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New polytope decompositions and Euler–Maclaurin formulas for simple integral polytopes

José Agapito [a,1], Leonor Godinho [b,*,2]

[a] *Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal*

[b] *Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal*

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Abstract

We give new weighted decompositions for simple polytopes, generalizing previous results of Lawrence–Varchenko and Brianchon–Gram. We start with Witten's non-abelian localization principle in equivariant cohomology for the norm-square of the moment map in the context of toric varieties to obtain a decomposition for Delzant polytopes. Then, by a purely combinatorial argument, we show that this formula holds for any simple polytope. As an application, we study Euler–Maclaurin formulas.

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* Corresponding author.
E-mail addresses: agapito@math.ist.utl.pt (J. Agapito), lgodin@math.ist.utl.pt (L. Godinho).

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1. Introduction

The interplay between symplectic geometry and combinatorics through the study of moment maps and the use of equivariant cohomology and the geometry of toric varieties is a well known and fertile theme in mathematics. See for example the work of Brion–Vergne [10], Cappell–Shaneson [12,13], Ginzburg–Guillemin–Karshon [21], Guillemin [18,19], Morelli [32], and Pommersheim and Thomas [34,35].

In this paper we use Witten's non-abelian localization principle in equivariant cohomology for the norm-square of the moment map [40], described by Paradan in [33], to motivate several new weighted polytope decomposition formulas. Indeed, applying this principle to the case of toric manifolds, we obtain weighted polytope decompositions for Delzant polytopes [14] (see Example 17 in Section 3). Then, by a purely combinatorial argument, we show, in Section 4.4, that they are in fact valid for *any* simple polytope not necessarily the moment map image of a toric manifold (recall that a polytope in $\mathbb{R}^d$ is called simple if each vertex is the intersection of exactly d facets). In Section 4.5, following the idea behind these decompositions, we construct new ones that generalize the Lawrence–Varchenko decomposition (see [39] and [29]) and, in some cases, the Brianchon–Gram formula (see [9,16,17,37]) (cf. Remark 37).

The well-known classical polytope decomposition formula of Brianchon–Gram expresses the characteristic function $\mathbf{1}_P$ of a convex polytope P as the alternating sum of the characteristic functions of all tangent cones to the faces of P. By flipping the edge vectors emanating from each vertex of P in a systematic way using a polarizing vector, we obtain the Lawrence–Varchenko decomposition also known as the polar decomposition. This formula expresses the characteristic function of a convex simple polytope (*only*) in terms of the characteristic functions of polarized cones supported at the vertices. Karshon, Sternberg and Weitsman [23] and Agapito [1] gave weighted versions of this decomposition by assigning weights to the faces of the polytope and of the cones in a consistent way. Our polytope decompositions combine the above two formulas. Like Brianchon–Gram they express $\mathbf{1}_P$ in terms of characteristic functions of cones with apices the different faces of the polytope. However, these cones may no longer be the ordinary tangent cones to the polytope. In our formulas we assign a *different* polarizing vector to each face and we flip the edges of the tangent cones accordingly. In the first decomposition presented, Theorem 4.1, these polarizing vectors are obtained by choosing a suitable starting point ε (the same for all vectors) and taking as end points its orthogonal projections $\beta(\varepsilon, F)$ onto the faces F of the polytope, whenever these projections are nonempty (cf. Fig. 3). In the second decomposition formula, Theorem 4.2, we take the vectors $\varepsilon - \beta(\varepsilon, F)$ as polarizing vectors (instead of $\beta(\varepsilon, F) - \varepsilon$ as above). Choosing ε appropriately we obtain the Lawrence–Varchenko and, in some cases, the Brianchon–Gram relations.

As an application, in Section 5 we use our decompositions to give new Euler–Maclaurin formulas with remainder similar to those of Karshon–Sternberg–Weitsman [23] and Agapito–Weitsman [3]. The classical Euler–Maclaurin formula computes the sum of the values of a function f over the integer points of an interval in terms of the integral of f over variations of that interval. This formula was generalized by Khovanskii and Pukhlikov (see [24] and [25]) to a formula for the sum of the values of an exponential or polynomial function on the lattice points of a regular integral polytope. In addition, Cappell and Shaneson [12,13,36], Guillemin [19] and Brion and Vergne [10] generalized it to *simple* integral polytopes, and Berline, Brion, Szenes and Vergne [6,11,38], to any rational polytope. Note that all these formulas are exact and valid for sums of values of exponential or polynomial functions. Moreover, the formula in [6] has the additional feature that it is *local*, in the sense that it is given as a sum of integrals over the faces of the polytope of maps $D(F) \cdot p$, for operators $D(F)$ depending *only* on a neighborhood of a generic point of the face F. In [23], Karshon, Sternberg and Weitsman prove a formula with remainder for the sum of an arbitrary smooth function f of compact support, on the integer points of a simple polytope. There, the remainder is given as a sum over the vertices of the polytope, of integrals over cones with those vertices, of bounded periodic functions times several partial derivatives of f. In our formulation (Theorem 5.1) both the Euler–Maclaurin formula and the remainder are given as a sum over *all* faces of the polytope (not only over vertices) of integrals over cones with apex the affine spaces generated by those faces. Since our formula generalizes to symbols (in the sense of Hormander [22]), we show that, in the case of a polynomial function p, we also obtain an exact Euler–Maclaurin formula for the sum of the values of p over the integer points of the polytope. This relation is a weighted version of the exact Euler–Maclaurin formulas obtained in [10] and in [23].

