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Decomposition of the Complete Directed Graph into k-Circuits

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We study the decomposition of K_n^* (the complete directed graph with n vertices) into arc-disjoint elementary k-circuits, primarily for the case k even. We solve the problem for many values of (n, k) and in particular for all n in the cases $k = 4, 6, 8$, and 16.

Let K_n^* be the complete directed graph with n vertices: Every ordered pair of vertices is connected by exactly one arc (directed edge). By a *k-circuit* we mean a (directed) elementary circuit of length k; we denote a k-circuit by $(x_1, x_2, \ldots, x_k, x_1)$ with $x_i \neq x_j$ unless $i = j$. We are interested in the following problem: *For what values of n is it possible to decompose K_n^* into pairwise arc-disjoint k-circuits* (that is, to partition the arcs of K_n^* into k-circuits)? A necessary condition is that the number of arcs of K_n^* be a multiple of k; thus we have Proposition 1.

PROPOSITION 1. *If K_n^* $(n \geqslant k)$ can be decomposed into k-circuits, then $n(n-1) \equiv 0 \pmod{k}$.*

In the case $k = 3$, it has been proven by the first author [3] that this condition is sufficient except for $n = 6$.

THEOREM 1. *K_n^* $(n \geqslant 3)$ can be decomposed into 3-circuits if and only if $n(n-1) \equiv 0 \pmod 3$ except for $n = 6$ (in this case the decomposition is not possible.)*

In this paper we give results on the decomposition of K_n^* into k-circuits, primarily in the case k even, and solve the problem completely in the cases $k = 4, 6, 8$, and 16.

The decomposition of the undirected complete graph, K_n, into (undirected) k-cycles has been discussed by Kotzig [10] and Rosa [12, 13]. We summarize their results in Section 5.

1. Decomposition of $K_n{}^*$ into n-circuits (Hamiltonian circuits)

The problem in this case is a generalization of a similar problem in the undirected case (due to Kirkman): When is it possible to decompose the edges of the complete (undirected) graph K_n into Hamiltonian cycles? The answer is given by Theorem 2.

Theorem 2 (see [2, p. 233]). *The complete (undirected) graph K_n can be decomposed into Hamiltonian cycles if and only if n is odd.*

Corollary 1. *If n is odd, $K_n{}^*$ can be decomposed into Hamiltonian circuits.*

Proof. We obtain such a decomposition by associating with each n-cycle, $x_1 x_2 \cdots x_n x_1$, of a decomposition of K_n the two opposing n-circuits $(x_1, x_2, ..., x_n, x_1)$ and $(x_1, x_n, ..., x_2, x_1)$.

This problem is also closely related to a problem of E. G. Strauss: When is it possible to decompose $K_n{}^*$ into Hamiltonian (spanning) paths? In fact, we have

Proposition 2. *$K_n{}^*$ can be decomposed into Hamiltonian paths if and only if K_{n+1}^* can be decomposed into Hamiltonian circuits.*

Proof. If $K_n{}^*$ is decomposed into n Hamiltonian paths P_i, but some vertex is not an initial vertex of one of the paths, it would be the terminal vertex of n arcs, an impossibility. Similarly, each vertex is a terminal vertex of one of the paths. To decompose K_{n+1}^* into Hamiltonian circuits, one needs only to consider K_{n+1}^* as $K_n{}^*$ together with a vertex x which is joined to each vertex of $K_n{}^*$. The required circuits are $C_i = (x, P_i, x)$. The converse is obvious.

The problem of E. G. Strauss has been studied in [1, 7, 8, 11, 16]. In [11] Mendelsohn has pointed out that a decomposition of $K_n{}^*$ into Hamiltonian paths is possible if there exists a *complete* latin square of order n (that is, a latin square in which every ordered pair of elements appears exactly once in the rows and once in the columns.) Mendelsohn also showed that if there was a finite sequencable group of order n (a finite group of order n is sequencable if its elements can be listed $g_1, g_2, ..., g_n$ so that the partial products $g_1, g_1 g_2, ..., g_1 g_2 \cdots g_n$ are distinct), then a complete latin square of order n exists (see Lemma 1);

he exhibited a sequencing of the nonabelian group of order 21. Gordon [8] has shown that an abelian group has a sequencing if and only if it has exactly one element of order 2. Wang [16] wrote a computer program which produced sequencings for the nonabelian groups of orders 39, 55 and 57. It follows that $K_n{}^*$ can be decomposed into Hamiltonian circuits for all odd n and $n = 22, 40, 56$, and 58.

