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A note on sum of powers of the Laplacian eigenvalues of bipartite graphs ☆

Gui-Xian Tian [a], Ting-Zhu Huang [a,*], Bo Zhou [b]

[a] School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China

[b] Department of Mathematics, South China Normal University, Guangzhou, Guangdong 510631, PR China

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ABSTRACT

For a graph G and a real number $\alpha \neq 0$, the graph invariant $s_\alpha(G)$ is the sum of the αth power of the non-zero Laplacian eigenvalues of G. In this note, we obtain some bounds of $s_\alpha(G)$ for a connected bipartite graph G, which improve some known results of Zhou [B. Zhou, On sum of powers of the Laplacian eigenvalues of graphs, Linear Algebra Appl. 429 (2008) 2239–2246].

1. Introduction

Let G be a simple finite undirected connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For $v_i \in V$, the degree of v_i and the sum of the degrees of the vertices adjacent to v_i are denoted by d_i and t_i, respectively. Note that t_i is also called the 2-degree of v_i. Denote by $(d_1, d_2, \ldots, d_n)$ the degree sequence of G. Denote by $i \sim j$ if the vertices v_i and v_j are adjacent.

A graph G is called regular if every vertex of G has equal degree, that is, $d_1 = d_2 = \cdots = d_n$. G is said a semiregular bipartite graph if there is a bipartition V_1, V_2 of $V(G)$ such that each vertex in the same part V_i has the same degree for $i = 1, 2$.

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* Corresponding author.
E-mail addresses: guixiantian@uestc.edu.cn, guixiantian@163.com (G.-X. Tian), tzhuang@uestc.edu.cn, tingzhuhuang@126.com (T.-Z. Huang).

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For a connected graph G, the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues and denoted by $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \mu_n = 0$. The eigenvalue μ_1 is called the Laplacian spectral radius of G. In addition, $L(G)$ and $D(G) + A(G)$ have the same eigenvalues if G is bipartite [1].

Let G be a simple connected graph. For a non-zero real number α, $s_\alpha(G)$, recently introduced in [13], is defined as

$$s_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha.$$

This definition was motivated by the graph energy [2,10] and the Laplacian energy [4–6,12]. The case $\alpha = 1$ is trivial as $s_1(G) = 2m$. Some properties for $s_2(G)$ and $s_{\frac{1}{2}}(G)$ have been established in [5,6], respectively. In fact, for a connected graph G with n vertices, $ns_{-1}(G)$ is equal to its Kirhhoff index and quasi-Wiener index, which have extensive applications in electric circuit, probabilistic theory and chemistry [3,9]. Recently, Zhou [13] obtained some bounds for sum of powers of the Laplacian eigenvalues of G, $s_\alpha(G)$ with $\alpha \neq 0, 1$. For a bipartite graph G, Zhou [13] gave:

Theorem A [13]. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices, t spanning trees. Then, for a real number $\alpha \neq 0, 1$,*

$$s_\alpha \geqslant \left(2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}\right)^\alpha + (n-2)\left(\frac{tn}{2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}}\right)^{\frac{\alpha}{n-2}}$$

with equality if and only if $G = K_{\frac{n}{2},\frac{n}{2}}$.

Theorem B [13]. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges:*

(i) *If $\alpha < 0$ or $\alpha > 1$, then*

$$s_\alpha(G) \geqslant \left(2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}\right)^\alpha + (n-2)^{1-\alpha}\left(2m - 2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}\right)^\alpha$$

with equality if and only if $G = K_{\frac{n}{2},\frac{n}{2}}$.

(ii) *If $0 < \alpha < 1$, then*

$$s_\alpha(G) \leqslant \left(2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}\right)^\alpha + (n-2)^{1-\alpha}\left(2m - 2\sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}\right)^\alpha$$

with equality if and only if $G = K_{\frac{n}{2},\frac{n}{2}}$.

In this note, we obtain a new lower bound on the Laplacian spectral radius for a bipartite graph G. Applying this result, we also present some bounds on sum of powers of the Laplacian eigenvalues of G, $s_\alpha(G)$ with $\alpha \neq 0, 1$ and determine the extremal graphs of these bounds. Theoretic analysis shows that these results improve Theorems A and B.

2. Main results

We first present a new lower bound of the Laplacian spectral radius of bipartite graphs, which improves some known results in [11].