2. Critical set of the function $\|\mu_\varepsilon\|^2$

Let (M, ω) be a compact connected symplectic manifold equipped with a Hamiltonian action of a torus T. Denoting by $\mathfrak{t}$ the Lie algebra of T and by $\mathfrak{t}^*$ its dual space, we consider the moment map $\mu : M \to \mathfrak{t}^*$ associated to this action. This T-equivariant map is determined, up to a constant, by the equation $d\langle \mu, X\rangle = \iota(X_M)\omega$ for all $X \in \mathfrak{t}$. Since the action of T on $\mathfrak{t}$ is trivial, the perturbed map $\mu_\varepsilon := \mu - \varepsilon$ for $\varepsilon \in \mathfrak{t}^*$ is also a moment map for the action of T.

Hereafter we will choose a scalar product on $\mathfrak{t}^*$ (inducing a linear isomorphism $j : \mathfrak{t} \to \mathfrak{t}^*$) and consider two kinds of *orthogonality* on $\mathfrak{t}^*$: the orthogonality resulting from the duality between $\mathfrak{t}$ and $\mathfrak{t}^*$ and the j-orthogonality defined by the scalar product. We will begin by reviewing the structure of the critical points of the moment map and then define an index set $\mathcal{B}$ which will enable us to define a partition of the set of critical points of the function $\|\mu_\varepsilon\|^2$.

Following Paradan in [33], we will consider a modified definition of critical point. Usually, for a smooth map $f : M \to N$, a critical point x is defined to be a point where the derivative $df_x : T_xM \to T_{f(x)}N$ is not surjective. In the case of the moment map, $d\mu_x$ is not surjective if it exists $X \in \mathfrak{t} \setminus \{0\}$ such that $\langle d\mu_x, X\rangle = 0$. This is the case when $X_M(x) = 0$. Hence, a point x is a critical point of the moment map μ if and only if its stabilizer, $\mathrm{Stab}(x)$, contains a subtorus of dimension 1. However, if the action is not effective, all points of M will be critical points of μ. To avoid this situation we take the subgroup $S_M := \bigcap_{x \in M} \mathrm{Stab}(x)$, called the *generic stabilizer*, and make the following definition:

Definition 1. The critical points of the moment map $\mu : M \to \mathfrak{t}^*$ are the points $x \in M$ for which $\mathrm{Stab}(x)/S_M$ is not finite.

These are the critical points (by the usual definition) of the restriction $\mu : M \to A_M$, where A_M is an affine subspace with the direction of

$$(\mathfrak{s}_M)^\perp = \{\xi \in \mathfrak{t}^*: \langle \xi, X\rangle = 0,\ \forall X \in \mathfrak{s}_M\},$$

where $\mathfrak{s}_M$ is the Lie algebra of S_M.

Let T' be a subtorus of T containing S_M such that T'/S_M is not finite. Every connected component F' of $M^{T'}$ is a symplectic submanifold of M. Its image $P := \mu(F')$ is a convex polytope in $\mathfrak{t}^*$ equal to the convex hull of the image of the fixed points of T contained in F' (cf. [4,20]). Moreover, the Lie algebra of T' is contained in $P^\perp$ (the set of vectors of $\mathfrak{t}$ orthogonal to P), and, denoting by T_P the subtorus of T generated by $\exp(P^{\perp})$, we have that $\mathfrak{t}_P$, the Lie algebra of T_P, is equal to $P^{\perp}$. Hence, F' is a connected component of M^{T_P} where T/T_P acts quasi-effectively, meaning that the generic stabilizer of F' is a subgroup of T with identity component equal to T_P. Knowing this and denoting by Δ_P the affine subspace of $\mathfrak{t}^*$ generated by P, we consider the following sets:

$$\mathcal{B}' := \big\{\text{convex polytopes } P \subset \mathfrak{t}^* \text{ for which there exists a connected component } F' \text{ of } M^{T_P} \text{ with } \mu(F') = P\big\}; \tag{2}$$