Bankes [1] wrote a computer program to decompose $K_n{}^*$ into Hamiltonian circuits; he found that $K_n{}^*$ can be decomposed into Hamiltonian circuits if $n = 8, 10, 12$, and 14. For example, a decomposition of $K_8{}^*$ is: (1, 2, 3, 4, 5, 6, 7, 8, 1), (1, 3, 2, 4, 6, 5, 8, 7, 1), (1, 4, 2, 5, 7, 3, 8, 6, 1), (1, 5, 2, 6, 8, 3, 7, 4, 1), (1, 6, 4, 7, 2, 8, 5, 3, 1), (1, 7, 6, 3, 5, 4, 8, 2, 1), (1, 8, 4, 3, 6, 2, 7, 5, 1).

For $n = 4$ and $n = 6$ we have verified by considering all possible cases that $K_4{}^*$ and $K_6{}^*$ cannot be decomposed into Hamiltonian circuits (this result was confirmed by the program of Bankes.) This was also announced in [11], where it was said that $K_n{}^*$ cannot be decomposed into Hamiltonian paths for $n = 3, 5$, and 7. (Apparently $K_7{}^*$ can be decomposed into Hamiltonian paths as the decomposition of $K_8{}^*$ into Hamiltonian circuits shows.)

With an improved version of the program of Bankes, we found that $K_n{}^*$ can be decomposed into Hamiltonian circuits for $n = 16$ and 18 before time limitations occurred. Thus the smallest undecided values are $n = 20$ and 24.

2. DECOMPOSITION OF K_{n+1}^* INTO n-CIRCUITS

THEOREM 3. *For every n, K_{n+1}^* can be decomposed into n-circuits.*

In order to prove this theorem we need the following lemma which is an adaptation of R. C. Bose's method of "symmetrically repeated differences" [6].

LEMMA 1. *Let Γ be a group. Let G be any directed graph on elements in Γ. If $e = (x, y)$ is a directed edge in G, let $w_e = yx^{-1}$. If all members of the collection $(w_e : e \in G)$ are distinct, then the graphs $\{Gg : g \in \Gamma\}$ are arc disjoint.*

Proof. Suppose Gg and Gh have an arc in common, that is, there exist $e = (x, y)$ and $f = (x_1, y_1)$, arcs in G, such that $eg = fh$. Then $(xg, yg) = (x_1 h, y_1 h)$, and so $xg = x_1 h$ and $yg = y_1 h$. Thus $yx^{-1} = y_1 x_1^{-1}$ which implies that $e = f$ and hence $g = h$.

Proof of Theorem 3. We shall apply the lemma with $\Gamma = \mathbb{Z}_n = \{0, 1,..., n-1\}$ and denote the vertices of $K_n{}^*$ by the elements of $\mathbb{Z}_n$.

Case 1: *n even.* Let the vertices of K_{n+1}^* be the elements of $\mathbb{Z}_n \cup \{\infty\}$ where ∞ is a new symbol satisfying $x + \infty = \infty$ for all $x \in \mathbb{Z}_n$. Define a circuit P by $P = (\infty, 0, n-1, 1, n-2, \ldots, n/2+1, \ n/2-1, \ \infty)$. Since among the differences on the edges of P, all nonzero residues modulo n except 1 occur precisely once, developing P according to $\mathbb{Z}_n$ produces a total of n n-circuits. If we adjoin one additional circuit $C = (0, 1, 2, \ldots, n-1, 0)$, we obtain $n+1$ circuits forming a decomposition of K_{n+1}^*.

Case 2: *n odd.* Again let the vertices of K_{n+1}^* be the elements of $\mathbb{Z}_n \cup \{\infty\}$. Define a circuit P by $P = (\infty, 0, n-1, 1, n-2, \ldots, [n/4]-1, [3n/4]+1, \ [n/4], \ [3n/4]-1, \ [n/4]+1, \ldots, [n/2]-1, \ [n/2], \infty)$. Since among the differences on the edges of P, all nonzero residues modulo n except $a = [n/2]+1$ occur precisely once, developing P according to $\mathbb{Z}_n$ produces a total of n n-circuits. If we adjoin one additional circuit $C = (0, a, 2a, 3a, \ldots, (n-1)a, 0)$, we obtain $n+1$ circuits forming a decomposition of K_{n+1}^*.

