k \otimes [b_k a_i] \otimes b_i + [a_k a_i] \otimes b_k \otimes b_i$$
$$- b_k \otimes [a_k a_i] \otimes b_i - [b_k a_i] \otimes a_k \otimes b_i$$
$$- a_l \otimes [b_l b_k] \otimes a_k - [a_l b_k] \otimes b_l \otimes a_k$$
$$+ b_l \otimes [a_l b_k] \otimes a_k + [b_l b_k] \otimes a_l \otimes a_k$$
$$+ b_j \otimes a_l \otimes [b_l a_j] + b_l \otimes b_j \otimes [a_l a_j]$$
$$+ a_l \otimes b_k \otimes [a_k b_l] + a_k \otimes a_j \otimes [b_k b_l] .$$

Inspection shows that these twelve terms sum to $c(r)$, the right-hand side of our (CYBE). Since $c(r) = 0$, our group of 16 terms sums to 0. Similarly, the remaining 32 terms can be split in 2 groups of 16 terms each, one of which is $(R_x \otimes 1 \otimes 1)c(r)$ and the other one of which is $(1 \otimes R_x \otimes 1)c(r)$.

Thus Proposition 1 is proved. □

We point out that our proof shows that for $r \in L \wedge L$, if $\delta(x) = [r, x]$ for $x \in L$, then $(1 + \sigma + \sigma^2)(\delta \otimes 1)\delta(x) = [c(r), x]$, where x acts on $L \otimes L \otimes L$ by the adjoint action $[a \otimes b \otimes c, x] = [ax] \otimes b \otimes c + a \otimes [bx] \otimes c + a \otimes b \otimes [cx]$. A coboundary Lie bialgebra L is said to be *triangular* if $\delta = \delta_r$ for r in $L \wedge L$ satisfying (CYBE), i.e., $c(r) = 0$.

3. Witt and Virasoro algebras

Let $W_1 = \mathrm{Der}\, F[x]$, the Lie algebra of derivations of the polynomial algebra $F[x]$. W_1 has a basis $\{e_i\}$ for $i \geq -1$, where $e_i = x^{i+1}\, \mathrm{d}/\mathrm{d}x$. The product is given by $[e_i, e_j] = (j-i)e_{i+j}$. We call W_1 the (one-sided) Witt algebra in one variable. We look for non-zero solutions of (CYBE) of the form $e_i \wedge e_j$, $i \neq j$.

Proposition 2. *Let* $i, j \geq -1$, $i - j \neq 0$ in F. *Then* $e_i \wedge e_j$ *satisfies* (CYBE) *if and only if* $i = 0$ for $j = 0$.

Proof. Using the notation of Section 2, let $r = e_i \wedge e_j$. Then it is easy to see that

$$\begin{aligned} c(r) = (i-j)(&e_{i+j} \otimes e_j \otimes e_i - e_{i+j} \otimes e_i \otimes e_j \\ &+ e_i \otimes e_{i+j} \otimes e_j - e_j \otimes e_{i+j} \otimes e_i \\ &+ e_j \otimes e_i \otimes e_{i+j} - e_i \otimes e_j \otimes e_{i+j}) . \end{aligned}$$

This is 0 if $i = 0$ or $j = 0$. If $i \neq 0$ and $j \neq 0$, then the six displayed terms in $W_1 \otimes W_1 \otimes W_1$ are linearly independent. This proves Proposition 2. □

For each $i \geq 0$, let $W_1^{(i)}$ be the triangular coboundary Lie bialgebra structure on W_1 with Lie comultiplication δ_i given by $\delta_i(w) = [e_0 \wedge e_i, w]$ for $w \in W_1^{(i)}$. Thus, $\delta_i(e_n) = [e_0 \wedge e_i, e_n] = [e_0, e_n] \wedge e_i + e_0 \wedge [e_i, e_n] = n(e_n \wedge e_i) + (n-i)(e_0 \wedge e_{n+i})$ for all $n \geq -1$.