Theorem 1. *Let G be a simple connected bipartite graph with degrees $d_1, d_2, \ldots, d_n$. Then*

$$\mu_1 \geqslant \sqrt{\frac{\sum_{i=1}^{n}(d_i(d_i^2+t_i)+\sum_{j\sim i}(d_j^2+t_j))^2}{\sum_{i=1}^{n}(d_i^2+t_i)^2}}. \tag{1}$$

The equality holds in (1) if and only if there exists a positive constant number t such that, for all $i \in \{1,2,\ldots,n\}$,

$$\frac{d_i(d_i^2+t_i)+\sum_{j\sim i}(d_j^2+t_j)}{d_i^2+t_i}=t.$$

In fact, $t=\mu_1$.

Proof. Since G is a bipartite graph, then $L(G)=D(G)-A(G)$ and $D(G)+A(G)$ have the same eigenvalues. Note that $D(G)+A(G)$ is a nonnegative irreducible symmetric matrix.

Now assume that $x=(x_1,x_2,\ldots,x_n)^T$ is the positive Perron eigenvector of $D(G)+A(G)$ corresponding to μ_1.

By the Raleigh principle, we have

$$\mu_1^2=\mu_1((D(G)+A(G))^2)=\frac{x^T(D(G)+A(G))^2x}{x^Tx}. \tag{2}$$

Take $C=(d_1^2+t_1,d_2^2+t_2,\ldots,d_n^2+t_n)^T$. Then

$$\begin{aligned}
&(D(G)+A(G))C\\
&\quad=D(G)C+A(G)C\\
&\quad=\left(d_1(d_1^2+t_1)+\sum_{j=1}^{n}a_{1j}(d_j^2+t_j),\ldots,d_n(d_n^2+t_n)+\sum_{j=1}^{n}a_{nj}(d_j^2+t_j)\right)^T\\
&\quad=\left(d_1(d_1^2+t_1)+\sum_{j\sim 1}(d_j^2+t_j),\ldots,d_n(d_n^2+t_n)+\sum_{j\sim n}(d_j^2+t_j)\right)^T
\end{aligned}$$

and $C^TC=\sum_{i=1}^{n}(d_i^2+t_i)^2$. It follows from (2) that

$$\begin{aligned}
\mu_1&=\sqrt{\frac{x^T(D(G)+A(G))^2x}{x^Tx}}\\
&\geqslant\sqrt{\frac{C^T(D(G)+A(G))^2C}{C^TC}}\\
&=\sqrt{\frac{\sum_{i=1}^{n}\left(d_i(d_i^2+t_i)+\sum_{j\sim i}(d_j^2+t_j)\right)^2}{\sum_{i=1}^{n}(d_i^2+t_i)^2}}.
\end{aligned}$$

Now suppose that the equality holds in (1). Then C is the positive Perron eigenvector of $(D(G)+A(G))^2$ corresponding to $\mu_1((D(G)+A(G))^2)$, that is, $(D(G)+A(G))^2C=\mu_1((D(G)+A(G))^2)C$. If the multiplicity of $\mu_1^2=\mu_1((D(G)+A(G))^2)$ is two, then $-\mu_1$ is also an eigenvalue of $D(G)+A(G)$. This is impossible as $D(G)+A(G)$ is a nonnegative irreducible positive semidefinite matrix . Hence, the multiplicity of $\mu_1^2=\mu_1((D(G)+A(G))^2)$ is one, and C is the positive Perron eigenvector of $D(G)+A(G)$ corresponding to μ_1, that is, $(D(G)+A(G))C=\mu_1 C$. This implies, for all $i\in\{1,2,\ldots,n\}$,

$$\frac{d_i(d_i^2+t_i)+\sum_{j\sim i}(d_j^2+t_j)}{d_i^2+t_i}=\mu_1.$$

Conversely, if there exists a positive constant number t such that, for all $i \in \{1, 2, \ldots, n\}$,

$$\frac{d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)}{d_i^2 + t_i} = t.$$

Then $(D(G) + A(G))C = tC$. By Perron–Frobenius Theorem, we have

$$\mu_1 = t = \sqrt{\frac{\sum_{i=1}^n \left(d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)\right)^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}.$$

This completes our proof. □

Corollary 2 [11]. *Let G be a simple connected bipartite graph with degrees $d_1, d_2, \ldots, d_n$. Then*

$$\mu_1 \geqslant \sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}}. \tag{3}$$

The equality holds in (3) if and only if G is a semiregular connected bipartite graph.