$$\mathcal{B} := \{\Delta_P \mid P \in \mathcal{B}'\}; \tag{3}$$

$$\mathcal{B}'_\Delta := \big\{P \in \mathcal{B}' \bigm| P \subset \Delta \text{ and } \dim(P) < \dim(\Delta)\big\}, \quad \text{for } \Delta \in \mathcal{B}; \tag{4}$$

$$\mathfrak{t}^*_{\mathrm{reg}} := \mathfrak{t}^* \setminus \bigcup_{P \in \mathcal{B}' \setminus \mu(M)} P, \qquad \Delta_{\mathrm{reg}} := \Delta \setminus \bigcup_{P \in \mathcal{B}'_\Delta} P; \tag{5}$$

$$W_\Delta := \Delta_{\mathrm{reg}} + j(\mathfrak{t}_\Delta), \quad \text{for } \Delta \in \mathcal{B}; \tag{6}$$

$$W := \bigcap_{\Delta \in \mathcal{B}} W_\Delta. \tag{7}$$

Here, for $\Delta \in \mathcal{B}$, we denoted by T_Δ the subtorus of T generated by $\exp(\Delta^{\perp})$ and by $\mathfrak{t}_\Delta$ its Lie algebra. Note that the set $\mathcal{B}'$ contains all the faces of the polytope $\mu(M)$ and that, if P is a polytope in $\mathcal{B}'$, all its faces are also in $\mathcal{B}'$. Moreover, note that in (5) the set $\mathfrak{t}^*_{\mathrm{reg}} \subset \mathfrak{t}^*$ is the set of regular points of the moment map and that, in (6), $j(\mathfrak{t}_\Delta)$ is the j-orthogonal complement of Δ in $\mathfrak{t}^*$.

With this notation we have the following proposition which characterizes the critical set of $\|\mu_\varepsilon\|^2$ (cf. [26] and [33] for details):

Proposition 8 *(Kirwan). For every $\varepsilon \in \mathfrak{t}^*$, the critical set of $\|\mu_\varepsilon\|^2$ is given by*

$$\mathrm{Cr}\big(\|\mu_\varepsilon\|^2\big) = \bigcup_{\Delta \in \mathcal{B}} M^{T_\Delta} \cap \mu^{-1}\big(\beta(\varepsilon, \Delta)\big),$$

where, for an affine subspace Δ of $\mathfrak{t}^*$, $\beta(\varepsilon, \Delta)$ is the orthogonal projection of ε on Δ. Moreover,

(i) *for every $\varepsilon \in W_\Delta$, the set*

$$C^\varepsilon_\Delta := M^{T_\Delta} \cap \mu^{-1}\big(\beta(\varepsilon, \Delta)\big) \tag{9}$$

is a submanifold of M on which T/T_Δ acts locally freely;

(ii) *the set W is dense in $\mathfrak{t}^*$ and, for every $\varepsilon \in W$, the submanifolds C^ε_Δ, for $\Delta \in \mathcal{B}$, form a partition of* $\mathrm{Cr}(\|\mu_\varepsilon\|^2)$.

Since the group T/T_Δ acts locally freely on the manifold C^ε_Δ, we can define the quotient $M^\varepsilon_\Delta := C^\varepsilon_\Delta/(T/T_\Delta)$ which will be an orbifold. Moreover, for each connected component F of C^ε_Δ, we will consider the subgroup $S^\Delta(F) := \bigcap_{x \in F} \mathrm{Stab}(x)$ and the map $F \mapsto |S^\Delta(F)|$ (defining a locally constant function on M^ε_Δ) which will be denoted by $|S^\Delta|$.

Remark 10. Note that, if $\dim(T) = \frac{1}{2}\dim(M)$ (i.e. if M is a toric manifold), then $M^{\varepsilon}_{\Delta} = \mu^{-1}(\beta(\varepsilon, \Delta))/T$ is either empty or a point.

From now on we will consider $\varepsilon \in W$.

3. Localization formulas

3.1. Equivariant cohomology

Let us begin by reviewing the different T-equivariant de Rham complexes on M (a compact connected symplectic T-manifold). We have three spaces of equivariant differential forms on M, $\Omega^*_T(M) \subset \Omega^{\infty}_T(M) \subset \Omega^{-\infty}_T(M)$, respectively with polynomial, smooth and generalized coefficients. The model for $\Omega^*_T(M)$ is due to Cartan while the other two were studied by Berline, Duflo, Kumar and Vergne (see [5,7,15,28]). Let us recall their definition.