We remark that the triangular coboundary Lie bialgebra structure on $W_1^{(i)}$, as well as that on $W^{(i)}$ and $V^{(i)}$ later in this section, were also presented in [9], in the case where F has characteristic zero, where the same formula for $\delta_i(e_n)$ is obtained. Here we also discuss the positive characteristic case.

We discuss some of these structures. For $i = -1$, $\delta_{-1}(e_n) = n(e_n \wedge e_{-1}) + (n+1)(e_0 \wedge e_{n-1})$. Thus, for each $n \geq -1$, $e_{-1}, e_0, \ldots, e_n$ span a finite-dimensional Lie subcoalgebra, so $\mathrm{Loc}\, W_1^{(-1)} = W_1^{(-1)}$. For $i = 0$, $\delta_0 = 0$. For $i = 1$, $\delta_1(e_n) = n(e_n \wedge e_1) + (n-1)(e_0 \wedge e_{n+1})$. Then $\delta_1(e_{-1}) = -(e_{-1} \wedge e_1)$, $\delta_1(e_0) = -(e_0 \wedge e_1)$ and $\delta_1(e_1) = 0$. If the characteristic of F is 0, then each e_n for $n \geq 2$ generates an infinite-dimensional Lie subcoalgebra, and $\mathrm{Loc}\, W_1^{(1)}$ is three-dimensional. If F has positive characteristic p, then for each $k > 0$, $\delta_1(e_{kp+1}) = e_{kp+1} \wedge e_1$, so that $e_{-1}, e_0, e_1, \ldots, e_{kp+1}$ span a finite-dimensional Lie subcoalgebra. Since every e_n is captured in this way, $\mathrm{Loc}\, W_1^{(1)} = W_1^{(1)}$ at positive characteristic.

Now fix $i \geq 2$. Note that $\delta_i(e_i) = 0$ and that $\delta_i(e_0) = -i(e_0 \wedge e_i)$ so that e_0 and e_i span a two-dimensional Lie subcoalgebra. If F has a positive characteristic p, then as in the case $i = 1$, $\delta_i(e_{kp+i}) = i(e_{kp+i} \wedge e_i)$. Suppose $(p, i) = 1$. Then for $n \geq -1$, $n, n+i, \ldots n + (p-1)i$ are mutually distinct modulo p, so one of them, say $n + li$, is 0 modulo p. Then $n + (l+1)i = kp + i$ for some k, and the formula for

$\delta_i(e_{kp+1})$ shows that e_n lies in a finite-dimensional Lie subcoalgebra. However, if $(p,i)=p$, then one can see that Loc $W_1^{(i)}$ is spanned by the e_n with $p \mid n$.

Now let F have characteristic zero, $i \geq 2$. Then $\delta_i(e_n) = n(e_n \wedge e_i) + (n-i)(e_0 \wedge e_{n+i})$ implies that each e_n with $n \neq 0, i$ generates an infinite-dimensional Lie subcoalgebra, and it is not hard to see that Loc $W_1^{(i)} = Fe_0 \oplus Fe_i$. The structure of Loc $W_1^{(i)}$ is also mentioned in [9] for characteristic zero.

We use the above remarks on Loc $W_1^{(i)}$ to show that the Lie coalgebra structures $(W_1^{(i)}, \delta_i)$ are mutually non-isomorphic for $i \geq -1$ when F is of characteristic zero. Thus take $i \neq j$, $i \geq -1$, $j \geq -1$. Clearly δ_i and δ_j give non-isomorphic Lie coalgebras if either i or j is -1, 0 or 1. So let $i \geq 2$ and $j \geq 2$. Let T be a Lie coalgebra isomorphism of $W_1^{(i)}$ to $W_1^{(j)}$. We can assume $j > i$. Since Loc $W_1^{(i)} = Fe_0 \oplus Fe_i$ and Loc $W_1^{(j)} = Fe_0 \oplus Fe_j$, one sees that the condition $\delta_j T = (T \otimes T)\delta_i$ requires the action of T on Loc $W_1^{(i)}$ to be given by a matrix $\left[\begin{smallmatrix} \alpha & 0 \\ \beta & j/i \end{smallmatrix}\right]$ for α, β in F, $\alpha \neq 0$. We use this to derive a contradiction.