Remark. Wilson [17] has proved that if $n \equiv 0$ or $1 \pmod k$ and $n > C(k)$, then the (undirected) complete graph K_n can be decomposed into edge disjoint K_k and K_{k+1}. By applying Theorem 3 for K_{k+1}^* and Corollary 1 for $K_k{}^*$ (k odd), we deduce that if k is odd, then $n \equiv 0$ or $1 \pmod k$ and $n > C(k)$ imply that $K_n{}^*$ can be decomposed into arc disjoint k-circuits. For example, if k is a power of a prime, the necessary condition in order to decompose $K_n{}^*$ into k-circuits, $n(n-1) \equiv 0 \pmod k$ reduces to $n \equiv 0$ or $1 \pmod k$ and so for large values of n the condition is also sufficient. Sotteau has recently shown [15] that if k is odd, then $K_n{}^*$ can be decomposed into arc disjoint k-circuits for all $n \equiv 0$ or $1 \pmod k$.

In the case k even, we can deduce a similar result whenever $K_k{}^*$ can be decomposed into k-circuits (see Theorems 5 and 6).

3. Two Lemmas

Definition. The *complete directed bipartite graph*, $K_{m,n}^*$, is the directed graph whose set of vertices is $X \cup Y$ with $|X| = m$ and $|Y| = n$ and whose arcs are all the ordered pairs (x, y) and (y, x) where $x \in X$ and $y \in Y$.

We have the following lemmas which will be useful in the case k even (all the circuits of $K_{m,n}^*$ being of even length).

Lemma 2. *If $K_m{}^*$, $K_n{}^*$ and $K_{m,n}^*$ can be decomposed into k-circuits, then K_{m+n}^* can be decomposed into k-circuits.*

Proof. The proof is obvious and is omitted.

LEMMA 3. *If K^*_{m+1}, K^*_{n+1}, and $K^*_{m,n}$ can be decomposed into k-circuits, then K^*_{m+n+1} can be decomposed into k-circuits.*

Proof. Let the set of vertices of K^*_{m+n+1} be $X \cup Y \cup \{x_0\}$ with $|X| = m$ and $|Y| = n$. The arcs of K^*_{m+n+1} can be partitioned into the arcs of the complete graph on $X \cup \{x_0\}$, isomorpnic to K^*_{m+1}, the arcs of the complete graph on $Y \cup \{x_0\}$, isomorphic to K^*_{n+1}, and the arcs of the complete bipartite graph on X and Y, isomorphic to $K^*_{m,n}$.

4. DECOMPOSITION OF $K^*_{m,n}$ INTO $2k$-CIRCUITS

THEOREM 4. *If $m \geqslant k$, $n \geqslant k$ and if k divides m or n, then $K^*_{m,n}$ can be decomposed into 2k-circuits.*

Proof. We shall suppose that k divides m.

Step 1. $K^*_{k,r}$ where $k \leqslant r \leqslant 2k-1$ can be decomposed into $2k$-circuits: If we denote the vertices of $K^*_{k,r}$ by $x_1, x_2, ..., x_k$ and $y_1, y_2, ..., y_r$, the decomposition is given by the following r $2k$-circuits: $(x_1, y_{1+j}, x_2, y_{2+j}, ..., x_i, y_{i+j}, ..., x_k, y_{k+j}, x_1)$ for $j = 0, 1, ..., r-1$ (the numbers which appear are to be taken modulo r.)

Step 2. $K^*_{k,n}$ can be decomposed into $2k$-circuits for all $n \geqslant k$: This has been proven for $k \leqslant r \leqslant 2k-1$. The arcs of $K^*_{k,n}$ can be partitioned into the arcs of q graphs isomorphic to $K^*_{k,k}$ and the arcs of a graph isomorphic to $K^*_{k,r}$, and so by Step 1 the arcs of $K^*_{k,n}$ can be partitioned into $2k$-circuits.

Step 3. $K^*_{m,n}$ can be decomposed into $2k$-circuits for $n \geqslant k$: This results from the partition of the arcs of $K^*_{m,n}$ into the arcs of m/k graphs isomorphic to $K^*_{k,n}$ and by Step 2.