Let $C^{\infty}(\mathfrak{t}, \Omega(M))$ be the algebra of forms $\alpha(X)$ on M which depend smoothly on $X \in \mathfrak{t}$. We will denote by $\Omega^{\infty}_T(M)$ the sub-algebra formed by its T-invariant elements (called equivariant forms). Let $S(\mathfrak{t}^*)$ be the space of polynomials on $\mathfrak{t}$ and denote by $\Omega^*_T(M) := (S(\mathfrak{t}^*) \otimes \Omega(M))^T$ the subalgebra of $\Omega^{\infty}_T(M)$ of equivariant forms with polynomial coefficients. On $\Omega^{\infty}_T(M)$ we have the differential d_T defined by

$$(d_T\alpha)(X) := \big(d - \iota(X_M)\big)\big(\alpha(X)\big)$$

for every $\alpha \in \Omega^{\infty}_T(M)$ and $X \in \mathfrak{t}$. The cohomologies associated to $(\Omega^{\infty}_T(M), d_T)$ and $(\Omega^*_T(M), d_T)$ are called the T-equivariant cohomology with C^{∞} and polynomial coefficients and denoted by $H^{\infty}_T(M)$ and $H^*_T(M)$.

Let $C^{-\infty}(\mathfrak{t}, \Omega(M))$ be the space of generalized functions on $\mathfrak{t}$ with values on $\Omega(M)$. This is, by definition, the space of continuous $\mathbb{R}$-linear maps $Hom(\mathcal{D}(\mathfrak{t}), \Omega(M))$ from the space of smooth compactly supported densities on $\mathfrak{t}$ to the space $\Omega(M)$, where $\mathcal{D}(\mathfrak{t})$ and $\Omega(M)$ are both endowed with the C^{∞}-topologies. An element $\alpha \in C^{-\infty}(\mathfrak{t}, \Omega(M))$ is denoted by $\alpha(X)$ although the value at $X \in \mathfrak{t}$ may not be defined. By definition, it is always defined in the distributional sense: if $\phi(X)$ is a C^{∞} function on $\mathfrak{t}$ with compact support (a test function) then $\langle \alpha, \phi\, dX\rangle$ is a well-defined differential form on M denoted by $\int_{\mathfrak{t}} \alpha(X)\phi(X)\, dX$. Let $\Omega^{-\infty}_T(M)$ be the subspace of T-invariant elements of $C^{-\infty}(\mathfrak{t}, \Omega(M))$. We have a canonical inclusion $\Omega^{\infty}_T(M) \subset \Omega^{-\infty}_T(M)$ and the differential d_T defined on $\Omega^{\infty}_T(M)$ extends to $\Omega^{-\infty}_T(M)$. The cohomology of the complex $(\Omega^{-\infty}_T(M), d_T)$ is called the T-equivariant cohomology on M with generalized coefficients and is denoted by $H^{-\infty}_T(M)$. We can also consider the sub-algebras $\Omega^{\infty}_{T,\mathrm{cpt}}(M)$ and $\Omega^{-\infty}_{T,\mathrm{cpt}}(M)$ of T-equivariant forms of compact support. These are stable with respect to d_T and the cohomologies associated to the corresponding complexes $(\Omega^{\infty}_{T,\mathrm{cpt}}(M), d_T)$ and $(\Omega^{-\infty}_{T,\mathrm{cpt}}(M), d_T)$ are denoted by $H^{\infty}_{T,\mathrm{cpt}}(M)$ and $H^{-\infty}_{T,\mathrm{cpt}}(M)$.

If M is oriented, integration over M defines a map $\int_M$ from $\Omega^{-\infty}_{T,\mathrm{cpt}}(M)$ to the space of T-invariant generalized functions on $\mathfrak{t}$:

$$\left\langle \int_M \alpha, \phi\, dX \right\rangle := \int_M \langle \alpha, \phi\, dX\rangle,$$

for any test function ϕ on $\mathfrak{t}$. This map induces a map from $H^{-\infty}_{T,\mathrm{cpt}}(M)$ to the space of T-invariant generalized functions on $\mathfrak{t}$.

3.2. Equivariant Euler classes

Let $p : E \to M$ be a T-equivariant real vector bundle. The vector bundle is said to have T-oriented fibers if the fibers are oriented with an orientation varying continuously and if the T-action on M preserves the orientation of all the fibers. Let us fix a T-orientation on the fibers of E and let $p_* : \Omega^{-\infty}_{T,\mathrm{cpt}}(E) \to \Omega^{-\infty}_T(M)$ be integration along the fibers (here $\Omega^{-\infty}_{T,\mathrm{cpt}}(E)$ denotes the space of equivariant forms with compact support on the total space E). This operator induces a map in cohomology (again denoted by p_*), $p_* : H^{-\infty}_{T,\mathrm{cpt}}(E) \to H^{-\infty}_T(M)$. This map is in fact an isomorphism since we are fixing a T-orientation on E (cf. Proposition 11 in [28]).