Proof. By a simple calculation, we have

$$\begin{aligned}
\left(\sum_{i=1}^n (d_i^2 + t_i)^2\right)^2 &= \left(\sum_{i=1}^n d_i^2(d_i^2 + t_i) + \sum_{j=1}^n (d_j^2 + t_j)t_j\right)^2 \\
&= \left(\sum_{i=1}^n d_i^2(d_i^2 + t_i) + \sum_{j=1}^n (d_j^2 + t_j)\sum_{i=1}^n a_{ij}d_i\right)^2 \\
&= \left(\sum_{i=1}^n d_i^2(d_i^2 + t_i) + \sum_{i=1}^n d_i \sum_{j=1}^n a_{ij}(d_j^2 + t_j)\right)^2 \\
&= \left(\sum_{i=1}^n (d_i^2(d_i^2 + t_i) + d_i \sum_{j \sim i}(d_j^2 + t_j))\right)^2 \\
&= \left(\sum_{i=1}^n (d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)) \cdot d_i\right)^2.
\end{aligned}$$

By the Cauchy–Schwartz inequality,

$$\left(\sum_{i=1}^n (d_i^2 + t_i)^2\right)^2 \leqslant \sum_{i=1}^n \left(d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)\right)^2 \cdot \sum_{i=1}^n d_i^2 \tag{4}$$

with equality if and only if there exists a positive constant number l such that, for all $i \in \{1, 2, \ldots, n\}$,

$$\frac{d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)}{d_i} = l.$$

Following from (4) and Theorem 1, the inequality (3) holds.

Now suppose that the equality holds in (3). By Theorem 1, there exists a positive constant number t such that, for all $i \in \{1, 2, \ldots, n\}$,

$$\frac{d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)}{d_i^2 + t_i} = t.$$

Thus we have, for all $i \in \{1, 2, \ldots, n\}$,

$$\frac{d_i^2 + t_i}{d_i} = \frac{l}{t}.$$

The rest of the proof is similar to that of Theorem 9 in [11].

Conversely, assume that G is a semiregular connected bipartite graph of order n with first n_1 vertices of degree Δ and the remaining n_2 vertices of degree δ. Then $\mu_1 = \Delta + \delta$ is the Laplacian spectral radius of $L(G)$ and the corresponding eigenvector is $(\Delta e_{n_1}^T, -\delta e_{n_2}^T)^T$, where $e_n^T \in R^n$ is the vector with each element 1. On the other hand, noting that $t_i = \Delta\delta$ for all $i \in \{1, 2, \ldots, n\}$, we have

$$\sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}} = \Delta + \delta.$$

Hence, the equality holds in (3). □

Applying Corollary 2, we can get the following result, which is exactly Corollary 10 in [11].

Corollary 3 [11]. *Let G be a simple connected bipartite graph with degrees $d_1, d_2, \ldots, d_n$. Then*

$$\mu_1 \geqslant 2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2}. \tag{5}$$

The equality holds in (5) if and only if G is a regular connected bipartite graph.

For a connected graph G, the diameter of G is the maximum distance between any two vertices of G. Denote the complement of the graph G by $\overline{G}$. Let K_n be the complete graph on n vertices; let $K_{r,n-r}$ be the complete bipartite graph with a bipartition V_1 and V_2, where $|V_1| = r$ and $|V_2| = n - r$ with $1 \leqslant r \leqslant n - 1$. The join $G_1 \vee G_2$ of two graphs G_1, G_2 is formed from G_1 and G_2 by adding edges joining every vertex of G_1 to every vertex of G_2.

Lemma 4 [7]. *Let G be a connected graph with diameter d. Then G has at least $d + 1$ distinct Laplacian eigenvalues.*

Lemma 5 [8]. *Let G_1 and G_2 be two graphs with r vertices and s vertices, respectively. If the Laplacian eigenvalues of G_1 and G_2 are $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $\mu_1, \mu_2, \ldots, \mu_s$, respectively, then the Laplacian eigenvalues of $G_1 \vee G_2$ are $r + s$; $\lambda_1 + s, \ldots, \lambda_{r-1} + s$; $\mu_1 + r, \ldots, \mu_{s-1} + r$; and 0.*

Lemma 6. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. Then G has exactly three distinct Laplacian eigenvalues if and only if $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.*

Proof. Suppose that G has exactly three distinct Laplacian eigenvalues. By Lemma 4, G has diameter at most 2. If the diameter of G equals to 1, then $G = K_n$, which contradicts with the condition that G is a connected bipartite graph of order $n \geqslant 3$. Thus, the diameter of G is 2, which implies that G is a complete bipartite graph $K_{r,n-r}$ with $1 \leqslant r \leqslant n - 1$. Since $K_{r,n-r} = \overline{K_r} \vee \overline{K_{n-r}}$, by Lemma 5, the Laplacian eigenvalues of $K_{r,n-r}$ are

$$n, \underbrace{n - r, \ldots, n - r}_{r-1}, \underbrace{r, \ldots, r}_{n-r-1}, 0$$

If $r = 1, n - 1$, then G has three distinct Laplacian eigenvalues n, 1 and 0. Thus, $G = K_{1,n-1}$; If $2 \leqslant r \leqslant n - 2$, then G has exactly three distinct Laplacian eigenvalues when $n - r = r$, thus forcing $r = \frac{n}{2}$. Hence, $G = K_{\frac{n}{2},\frac{n}{2}}$.