Thus, let $T(e_0) = \alpha e_0 + \beta e_j$ and $T(e_i) = \gamma e_0 + \delta e_j$, with $\Delta = \alpha\delta - \beta\gamma \neq 0$. Since $\delta_i(e_0) = -i(e_0 \wedge e_i)$, we get that $(T \otimes T)\delta_i(e_0) = -i\Delta(e_0 \wedge e_j)$ and $\delta_j(T(e_0)) = -j\alpha(e_0 \wedge e_j)$, so $i\Delta = j\alpha$. Also $(T \otimes T)\delta_i(e_i) = 0$ and $\delta_j(T(e_i)) = -j\gamma(e_0 \wedge e_j)$, so $\gamma = 0$. This $\Delta = \alpha\delta \neq 0$ so $i\Delta = j\alpha$ gives $\delta = j/i$. So $T(e_i) = \alpha e_0 + \beta e_j$, $\alpha \neq 0$ and $T(e_i) = (j/i)e_j$.

Let $T(e_s) = \sum_{k \geq -1} \alpha_{sk} e_k$ for $s \geq -1$. We will show that $T(e_{-1}) \in Fe_0 \oplus Fe_j = \text{Loc}\, W_1^{(j)}$, so that T is not injective, a contradiction. Since $\delta_i(e_{-1}) = -(e_{-1} \wedge e_i) + (-1-i)e_0 \wedge e_{i-1}$, $(T \otimes T)\delta_i(e_{-1}) = \delta_j T(e_{-1})$ yields

$$-\sum_k \alpha_{-1,k}(e_k \wedge (j/i)e_j) + (-1-i)\Big[(\alpha e_0 + \beta e_j) \wedge \sum_l \alpha_{i-1,l} e_l\Big]$$
$$= \sum_r \alpha_{-1,r}(r(e_r \wedge e_j) + (r-j)(e_0 \wedge e_{r+j}))\,.$$

Comparing coefficients of $e_s \wedge e_j$ for $s \neq 0, j$, we get

$$-\alpha_{-1,s}(j/i) + (-1)(-1-i)\beta\alpha_{i-1,s} = s\alpha_{-1,s} \quad \text{for } s \neq 0, j\,. \tag{1}$$

If $\beta = 0$, this says $(s + j/i)\alpha_{-1,s} = 0$ if $s \neq 0, j$. Since $j > i$, $s \geq -1$, we have that $s + j/i > 0$ so that $\alpha_{-1,s} > 0$ if $s \neq 0, j$, and $T(e_{-1}) \in Fe_0 \oplus Fe_j$, a contradiction. Hence assume that $\beta \neq 0$, and we rewrite (1) as

$$(s + j/i)\alpha_{-1,s} = (i+1)\beta\alpha_{i-1,s} \quad \text{for } s \neq 0, j\,. \tag{1'}$$

Next we compare coefficients of $e_0 \wedge e_s$ for $s \neq 0, j$ to get

$$(-1-i)\alpha\alpha_{i-1,s} = (s-2j)\alpha_{-1,s-j} \quad \text{for } s \neq j,\ s-j \geq -1\,. \tag{2}$$

Comparison with (1′) yields

$$\alpha_{-1,s} = \frac{-(s-2j)\beta}{(s+j/i)\alpha}\, \alpha_{-1,s-j} \quad \text{for } s \neq j,\ s \geq j-1\,. \tag{3}$$

In particular, $\alpha_{-1,2j} = 0$ which then implies that $\alpha_{-1,kj} = 0$ for $k \geq 2$. Suppose $\alpha_{-1,l\neq 0}$ for $l \not\equiv 0 \pmod{j}$ with $l \geq -1$. Then (3) yields $\alpha_{-1,l+tj} \neq 0$ for all $t \geq 0$, which is impossible. Thus again $T(e_{-1}) \in Fe_0 \oplus Fe_j$, a contradiction.