Still fixing a T-orientation on the vector bundle $p : E \to M$, there is a unique equivariant class $u \in H^{\infty}_{T,\mathrm{cpt}}(E)$ such that $p_* u = 1_M$, where 1_M is the constant function equal to 1 on M. This class is called the equivariant Thom class of E and denoted by $\mathrm{Thom}_T(E)$. The uniqueness of this form for a T-oriented, T-equivariant vector bundle follows from [31]. Moreover, its restriction to M is the equivariant Euler class of the bundle E, i.e. $e_T(E) := i^* \mathrm{Thom}_T(E)$, where $i : M \to E$ is the inclusion map.

Fixing a T-invariant pair of a scalar product on the fibers and an Euclidean connection ∇^E, Berline, Getzler and Vergne construct in [7] an explicit representative element of the equivariant Euler class which, to simplify notation, we will also denote by $e_T(E)$.

Let us assume now that there is an element $\beta \in \mathfrak{t}$ for which the zero set of the vector field on E generated by β is equal to M. For every $s > 0$ and every $X \in \mathfrak{t}$ the form $e_T(E)(X + is\beta)$ has a 0-degree component which does not vanish on M and so we can take its inverse $(e_T(E)(X + is\beta))^{-1}$ in $\Omega(M)$. Then, taking the limit

$$e_\beta^{-1}(E) := \lim_{s \to 0^+} \big(e_T(E)(X + is\beta)\big)^{-1},$$

we obtain an equivariant closed form on M that satisfies $e_T(E) \cdot e_\beta(E) = 1_M$ (see [33] for details).

Example 11. Consider the trivial bundle $E := M \times \mathbb{C}$ equipped with a T-action which is trivial on M and which, on $\mathbb{C}$, is determined by the weight $\alpha \in \mathfrak{t}^*$ (that is, $\exp(X) \cdot z := e^{i\langle \alpha, X\rangle} z$, for $X \in \mathfrak{t}$ and $z \in \mathbb{C}$). Choosing $\beta \in \mathfrak{t}$ such that $\langle \alpha, \beta\rangle \neq 0$, we have $e_T(E)(X) = -\frac{1}{2\pi}\langle \alpha, X\rangle$ and, "polarizing," that is, taking $\alpha^+ := \epsilon_\beta \alpha$ with $\langle \alpha^+, \beta\rangle > 0$ and $\epsilon_\beta = \pm 1$, we have

$$e_\beta^{-1}(E)(X) = -2\pi \lim_{s \to 0^+} \frac{1}{\langle \alpha, X + is\beta\rangle} = 2\pi i \epsilon_\beta \int_0^\infty e^{i\langle \alpha^+, X\rangle t}\, dt$$

as generalized functions. Taking the Fourier transform we obtain the equality of measures on $\mathfrak{t}^*$, $\mathcal{F}(e_\beta^{-1}(E)) = 2\pi i \epsilon_\beta H_{\alpha^+}$, where H_{α^+} is the Heaviside distribution associated to α^+ defined by

$$\langle H_{\alpha^+}, \phi\rangle = \int_0^\infty \phi\big(u\alpha^+\big)\, du,$$

for every ϕ in the Schwartz space of rapidly decreasing functions on M.

Example 12. If $M = \{F\}$ is a single point, fixed by the action of T, the bundle E decomposes as a sum of non-trivial 2-dimensional real representations of T, $E := L_1 \oplus \cdots \oplus L_k \to F$, with the action of T on each L_j determined by a weight $\alpha_j \in \mathfrak{t}^*$. Following Paradan (cf. Proposition 4.8 in [33]) we obtain the expression for the Fourier transform of $e_\beta^{-1}(E)$:

$$\mathcal{F}\big(e_\beta^{-1}(E)\big) = (2\pi i)^k \epsilon_\beta H_{\alpha_1^+} * \cdots * H_{\alpha_k^+},$$

where we polarize each α_j according to some $\beta \in \mathfrak{t}$ (such that $\langle \alpha_j, \beta\rangle \neq 0$ for $j = 1, \ldots, k$), obtaining $\alpha_j^+ := \epsilon_\beta^j \alpha_j$ with $\epsilon_\beta^j = \pm 1$, and we take $\epsilon_\beta := \prod_{j=1}^k \epsilon_\beta^j$. Note that $*$ denotes the convolution product. This measure, supported on the cone $\mathbb{R}^+\alpha_1^+ + \cdots + \mathbb{R}^+\alpha_k^+$, is defined by

$$\langle H_{\alpha_1^+} * \cdots * H_{\alpha_k^+}, \phi\rangle = \int_0^\infty \cdots \int_0^\infty \phi\left(\sum_{i=1}^k u_i \alpha_i^+\right) du_1 \ldots du_k,$$

for every rapidly decreasing function ϕ on M.