Conversely, assume that $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$. It is easy to verify that G has exactly three distinct Laplacian eigenvalues. □

Now we shall present some bounds on sum of powers of the Laplacian eigenvalues of a bipartite graph G and determine the extremal graphs of these bounds.

Let G be a simple connected graph with n vertices. For convenience, let $w_i = d_i(d_i^2 + t_i) + \sum_{j\sim i}(d_j^2 + t_j)$ for all $i \in \{1, 2, \ldots, n\}$.

Theorem 7. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices, t spanning trees. Then, for a real number $\alpha \neq 0, 1$,*

$$s_\alpha(G) \geqslant \left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}\right)^\alpha + (n-2)(tn)^{\frac{\alpha}{n-2}} \left(\sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n w_i^2}}\right)^{\frac{\alpha}{n-2}}. \tag{6}$$

The equality holds in (6) if and only if $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

Proof. Note that our proof is similar to the proof of Theorem 4 in [13]. By the Matrix Tree Theorem [1], we get $\prod_{i=1}^{n-1} \mu_i = tn$. Hence,

$$s_\alpha(G) = \mu_1^\alpha + \sum_{i=2}^{n-1} \mu_i^\alpha \geqslant \mu_1^\alpha + (n-2)\left(\prod_{i=2}^{n-1} \mu_i^\alpha\right)^{\frac{1}{n-2}} = \mu_1^\alpha + (n-2)(tn)^{\frac{\alpha}{n-2}} \mu_1^{-\frac{\alpha}{n-2}},$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. Take a function

$$f(x) = x^\alpha + (n-2)(tn)^{\frac{\alpha}{n-2}} x^{-\frac{\alpha}{n-2}}$$

for $x \geqslant (tn)^{\frac{1}{n-1}}$. Solving

$$f'(x) = \alpha x^{\alpha-1} - \alpha(tn)^{\frac{\alpha}{n-2}} x^{-\frac{\alpha}{n-2}-1} \geqslant 0,$$

we get $x \geqslant (tn)^{\frac{1}{n-1}}$. Hence, $f(x)$ is increasing for $x \geqslant (tn)^{\frac{1}{n-1}}$. By Theorem 1, Corollaries 2, 3 and the proof of Theorem 4 in [13], we have

$$\mu_1 \geqslant \sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}} \geqslant \sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}} \geqslant 2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2} \geqslant (tn)^{\frac{1}{n-1}}, \tag{7}$$

where the last inequality holds as

$$2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2} \geqslant \frac{4m}{n} > \frac{2m}{n-1} = \frac{\sum_{i=1}^{n-1} \mu_i}{n-1} \geqslant \left(\prod_{i=1}^{n-1} \mu_i\right)^{\frac{1}{n-1}} = (tn)^{\frac{1}{n-1}}. \tag{8}$$

Hence,

$$s_\alpha(G) \geqslant f(\mu_1) \geqslant f\left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}\right),$$

which implies that the inequality (6) holds, and the equality holds in (6) if and only if $\mu_1 = \sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}$ and $\mu_2 = \cdots = \mu_{n-1}$.

Now assume that the equality holds in (6). Then G has at most three distinct Laplacian eigenvalues. If G has two distinct Laplacian eigenvalues, then $G = K_n$, which contradicts with the condition that G is a connected bipartite graph of order $n \geqslant 3$. Thus G has exactly three distinct Laplacian eigenvalues. By Lemma 6, we obtain that $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

Conversely, assume that $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$. It is easy to verify that the equality holds in (6). □

Remark 1. By the proof of Theorem 7, we may get

$$s_\alpha(G) \geqslant f(\mu_1) \geqslant f\left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right) \geqslant f\left(\sqrt{\frac{\sum_{i=1}^n (d_i^2+t_i)^2}{\sum_{i=1}^n d_i^2}}\right) \geqslant f\left(2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2}\right),$$

which implies that Theorem 7 improves Theorem A, that is, Theorem 4 in [13].