Remark. Necessary conditions in order to decompose $K^*_{m,n}$ into $2k$-circuits are that $m \geqslant k$, $n \geqslant k$ and that the number of arcs, $2mn$, is a multiple of $2k$. Thus we pose the following conjecture (which holds by Theorem 4 in case k is prime).

Conjecture 1. $K^*_{m,n}$ can be decomposed into $2k$-circuits if and only if $m \geqslant k$, $n \geqslant k$ and k divides the product mn.

5. DECOMPOSITION OF K^*_{qk+1} AND K^*_{qk} INTO k-CIRCUITS (k EVEN)

If the undirected complete graph, K_n, can be decomposed into (undirected) k-cycles, then necessarily n is odd and k divides $n(n-1)/2$. Kotzig and Rosa have shown that (∗) if $n \geqslant k$ is odd and either $n \equiv 0$

(mod k) or $\frac{1}{2}(n-1) \equiv 0 \pmod{k}$, then K_n can be decomposed into k-cycles ($k \geqslant 3$). More specifically, Kotzig [10] showed (*) for $k \equiv 0 \pmod{4}$, while Rosa showed (*) for odd k in [12] and for $k \equiv 2 \pmod{4}$ in [13]. The methods employed in this section can also be used to show (*) when k is even.

THEOREM 5. *If k is even, K^*_{qk+1} can be decomposed into k-circuits.*

Proof. The theorem is true for $q = 1$ (Theorem 3). Suppose the theorem holds for q, then by Lemma 3 applied to K^*_{qk+1}, K^*_{k+1} and $K^*_{qk,k}$ (decomposable by Theorem 4) we deduce that $K^*_{(q+1)k+1}$ is decomposable into k-circuits; and so the theorem holds by induction.

The following proposition is a particular case of a theorem found independently by Hartnell and Milgram [9].

PROPOSITION 3. *Let k be an even positive integer. If $p \geqslant k$ is a prime, $K_p{}^*$ can be decomposed into k-circuits if and only if $p(p-1) \equiv 0 \pmod{k}$.*

Proof. The necessary condition is Proposition 1. If $p(p-1) \equiv 0 \pmod{k}$, since k is even, k divides $p - 1$ (the case $p = 2$ is trivial). Thus $p = qk + 1$ and the condition is sufficient by Theorem 5.

THEOREM 6. *If k is even and if $K_k{}^*$ is decomposable into k-circuits, then K^*_{qk} can be decomposed into k-circuits.*

Proof. By hypothesis the theorem holds for $q = 1$. Suppose the theorem is true for q, then by Lemma 2 with $m = qk$ and $n = k$, $K^*_{(q+1)k}$ can be decomposed into k-circuits and the theorem follows by induction.

Theorem 6 can also be proved with Theorem 5 and Lemma 3 as we see in the following generalization.

THEOREM 7. *If k is even and if $K^*_{n_0}$ is decomposable into k-circuits, then $K^*_{n_0+qk}$ can be decomposed into k-circuits.*

Proof. We apply Lemma 3 with $m = qk$ and $n = n_0 - 1$; $K^*_{n_0}$ is decomposable by hypothesis, K^*_{qk+1} by Theorem 5 and K^*_{qk,n_0-1} by Theorem 4.

Theorem 7 will be useful in proving Conjecture 2 in the case where k is even.

Conjecture 2. $K_n{}^*$ ($n \geqslant k$) can be decomposed into k-circuits if and only if $n(n-1) \equiv 0 \pmod{k}$, except for $n = 6 = 2k$, $n = 4 = k$, and $n = 6 = k$.

To prove that this conjecture holds for all values of $n \geqslant n_0(k)$, in the case where k is even, it will suffice to prove that $K_n{}^*$ is decomposable into k-circuits for all n such that $n(n-1) \equiv 0 \pmod k$ with $n_0 \leqslant n < n_0 + k$. We shall see applications in the cases $k = 4, 6, 8, 16$.

6. Decomposition of K_{2k}^* into k-Circuits (k even).

Since $K_k{}^*$ is decomposable into k-circuits for $k = 8, 10, 12, 14, 16, 18, 22, 40, 56$ and 58, K_{2k}^* is decomposable into k-circuits for those values of k by Theorem 6. We shall prove in Lemmas 4 and 5 that K_{2k}^* is decomposable into k-circuits for $k = 4$ and 6 (values for which $K_k{}^*$ is not decomposable into k-circuits.)