3.3. Localization

Using the sets $\mathcal{B}$, W and C_Δ^ε defined in (3), (7) and (9) of Section 2, and the orbifold $M_\Delta^\varepsilon = C_\Delta^\varepsilon/(T/T_\Delta)$, Paradan proves the following localization theorem:

Theorem 3.1 *(Paradan). Let $\varepsilon \in W$ and let $\eta \in \Omega_T^\infty(M)$ be a closed form. Then, on $C^{-\infty}(\mathfrak{t})$ we have*

$$\int_M \eta = \sum_{\Delta \in \mathcal{B}} I_\Delta^\varepsilon(\eta),$$

where $I_\Delta^\varepsilon(\eta)$ is the generalized function supported on $\mathfrak{t}_\Delta$ defined by

$$\begin{aligned} &I_\Delta^\varepsilon(\eta)(X_1 + X_2) \\ &\quad = (2\pi i)^{\dim \Delta} \int_{M_\Delta^\varepsilon} \frac{1}{|S^\Delta|} k_\Delta(\eta)(X_1) e_{\beta_\Delta}^{-1}(E_\Delta)(X_1) \diamond \delta(X_2 - w_\Delta). \end{aligned} \tag{13}$$

Here,

(i) *the variables X_1, X_2 are respectively in $\mathfrak{t}_\Delta$ and $\mathfrak{t}/\mathfrak{t}_\Delta$ (note that, for each $\Delta \in \mathcal{B}$, $\mathfrak{t}$ decomposes as a sum of vector spaces t_Δ and $\mathfrak{t}/\mathfrak{t}_\Delta$, where $\mathfrak{t}_\Delta$ is the Lie algebra of the subtorus T_Δ generated by* $\exp(\Delta^\perp)$);

(ii) *$\beta_\Delta := j^{-1}(\beta(\varepsilon, \Delta) - \varepsilon)$, where $\beta(\varepsilon, \Delta)$ is the orthogonal projection of ε on Δ;*

(iii) *$E_\Delta := N_\Delta/(T/T_\Delta)$, where N_Δ is the normal bundle of M^{T_Δ} inside M, restricted to C_Δ^ε;*

(iv) *the operator $\diamond$:*

$$\begin{aligned} \Omega_{T_\Delta}^{-\infty}\big(C_\Delta^\varepsilon\big) \times \Omega_{T/T_\Delta}^{-\infty}\big(C_\Delta^\varepsilon\big) &\to \Omega_T^{-\infty}\big(C_\Delta^\varepsilon\big) \\ (\eta, \nu) &\mapsto \eta \diamond \nu \end{aligned}$$

is defined by

$$\langle \eta \diamond \nu, \phi(X)\, dX \rangle := \langle \eta, \langle \nu, \phi(X_1 + X_2)\, dX_2 \rangle dX_1 \rangle$$

for every density $\phi(X)\, dX$ of compact support on $\mathfrak{t}$, with the Lebesgue measure dX decomposed as a product $dX_1\, dX_2$ of two Lebesgue measures on $\mathfrak{t}_\Delta$ and $\mathfrak{t}/\mathfrak{t}_\Delta$;

(v) *w_Δ is the curvature of the principal orbibundle $C^\varepsilon_\Delta \to M^\varepsilon_\Delta$;*

(vi) *$k_\Delta : H^\infty_T(M) \to H^\infty_{T_\Delta}(M^\varepsilon_\Delta)$ is the map defined by Kirwan in* [26] *as the composition of the restriction morphism $i^*_\Delta : H^\infty_T(M) \to H^\infty_{T_\Delta}(M^\varepsilon_\Delta)$ and the Chern–Weil isomorphism $W_\Delta : H^\infty_T(C^\varepsilon_\Delta) \to H^\infty_{T_\Delta}(M^\varepsilon_\Delta)$ determined by w_Δ (see* [28] *for details);*

(vii) *the equivariant form $\delta(X_2 - w_\Delta) \in \Omega^{-\infty}_{T/T_\Delta}(M^\varepsilon_\Delta)$ is defined by*

$$\langle \delta(X_2 - w_\Delta), \phi(X_2)\, dX_2 \rangle = \phi(w_\Delta)\, \mathrm{vol}(T/T_\Delta, dX_2),$$

for every function $\phi \in C^\infty(\mathfrak{t}/\mathfrak{t}_\Delta)$, where $\mathrm{vol}(T/T_\Delta, dX_2)$ is the volume of the group T/T_Δ with respect to the Haar measure compatible with the Euclidean measure[3] *on $\mathfrak{t}/\mathfrak{t}_\Delta$, dX_2.*

For every compactly supported function $\phi \in C^\infty(\mathfrak{t})$ we have

$$\begin{aligned}\left\langle \int_M \eta, \phi\, dX \right\rangle &= \int_M \langle \eta, \phi\, dX \rangle = \int_{M\times\mathfrak{t}} \eta(X)\phi(X)\, dX \\ &= \sum_{\Delta\in\mathcal{B}} c_\Delta \int_{M^\varepsilon_\Delta \times \mathfrak{t}_\Delta} \frac{1}{|S^\Delta|} k_\Delta(\eta\cdot\phi)(X_1) e^{-1}_{\beta_\Delta}(E_\Delta)(X_1)\, dX_1,\end{aligned}$$

with $c_\Delta = (2\pi i)^{\dim\Delta}\, \mathrm{vol}(T/T_\Delta, dX_2)$ and $dX = dX_1\, dX_2$.