We have not determined other solutions of (CYBE) in $W_1 \wedge W_1$ other than those of the form $e_i \wedge e_j$, for F of characteristic zero. At positive characteristic p, there will also be solutions $e_i \wedge e_j$ for $p \mid (j-i)$, as shown by the proof of Proposition 2.

Now we discuss the (full) Witt algebra W (or Virasoro algebra) in one variable, with basis $\{e_i\}$ for $i \in \mathbb{Z}$, and multiplication $[e_i, e_j] = (j-i)e_{i+j}$. This is the Lie algebra of derivations of the algebra $F[x, x^{-1}]$ of Laurent polynomials, with $e_i = x^{i+1}\,\mathrm{d}/\mathrm{d}x$. The proof of Proposition 2 is valid for i, j in $\mathbb{Z}$, and for each $i \in \mathbb{Z}$, $\delta_i(w) = [e_0 \wedge e_i, w]$ defines a triangular coboundary Lie bialgebra structure $W^{(i)}$ on W. The same formula $\delta_i(e_n) = n(e_n \wedge e_i) + (n-i)(e_0 \wedge e_{n+i})$ holds. For $i = -1$, we note that for $n \geq -1$, $e_0, \ldots, e_n$ is a finite-dimensional Lie subcoalgebra of $W^{(-1)}$, but that for $n \leq -2$, e_n generates an infinite-dimensional Lie subcoalgebra at characteristic zero, and $\operatorname{Loc} W^{(-1)} = \bigoplus_{n \geq -1} Fe_n$. At characteristic p, $\delta_{-1}(e_{-sp-1}) = -(e_{-sp-1} \wedge e_{-1})$ shows that $\operatorname{Loc} W^{(-1)} = W^{(-1)}$. Of course δ_0 is still 0. For $i = 1$, we note that for each $n \leq 1$, $e_n, \ldots, e_1$ span a finite-dimensional Lie subcoalgebra. So at characteristic zero, $\operatorname{Loc} W^{(1)} = \bigoplus_{n \leq 1} Fe_n$. At positive characteristic, the same argument used for $W_1^{(1)}$ shows that $\operatorname{Loc} W^{(1)} = W^{(1)}$. Now let $i \geq 2$. For each $n = ki \leq 1$, note that $e_{ki}, e_{(k+1)i}, \ldots, e_i$ span a Lie subcoalgebra. At characteristic zero, each e_n with $n > i$, or $n \leq i$ and $i \mid n$, generates an infinite-dimensional Lie subcoalgebra. So $\operatorname{Loc} W^{(i)} = \bigoplus_{k \leq 1} Fe_{ki}$. (This was also valid for $i = -1$ and $i = 1$.) At characteristic p, the same discussion as for $W_1^{(i)}$ shows that $\operatorname{Loc} W^{(i)} = W^{(i)}$ if $(p, i) = 1$, and $\operatorname{Loc} W^{(i)} = \bigoplus_{p \mid n} Fe_n$ if $(p, i) = p$. Now let $i \leq -2$. Then symmetric arguments show that at characteristic zero, $\operatorname{Loc} W^{(i)} = \bigoplus_{k \leq 1} Fe_{ki}$. At positive characteristic p, $\operatorname{Loc} W^{(i)} = W^{(i)}$ if $(p, i) = 1$ and $\operatorname{Loc} W^{(i)} = \bigoplus_{p \mid n} Fe_n$ if $(p, i) = p$.