Remark. By Theorem 7, if K_{2K}^* were decomposable into k-circuits then K_{qk}^* $(q \geqslant 2)$ would be decomposable into k-circuits, and in the case $k = 2^r$ the decomposition of $K_n{}^*$ would be solved (except for $n = k$) because in the case $k = 2^r$ the condition $n(n-1) \equiv 0 \pmod k$ reduces to $n \equiv 0$ or $1 \pmod k$.

Lemma 4. *$K_8{}^*$ is decomposable into 4-circuits.*

Proof. The decomposition is given by the following 4-circuits: (1, 2, 3, 4, 1), (1, 4, 3, 2, 1), (5, 6, 7, 8, 5), (5, 8, 7, 6, 5), (1, 3, 7, 5, 1), (2, 4, 8, 6, 2), (1, 6, 8, 3, 1), (5, 7, 4, 2, 5), (1, 5, 2, 6, 1), (3, 8, 4, 7, 3), (1, 7, 2, 8, 1), (1, 8, 2, 7, 1), (3, 5, 4, 6, 3), and (3, 6, 4, 5, 3).

Lemma 5. *K_{12}^* is decomposable into 6-circuits.*

Proof. The decomposition is given by the following 22 6-circuits: 4 6-circuits of the complete graph on {1, 2, 3, 4, 5, 6}: (1, 2, 6, 3, 5, 4, 1), (1, 4, 5, 3, 6, 2, 1), (1, 3, 2, 4, 6, 5, 1), (1, 5, 6, 4, 2, 3, 1); 4 6-circuits of the complete graph on {7, 8, 9, 10, 11, 12}: (7, 8, 12, 9, 11, 10, 7), (7, 10, 11, 9, 12, 8, 7), (7, 9, 8, 10, 12, 11, 7), (7, 11, 12, 10, 8, 9, 7); 6 6-circuits using the remaining arcs of these complete graphs: (1, 6, 7, 12, 2, 11, 1), (2, 5, 8, 11, 3, 10, 2), (3, 2, 9, 10, 1, 12, 3), (6, 1, 10, 9, 5, 7, 6), (5, 2, 12, 7, 4, 8, 5), (4, 3, 11, 8, 6, 9, 4); the 6 6-circuits of $K_{\{1,2,3\},\{7,8,9\}}^*$ and $K_{\{4,5,6\},\{10,11,12\}}^*$; and 2 6-circuits using the remaining arcs: (1, 11, 2, 10, 3, 12, 1), (6, 8, 4, 7, 5, 9, 6).

Remark. The first author [3] has shown that $K_6{}^*$ cannot be decomposed into 3-circuits. In [5], it is shown that K_{2k}^* can be decomposed into k circuits for all $k > 3$.

7. Decomposition of $K_n{}^*$ into k-Circuits for $k = 4, 6, 8, 16$

THEOREM 8. *$K_n{}^*$ is decomposable into 4-circuits if and only if $n(n-1) \equiv 0 \pmod 4$ and $n > 4$.*

Proof. The equation $n(n-1) \equiv 0 \pmod 4$ reduces to $n \equiv 0$ or $1 \pmod 4$. $K_4{}^*$ is not decomposable; $K_5{}^*$ (by Theorem 3) and $K_8{}^*$ (by Lemma 4) are decomposable and thus the theorem follows from Theorem 7.

Remark. This theorem has been proved independently by Schonheim [14].