Theorem 8. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges. Then, for a real number $\alpha \neq 0, 1$,*

(i) *If $\alpha < 0$ or $\alpha > 1$, then*

$$s_\alpha(G) \geqslant \left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right)^\alpha + (n-2)^{1-\alpha}\left(2m - \sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right)^\alpha. \tag{9}$$

The equality holds in (9) if and only if $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

(ii) *If $0 < \alpha < 1$, then*

$$s_\alpha(G) \leqslant \left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right)^\alpha + (n-2)^{1-\alpha}\left(2m - \sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right)^\alpha. \tag{10}$$

The equality holds in (10) if and only if $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

Proof. Note that our proof is similar to the proof of Theorem 5 in [13].

To prove (i), assume that $\alpha < 0$ or $\alpha > 1$. Since $x^\alpha\,(x > 0)$ is a strictly convex function when $\alpha < 0$ or $\alpha > 1$, then

$$\sum_{i=2}^{n-1} \frac{1}{n-2}\mu_i^\alpha \geqslant \left(\sum_{i=2}^{n-1} \frac{1}{n-2}\mu_i\right)^\alpha$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. Hence,

$$s_\alpha(G) = \mu_1^\alpha + \sum_{i=2}^{n-1}\mu_i^\alpha \geqslant \mu_1^\alpha + (n-2)^{1-\alpha}\left(\sum_{i=2}^{n-1}\mu_i\right)^\alpha = \mu_1^\alpha + (n-2)^{1-\alpha}(2m-\mu_1)^\alpha.$$

Take a function

$$g(x) = x^\alpha + (n-2)^{1-\alpha}(2m-x)^\alpha$$

for $x \geqslant \frac{2m}{n-1}$. Solving

$$g'(x) = \alpha x^{\alpha-1} - \alpha(n-2)^{1-\alpha}(2m-x)^{\alpha-1} \geqslant 0,$$

we get $x \geqslant \frac{2m}{n-1}$, which implies that $g(x)$ is increasing for $x \geqslant \frac{2m}{n-1}$. Hence, from (7) and (8), we have

$$s_\alpha(G) \geqslant g(\mu_1) \geqslant g\left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}\right),$$

which implies that the inequality (9) holds, and the equality holds in (9) if and only if $\mu_1 = \sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2+t_i)^2}}$ and $\mu_2 = \cdots = \mu_{n-1}$.

In the following, using similar arguments as in Theorem 7, we get that the equality holds in (9) if and only if $G = K_{1,n-1}$ or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even.

If $0 < \alpha < 1$, then $x^\alpha (x > 0)$ is a strictly concave function and $g(x)$ is decreasing for $x \geqslant \frac{2m}{n-1}$. By a parallel argument, we may prove (ii). □

Remark 2. From the proof of Theorem 8, we may see

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geqslant g(\mu_1) \geqslant g\left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}\right) \geqslant g\left(\sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}}\right) \geqslant g\left(2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2}\right);$$

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leqslant g(\mu_1) \leqslant g\left(\sqrt{\frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n (d_i^2 + t_i)^2}}\right) \leqslant g\left(\sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}}\right) \leqslant g\left(2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2}\right).$$

Hence, Theorem 8 improves Theorem B, that is, Theorem 5 in [13].

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References

[1] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (2) (1990) 218–238.
[2] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.
[3] I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, J. Chem. Inf. Comput. Sci. 36 (1996) 982–985.
[4] I. Gutman, B. Zhou, Laplacian energy of a graph, Linear Algebra Appl. 414 (2006) 29–37.
[5] M. Lazić, On the Laplacian energy of a graph, Czech. Math. J. 56 (2006) 1207–1213.
[6] J. Liu, B. Liu, A Laplacian-energy-like invariant of a graph, MATCH Commun. Math. Comput. Chem. 59 (2008) 355–372.
[7] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[8] R. Merris, Laplacian graph eigenvectors, Linear Algebra Appl. 278 (1998) 221–236.
[9] J. Palacios, Foster's formulas via probability and the Kirchhoff index, Methodol. Comput. Appl. Probab. 6 (2004) 381–387.
[10] J. Rada, A. Tineo, Upper and lower bounds for the energy of bipartite graphs, J. Math. Anal. Appl. 289 (2004) 446–455.
[11] A. Yu, M. Lu, F. Tian, On the spectral radius of graphs, Linear Algebra Appl. 387 (2004) 41–49.
[12] B. Zhou, I. Gutman, A connection between ordinary and Laplacian spectra of bipartite graphs, Linear and Multilinear Algebra 56 (2008) 305–310.
[13] B. Zhou, On sum of powers of the Laplacian eigenvalues of graphs, Linear Algebra Appl. 429 (2008) 2239–2246.