Finally, let $V = W \oplus Fc$ be the Virasoro algebra with central charge, where c is a central element, and W has a basis $\{e_i\}$ for i in $\mathbb{Z}$, and multiplication $[e_i, e_j] = (j-i)e_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}c$. Here we let F have characteristic zero. The proof of Proposition 2 is valid for i, j in $\mathbb{Z}$, and one also checks that $e_i \wedge e_{-i}$ is not a solution of (CYBE) for $i \neq 0$. (At characteristic $p \geq 5$, $e_i \wedge e_{-i}$ is a solution of (CYBE) if $p \mid i$.) For any fixed w in W, $c \wedge w$ trivially satisfies (CYBE). So for $r = c \wedge w$, $\delta(x) = [c \wedge w, x] = c \wedge [wx]$ defines a triangular coboundary Lie bialgebra structure on V. In particular, for $w = e_i$, let $\delta_i'(x) = c \wedge [e_i x]$, so $\delta_i'(e_n) = (n-i)(c \wedge e_{i+n})$ and $\delta_i'(c) = 0$. Note that for $i \neq 0$, there is a Lie coalgebra map of $W^{(i)}$ into (V, δ_i') taking e_i to 0, e_0 to c and e_n to e_n for $n \neq 0, i$. For $i \neq 0$, $\operatorname{Loc}(V, \delta_i')$ is similar to $\operatorname{Loc}(W, \delta_i)$, i.e., $\operatorname{Loc}(V, \delta_i') =$

$(\bigoplus_{k\leq 1} Fe_{ki})\oplus Fc$. For $i=0$, $Fe_n \oplus Fc$ is a Lie subcoalgebra of (V, δ_0') for each $n\in\mathbb{Z}$, so $\operatorname{Loc}(V,\delta_0')=V$. For each $i\in\mathbb{Z}$, $e_0\wedge e_i$ satisfies (CYBE), so let $V^{(i)}=(V,\delta_i)$, where $\delta_i(v)=[e_0\wedge e_i, v]$ for $v\in V$. Then $\delta_i(c)=0$, so $\operatorname{Loc} V^{(i)}=(\operatorname{Loc} W^{(i)})\oplus Fc$. If $i=0$, then $\delta_0=0$ (on W_1, W and V). However, the formula $\delta_0'(e_n)=n(c\wedge e_n)$, suggests a replacement ρ_0 for δ_0 in W_1 and W. Set $\rho_0(e_n)=n(e_0\wedge e_n)$. This gives a Lie coalgebra structure, but not a Lie bialgebra structure. The invariance condition fails for $\rho_0([e_ie_j])$ for $i,j\neq 0$, $i\neq j$.

4. The continuous dual Lie bialgebra of a Lie bialgebra

Let (M,δ) be a Lie coalgebra. Then $M^*=\operatorname{Hom}_F(M,F)$ is a Lie algebra under the convolution product $(f*g)(m)=(f\otimes g)(\delta m)$, i.e., if $\delta m=\sum_m m_1\otimes m_2$ in $M\wedge M\subset M\otimes M$, then $(f*g)(m)=\sum_m f(m_1)g(m_2)$ for f,g in M^*. (Sweedler notation.)

Let L be a Lie algebra with multiplication $m: L\otimes L\to L$. Then $m^*: L^*\to(L\otimes L)^*$. A subspace V of L^* is called *good* if $m^*(V)\subseteq V\otimes V$, where we identify $V\otimes V\subseteq L^*\otimes L^*\subseteq (L\otimes L)^*$. This means that there exists a linear map $\delta: V\to V\otimes V$ such that if $\delta f=\sum_f f_1\otimes f_2$ for f,f_1,f_2 in V, then $f([xy])=\sum_f f_1(x)f_2(y)$ for all x,y in L.

Let $L^\circ=\sum V$, over all good subspaces V of L^*. Then L° is a Lie coalgebra, where for each V, $\delta(V)\subseteq V\wedge V$. See [4] for the original development of this idea.

Now let L be a Lie bialgebra. Then L^* is a Lie algebra, and $L^\circ\subseteq L^*$ is a Lie coalgebra.