LEMMA 6. *$K_9{}^*$ and K_{10}^* are decomposable into 6-circuits.*

Proof. The decomposition of $K_9{}^*$ is given by the following 12 6-circuits: 4 6-circuits of the complete graph on {1, 2, 3, 4, 5, 6}: (1, 2, 6, 3, 5, 4, 1), (1, 4, 5, 3, 6, 2, 1), (1, 3, 2, 4, 6, 5, 1), (1, 5, 6, 4, 2, 3, 1); and (1, 6, 7, 8, 2, 9, 1), (2, 5, 8, 9, 3, 7, 2), (3, 4, 9, 7, 1, 8, 3), (6, 1, 7, 9, 4, 8, 6), (5, 2, 8, 7, 6, 9, 5), (4, 3, 9, 8, 5, 7, 4), (1, 9, 2, 7, 3, 8, 1), (6, 8, 4, 7, 5, 9, 6). The decomposition of K_{10}^* is given by the following 15 6-circuits: 6 6-circuits of the complete graph on {0, 1, 2, 3, 4, 5, 6} (see Theorem 3): (6, 0, 5, 1, 4, 2, 6,), (6, 1, 0, 2, 5, 3, 6), (6, 2, 1, 3, 0, 4, 6), (6, 3, 2, 4, 1, 5, 6), (6, 4, 3, 5, 2, 0, 6), (6, 5, 4, 0, 3, 1, 6); and (0, 1, 7, 8, 2, 9, 0), (2, 3, 8, 9, 4, 7, 2), (4, 5, 9, 7, 0, 8, 4), (1, 2, 7, 9, 5, 8, 1), (3, 4, 8, 7, 1, 9, 3), (5, 0, 9, 8, 3, 7, 5), (6, 7, 4, 9, 2, 8, 6), (6, 8, 5, 7, 3, 9, 6), (6, 9, 1, 8, 0, 7, 6).

THEOREM 9. *$K_n{}^*$ is decomposable into 6-circuits if and only if $n(n-1) \equiv 0 \pmod 6$ and $n > 6$.*

Proof. The equation $n(n-1) \equiv 0 \pmod 6$ reduces to $n \equiv 0, 1, 3, 4 \pmod 6$. $K_6{}^*$ is not decomposable; $K_7{}^*$ (by Theorem 3), $K_9{}^*$ and K_{10}^* (by Lemma 6) and K_{12}^* (by Lemma 5) are decomposable into 6-circuits and thus the theorem follows from Theorem 7.

THEOREM 10. *$K_n{}^*$ ($n \geqslant 8$) is decomposable into 8 circuits if and only if $n(n-1) \equiv 0 \pmod 8$.*

Proof. The equation $n(n-1) \equiv 0 \pmod 8$ reduces to $n \equiv 0$ or $1 \pmod 8$. $K_8{}^*$ (see Section 1) and $K_9{}^*$ (by Theorem 3) are decomposable into 8-circuits and thus the theorem follows from Theorem 7.

THEOREM 11. *$K_n{}^*$ ($n \geqslant 16$) is decomposable into 16-circuits if and only if $n(n-1) \equiv 0 \pmod{16}$.*

Proof. As in Theorem 10, it is only necessary to decompose K_{16}^* into 16-circuits. The computer took 12.376 seconds to produce the following list: (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 1), (1, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1), (1, 4, 7, 10, 13, 16, 3, 6, 9, 12, 15, 2, 5, 8, 11, 14, 1), (1, 14, 11, 8, 5, 2, 15, 12, 9, 6, 3, 16, 13, 10, 7, 4, 1), (1, 6, 11, 16, 5, 10, 15, 4, 9, 14, 3, 8, 13, 2, 7, 12, 1), (1, 12, 7, 2, 13, 8, 3, 14, 9, 4, 15, 10, 5, 16, 11, 6, 1), (1, 3, 5, 7, 9, 2, 4, 6, 10, 12, 14, 16, 8, 15, 11, 13), (1, 15, 3, 7, 5, 9, 11, 2, 6, 13, 4, 8, 12, 16, 14, 10, 1), (1, 5, 3, 9, 7, 11, 4, 13, 15, 6, 2, 16, 10, 14, 12, 8, 1), (1, 13, 3, 10, 2, 8, 4, 11, 5, 12, 6, 14, 7, 16, 9, 15, 1), (1, 7, 3, 11, 9, 13, 6, 15, 8, 2, 10, 16, 12, 4, 14, 5, 1), (1, 11, 3, 12, 2, 14, 4, 10, 6, 8, 16, 7, 13, 5, 15, 9, 1), (1, 8, 10, 4, 16, 2, 9, 3, 15, 13, 7, 14, 6, 12, 5, 11, 1), (1, 10, 8, 14, 2, 11, 7, 15, 5, 13, 9, 16, 6, 4, 12, 3, 1), (1, 9, 5, 14, 8, 6, 16, 4, 2, 12, 10, 3, 13, 11, 15, 7, 1).

Remark. In [5], it is shown that K_n^* $(n \geqslant k)$ is decomposable into k-circuits if and only if $n(n-1) \equiv 0 \pmod{k}$ for $k = 10, 12$, and 14. In [4], this is shown for $k = 5$.

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