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Asymptotics for recurrence coefficients in the generalized Meixner case

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Polynomials $P_n(x)$ orthogonal with respect to the weight

$$\rho_M^{(\vec{\alpha},\mu)}(x) = \frac{\mu^x \prod_{j=1}^{q} \Gamma(x+\alpha_j)}{(x!)^q}, \qquad x = i = 0,1,2,\ldots; \quad \vec{\alpha} = (\alpha_1,\ldots,\alpha_q), \quad \alpha_j > 0; \;\; 0 < \mu < 1; \tag{1}$$

are semi-classical of class s $(0 \leqslant s \leqslant q-1)$, and are called Generalized Meixner [1].

The recurrence relation satisfied by the monic polynomial $P_n(x)$ is

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 0 \tag{2}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$.

When $\alpha_j = 1$ for all j, weight (1) reduces to μ^x, and

$$\beta_n = \frac{1+\mu}{1-\mu}\, n, \qquad \gamma_n = \frac{\mu}{(1-\mu)^2}\, n^2. \tag{3}$$

For $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = \cdots = \alpha_q = 1$, the weight (1) is the classical Meixner [2] weight for which β_n and γ_n are known explicitly and satisfy:

$$\lim_{n\to\infty} \frac{\beta_n}{n} = \frac{1+\mu}{1-\mu}, \quad \lim_{n\to\infty} \frac{\gamma_n}{n^2} = \frac{\mu}{(1-\mu)^2}. \tag{4}$$

These limits are independent of α, and the proposed problem is to prove that these results are true in general for any positive value of α_j in (1).

In the general case, coefficients β_n and γ_n can be computed recurrently from the two nonlinear algebraic coupled equations (Laguerre–Freud equations) [3]

$$\begin{cases} \beta_{n+1} = F_1(\beta_0,\ldots,\beta_n;\gamma_0,\gamma_1,\ldots,\gamma_n), \\ \gamma_{n+1} = F_2(\beta_0,\ldots,\beta_n;\gamma_0,\gamma_1,\ldots,\gamma_n). \end{cases} \tag{5}$$

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Even in the case $q=2$, these equations are already difficult to manage but numerical computations confirm the behaviour given in Eq. (4) [3,4].

It is therefore hopeless to investigate in the general case from Eq. (5) and a more general approach should be used.

Exact results given in (4) seem to show that the nonpolynomial modification $\Gamma(\alpha+x)/x!$ of the weight μ^x does not change the asymptotic behaviour of β_n/n and γ_n/n^2. So it is natural to guess that the asymptotic behaviour stays the same each time we multiply by $\Gamma(\alpha+x)/x! \dots$.

Let us squeeze any α_j, called α, between two consecutive integers

$$N_\alpha \leqslant \alpha \leqslant N_\alpha + 1, \quad N_\alpha \text{ integer}. \tag{6}$$

From the monotonicity of the $\Gamma(z)$ function $(1.462 < z)$, lower and upper bounds on the factor $\Gamma(\alpha+x)/x!$ can therefore be given:

$$\frac{\Gamma(N_\alpha+x)}{x!} \leqslant \frac{\Gamma(\alpha+x)}{x!} \leqslant \frac{\Gamma(N_\alpha+1+x)}{x!} \quad (2 \leqslant N_\alpha). \tag{7}$$

For $0<x<\infty$, the function $\Gamma(\alpha+x)/x!$ is bounded by the rising factorial polynomials $(x)_n \equiv x(x+1)\cdots(x+n-1)$

$$(x+1)_{N_\alpha-1} \leqslant \frac{\Gamma(\alpha+x)}{x!} \leqslant (x+1)_{N_\alpha+1}. \tag{8}$$

So the first step is to prove that polynomial modifications of the exponential weight μ^x does not change the limit given in Eq. (4). The second step should extend this result to the function $\Gamma(\alpha+x)/x!$ using the first step proof and Eq. (8). The third step should cover $0<\alpha<2$.

We are looking for such proofs.

References

[1] A. Ronveaux, Discrete semi-classical orthogonal polynomials: Generalized Meixner, J. Approx. Theory 46 (4) (1986) 403–407.

[2] A.F. Nikiforov, V.B. Uvarov, Fonctions Spéciales de la Physique Mathématique, MIR, Moscow, 1983.

[3] M. Foupouagnigni, M.N. Hounkonnou, A. Ronveaux, Laguerre–Freud Equations for the recurrence coefficients of D_ω-Semi-classical Orthogonal Polynomials of class one, J. Comput. Appl. Math. 99 (1998) 143–154.

[4] W. Van Assche, private communication.