Proposition 3. *Let L be a bialgebra. Then L° is a Lie subalgebra of L^*, and L° is a Lie bialgebra.*

Proof. To see that L° is closed under the convolution product on L^*, we claim that $L^\circ+(L^\circ * L^\circ)$ is a good subspace of L^*, so that $L^\circ * L^\circ\subseteq L^\circ$. Let V,W be good subspaces of L^*. Using δ for the Lie comultiplication in L, and γ for the Lie comultiplication in L°, let $f\in V$, $g\in W$, $\gamma(f)=\sum_f f_1\otimes f_2$ in $V\otimes V$ and $\gamma(g)=\sum_g g_1\otimes g_2$ in $W\otimes W$. The idea is that

$$\gamma(f*g)=\sum_f f_1\otimes(f_2*g)+\sum_g g_1\otimes(f*g_2)$$
$$+\sum_f (f_1*g)\otimes f_2+\sum_g (f*g_1)\otimes g_2$$

defines the appropriate map on $L^\circ * L^\circ$. To see this, evaluate both sides on $x\otimes y$ in $L\otimes L$. The left-hand side gives $(f*g)([xy])$. Using the invariance axiom on $\delta([xy])$, and dropping the summation signs, this is equal to

$$(f \otimes g)([x_1 y] \otimes x_2 + x_1 \otimes [x_2 y] + [xy_1] \otimes y_2 + y_1 \otimes [xy_2])$$
$$= f_1(x_1)f_2(y)g(x_2) + f(x_1)g_1(x_2)g_2(y)$$
$$+ f_1(x)f_2(y_1)g(y_2) + f(y_1)g_1(x)g_2(y_2) .$$

Inspection, using the definition of convolution product, shows that this agrees with the right-hand side acting on $x \otimes y$. The displayed formula for $\gamma(f * g)$ is precisely the invariance condition for L°, so that L° is a Lie bialgebra. (Proposition 3 is also in [8]. Since [8] has yet to appear, we included a proof here.) □

5. The continuous duals of the Witt and Virasoro algebras

In this section, F will be of characteristic zero. It is well-known that W_1 and W are simple Lie algebras. Now if L is any Lie algebra, then it is known (see [4]) that $\mathrm{Loc}(L^\circ) = \{f \in L^* \mid f(I) = 0$ for some cofinite ideal I of $L\}$. Hence if L is an infinite-dimensional simple Lie algebra, then $\mathrm{Loc}(L^\circ) = 0$. In particular, $\mathrm{Loc}(W_1^\circ) = 0$ and $\mathrm{Loc}(W^\circ) = 0$. Since V has Fc as its only non-trivial ideal, also $\mathrm{Loc}(V^\circ) = 0$.

W_1° has recently been identified as the space of F-linearly recursive sequences $f = (f_i)_{i \geq -1}$ [10] (see also [6] for a partial result). This means that f satisfies a recursive relation of the form $f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r}$ for all $n \geq r - 1$ for some $r \geq 1$. The linear identification of $f = (f_i)$ in W_1^* is via $f_i = f(e_i)$ for $i \geq -1$. We give some examples of the Lie bialgebra structures on W_1°. Note that the elements of the dual basis $\{e_i^*\}$ are in W_1°, i.e., W_1° contains all finite sequences. Of course, these are linearly recursive. To explain this in terms of the Lie comultiplication γ on W_1°, note that $\gamma(e_n^*) = \sum_{i+j=n}(j-i)e_i^* \otimes e_j^*$, i.e., $\bigoplus_{n \geq -1} Fe_n^*$ is a good subspace of W_1^*, so is contained in W_1°. Such a comultiplication formula is proved by acting on $e_k \otimes e_l$ for any $k,l \geq -1$. Both sides yield $(l-k)\delta_{n,k+l}$. Before giving further examples of γ, we note that γ depends only on the Lie algebra structure of W_1. Now W_1° contains the geometric (or exponential) sequences $(a^i)_{i \geq -1}$ for any $a \neq 0$ in F. More generally, it contains the sequences $(a^i i^n)_{i \geq -1}$ for $a \neq 0$ in F and n a fixed non-negative integer. A binomial calculation will show that

$$\gamma(a^i i^n) = \sum_{j=0}^{n+1} \left[\binom{n}{j} - \binom{n}{j-1} \right] (a^i i^j) \otimes (a^i i^{n+1-j}) .$$