Example 14. If $\Delta = \{p\}$ is a vertex of the polytope $\mu(M)$, then $C^\varepsilon_\Delta = \mu^{-1}(p)$ is a connected component F of M^T and

$$I^\varepsilon_{\{p\}}(\eta)(X) = \int_F i^*_F(\eta)(X) e^{-1}_{\beta_p}(N_F)(X),$$

where N_F is the normal bundle of F inside M and $\beta_p = j^{-1}(p - \varepsilon)$. If, in addition, the action of T is toric, then F is an isolated point. Moreover, taking $\eta = e^{i\omega^\sharp}$, where $\omega^\sharp$ is the equivariant symplectic form on M defined by $\omega^\sharp(X) := \omega + \langle \mu, X\rangle$, we obtain

$$I^\varepsilon_{\{p\}}\big(e^{i\omega^\sharp}\big)(X) = \epsilon_p e^{i\langle p, X\rangle} \prod_{j=1}^{\dim(M)/2} \int_0^\infty e^{i\langle \alpha^+_j, X\rangle t}\, dt, \tag{15}$$

[3] Given a Lebesgue measure dX on the Lie algebra $\mathfrak{g}$ of a compact Lie group G we can construct a translation-invariant measure on a neighborhood of G at e, and then extend this by translation in G to a Haar measure on G. For a general formula on how to compute the volume of G relative to this measure see, for instance, [30].

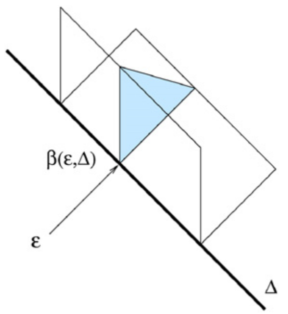

Fig. 1.

where the α_j^+'s are the weights of the action of T on the normal bundle of F polarized with respect to β_p (i.e. the polarized edge vectors at p) and ϵ_p is the sign obtained by polarization. Taking its Fourier transform we get

$$\mathcal{F}\big(I^{\varepsilon}_{\{p\}}\big(e^{i\omega^\sharp}\big)\big) = \epsilon_p \delta_p * H^+_{\alpha_1} * \cdots * H^+_{\alpha_n}, \tag{16}$$

where δ_p is the Dirac measure on $p \in \mathfrak{t}^*$. Moreover, the measure (16) is supported on the polarized cone $\mathbf{C}^\sharp_{\{p\}} := p + \mathbb{R}^+\alpha_1^+ + \cdots + \mathbb{R}^+\alpha_n^+$.

Example 17. For a toric manifold M^{2n} with moment map μ and for $\eta = e^{i\omega^\sharp}$, where again $\omega^\sharp$ is the equivariant symplectic form on M (cf. Example 14), the reduced space M^ε_Δ for $\varepsilon \in \mathfrak{t}^*$ is either empty or a single point $\{x\}$. Moreover, T_Δ acts (toricaly) on the manifold $E_\Delta := N_\Delta/(T/T_\Delta)$, where N_Δ is the normal bundle of M^{T_Δ} inside M restricted to C^ε_Δ. The moment map image for this T_Δ-action can be identified with a neighborhood of $\beta(\varepsilon, \Delta)$ inside the "slice" of the moment map polytope $\mu(M)$ that passes through $\beta(\varepsilon, \Delta)$ and is perpendicular to Δ (represented by the shaded region in Fig. 1).

Denoting by $\beta_1(\varepsilon, \Delta)$ and ε_1 the orthogonal projections of $\beta(\varepsilon, \Delta)$ and ε onto $\mathfrak{t}^*_\Delta$, formula (13) becomes

$$\begin{aligned} &I^\varepsilon_\Delta(X_1 + X_2) \\ &\quad = (2\pi i)^{\dim \Delta} \epsilon_\Delta \left(e^{i\langle \beta_1(\varepsilon,\Delta), X_1\rangle} \prod_{j=1}^{r_\Delta} \int_0^\infty e^{i\langle \alpha^+_{\Delta,j}, X_1\rangle t}\, dt \right) \diamond \delta_0(X_2), \end{aligned} \tag{18}$$

whenever $\beta(\varepsilon, \Delta) \cap \mu(M)$ is nonempty. Here,

(i) r_Δ is the codimension of Δ;
(ii) the $\alpha^+_{\Delta,j}$'s are the weights of the T_Δ-action on the manifold E_Δ restricted to the normal bundle of the fixed point x, polarized with respect to

$$j_1^{-1}\big(\beta_1(\varepsilon, \Delta) - \varepsilon_1\big);$$

(iii) $\epsilon_\Delta = \prod_{j=1}^{r_\Delta} \epsilon^j_\Delta$, with $\alpha^+_{\Delta,j} = \epsilon^j_\Delta \alpha_{\Delta,j}$, is the sign obtained by polarization.