For example $\gamma(a^i) = (a^i) \otimes (a^i i) - (a^i i) \otimes (a^i)$, which is proved by evaluating on $e_k \otimes e_l$, obtaining $(l-k)a^{k+l} = a^k(la^l) - (ka^k)(a^l)$. Thus, as a and n vary, the $(a^i i^n)$ span a good subspace of W_1°. When F is algebraically closed, it is well-known that the $(a^i i^n)$ and the e_j^* are a basis for the space of linearly recursive sequences, which shows that W_1° contains all linearly recursive sequences. See [10] for the converse, and for an explanation of why the algebraically closed condition can be assumed without loss of generality.

It is possible to give an alternate identification of W_1° as linearly recursive sequences. This starts with the associative algebra $F[x]$ of polynomials in x. It has a continuous (coassociative) coalgebra dual $F[x]^\circ$ (see [13]), which was identified in [12] as linearly recursive sequences. Nichols has recently shown (see [11, Corollary 3]), that W_1° can be identified as a vector space with $F[x]^\circ$, and that the Lie comultiplication γ on W_1° can be described in terms of the coassociative comultiplication on $F[x]^\circ$. The same technique will identify W° with $F[x, x^{-1}]^\circ$, which is the space of linearly recursive sequences $f=(f_i)_{i\in\mathbb{Z}}$. Here the minimal linearly recursive relation of f must be of the form $x^r - h_1x^{r-1} - \cdots - h_r$, where $h_r \neq 0$, i.e., from each coordinate, the sequence can be solved to the left as well as to the right. This is the space of sequences spanned by the $(a^i i^m)_{i\in\mathbb{Z}}$ for $a\neq 0$ in F, $m\geq 0$. The reason that the e_n^* do not belong to W° is that the formula $\gamma e_n^* = \sum_{i+j=n}(j-i)e_i^*\otimes e_j^*$ which worked for W_1° (where $i,j\geq -1$) does not make sense for $i,j\in\mathbb{Z}$. (It is not a finite sum.) The same formula for $\gamma(a^i i^m)$ in W_1° is valid in W°. As with W_1°, the Lie comultiplication on W° can be described in terms of the coassociative comultiplication on $F[x, x^{-1}]^\circ$ (see [11]). We have seen that $\operatorname{Loc} W_1^\circ = 0$ and $\operatorname{Loc} W^\circ = 0$. We remark that coassociative coalgebras are locally finite [13]. Thus in [12], we were able to give an algorithm to compute the comultiplication of a linearly recursive sequence in terms of an easily computable basis of the (finite-dimensional) subcoalgebra it generates. In [12], this was done for $F[x]^\circ$, but the same technique works for f in $F[x, x^{-1}]^\circ$ by taking its restriction to $F[x]$ in $F[x]^\circ$, applying the algorithm, and extending back to $F[x, x^{-1}]^\circ$. See [11] for an example of how to compute the Lie comultiplication of a Fibonacci sequence based on its coassociative comultiplication.

For V°, we note that $c^*\not\in V^\circ$ since $\frac{1}{12}\sum_{i\in\mathbb{Z}}(i-i^3)e_i\otimes e_{-i}$ is not a finite sum (it would be $\gamma(c^*)$). Since $V\to W$ $(c\mapsto 0)$ is a surjection of Lie algebras, it follows that $W^\circ \hookrightarrow V^\circ$ is a Lie coalgebra injection. To see that this is an isomorphism, we need to show that if $f\in V^\circ$, then $f(c)=0$. This is done in [11]. Hence $V^\circ \cong W^\circ$. The same formula for $\gamma(a^i i^n)$ in W_1° or W° works for V°, and the algorithmic approach to computing $\gamma(f)$ for $f\in V^\circ$ is still possible since $V^\circ\cong W^\circ$.

Now we comment on the Lie algebra structures on W_1°, W° and V° induced by our various Lie coalgebra (Lie bialgebra) structures on W_1, W and V respectively.

Starting with W_1, for each $i\geq -1$, we have a Lie bialgebra structure with Lie comultiplication $\delta_i(w)=[e_0\wedge e_i, w]$, so that $\delta_i(e_n)=n(e_n\wedge e_i)+(n-i)(e_0\wedge e_{n+i})$. We assume $i\neq 0$, as $\delta_0=0$ gives $(W_1^{(0)})^*$ the structure of an abelian Lie algebra. For the dual basis $\{e_n^*\}$, this means that the Lie multiplication in $(W_1^{(i)})^\circ$ is given by

$$[e_0^*, e_n^*] = (n-2i)e_{n-i}^* \quad \text{for } n\neq 0\,,$$
$$[e_n^*, e_i^*] = ne_n^* \qquad \text{for } n\neq 0,i\,,$$

and all other products are zero. These formulas describe in a sense the Lie product in $(W_1^{(i)})^*$, i.e.,

$$\left[\sum a_n e_n^*, \sum b_m e_m^*\right] = \sum c_p e_p^*,$$

where $c_p = p(a_0 b_{p+i} - b_0 a_{p+i} + b_i a_p - a_i b_p) + i(a_{p+i} b_0 - a_0 b_{p+i})$. If f and g are given linearly recursive sequences, one would like to express $[f, g]$ in some 'natural' way, rather than express f and g in terms of the $(a^j j^n)$ and the e_k^*, and develop formulas for the products $[(a^j j^n), (b^k k^m)]$ and $[(a^j j^n), e_k^*]$.

We note that $(W_1^{(i)})^*$ has its derived algebra with pseudo-basis $\{e_j^*\}$ for $j \neq i$, and the second derived algebra has pseudo-basis $\{e_k^*\}$ for $k \neq 0, i$, so that the third derived algebra is 0, i.e., $(W_1^{(i)})^*$ (and so also $(W_1^{(i)})^\circ$) are solvable.

The same multiplication rule for the e_k^* in $(W_1^{(i)})^*$ holds in $(W^{(i)})^* = \{(f_i)_{i \in \mathbb{Z}}\}$, only now $k \in \mathbb{Z}$. Of course, e_k^* is not in $(W_1^{(i)})^\circ$, and the same remarks apply concerning the need for an algorithm to describe a given product in $(W_1^{(i)})^\circ$.

Finally, for $V^{(i)} = W^{(i)} \oplus Fc$, $\delta_i(c) = 0$. In $(V^{(i)})^*$, the same rule holds for multiplying the e_k^*. So $(V^{(i)})^\circ \cong (W^{(i)})^\circ$ as Lie bialgebras. Recall that V also has a Lie bialgebra structure with comultiplication $\delta_i'(e_n) = (n - i)(c \wedge e_{i+n})$ and $\delta_i'(c) = 0$, resulting from the solution $c \wedge e_i$ of (CYBE). Fix i in $\mathbb{Z}$. Then the multiplication for the e_k^* and c^* in V^* is given by $[c^*, e_n^*] = (n - 2i)e_{n-i}^*$, and all other products are zero. So the rule in (V, δ_i') is

$$\left[\sum a_j e_j^* + \alpha c^*, \sum b_k e_k^* + \beta c^*\right] = \sum c_p e_p^*,$$

where $c_p = (p - 2i)(\alpha b_p - \beta a_p)$. This is a metabelian (2-step solvable) Lie algebra. Again, we have no algorithm for multiplying two given linearly recursive sequences under this product.

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