Taking the Fourier transform of (18) we obtain

$$(2\pi i)^n \epsilon_\Delta \big((\delta_{\beta_1(\varepsilon,\Delta)} * H_{\alpha^+_{\Delta,1}} * \cdots * H_{\alpha^+_{\Delta,r}}) \diamond \mathbf{1}_{(\mathfrak{t}/\mathfrak{t}_\Delta)^*} \big), \tag{19}$$

which is supported on the polarized cone $\mathbf{C}^\sharp_{\beta_1(\varepsilon,\Delta)} := \beta_1(\varepsilon,\Delta) + \mathbb{R}^+\alpha^+_{\Delta,1} + \cdots + \mathbb{R}^+\alpha^+_{\Delta,r_\Delta}$. Moreover, changing variables, we obtain

$$\begin{aligned}\langle H_{\alpha^+_{\Delta,1}} * \cdots * H_{\alpha^+_{\Delta,r_\Delta}}, \phi\rangle &= \int_0^\infty \cdots \int_0^\infty \phi\left(\sum_{i=1}^{r_\Delta} u_i \alpha^+_{\Delta,i} \right) du_1 \dots du_{r_\Delta} \\ &= \frac{1}{|\det(\alpha^+_{\Delta,i})_i|} \int_{\mathbf{C}^\sharp_0} \phi = \frac{1}{|\det(\alpha^+_{\Delta,i})_i|} \langle \mathbf{1}_{\mathbf{C}^\sharp_0}, \phi\rangle,\end{aligned}$$

for any rapidly decreasing function ϕ, where $\mathbf{1}_{\mathbf{C}^\sharp_0}$ is the characteristic function of the cone

$$\mathbf{C}^\sharp_0 := \mathbb{R}^+\alpha^+_{\Delta,1} + \cdots + \mathbb{R}^+\alpha^+_{\Delta,r_\Delta}.$$

However, since the $\alpha_{\Delta,i}$'s are the weights of the action of T_Δ on the toric manifold E_Δ at the fixed point, we have $|\det(\alpha^+_{\Delta,i})_i| = 1$ and so (19) becomes $(2\pi i)^n \epsilon_\Delta \mathbf{1}_{\mathbf{C}^\sharp_0} \diamond \mathbf{1}_{(\mathfrak{t}/\mathfrak{t}_\Delta)^*}$. On the other hand, since the Fourier transform of $\int_M e^{i\omega^\sharp t}$ is the direct image $\mu_*(dm_L)$ of the Liouville measure $dm_L := \omega^n/n!$ on M (which is supported on $\mu(M)$), we obtain (up to a factor of $(2\pi i)^n$)

$$\mathbf{1}_{\mu(M)} = \sum_{\Delta\in\mathcal{B}} \varphi(\varepsilon,\Delta)\epsilon_\Delta \mathbf{1}_{\mathbf{C}^\sharp_\Delta} \tag{20}$$

almost everywhere on the polytope $\mu(M)$, where $\varphi(\varepsilon,\Delta)$ is equal to 1 when $\beta(\varepsilon,\Delta) \cap \mu(M)$ is nonempty, and zero otherwise.

4. Polytope decompositions

In this section we will show that the polytope decomposition for Delzant polytopes that was obtained in (20), remains valid for any compact convex simple polytope. Moreover, we will give a weighted version of this decomposition that also holds on the boundary of the polytope.

Hereafter, we will consider the usual Euclidean inner product $\langle\,,\rangle$ of $\mathbb{R}^d$. Let P be a compact convex simple polytope in $\mathbb{R}^d$ and let $\mathcal{B}'$ be the set of faces of P. For each $F \in \mathcal{B}'$ we write Δ_F for the affine subspace of $\mathbb{R}^d$ generated by F. Then, just as in Section 2, we have the following sets:

$$\mathcal{B} := \{\Delta_F \mid F \in \mathcal{B}'\}; \tag{21}$$

$$\mathcal{B}'_\Delta := \big\{F \in \mathcal{B}' \bigm| F \subset \Delta \text{ and } \dim(F) < \dim(\Delta)\big\}, \quad \text{for } \Delta \in \mathcal{B}; \tag{22}$$

$$\Delta_{\mathrm{reg}} := \Delta \setminus \bigcup_{F\in\mathcal{B}'_\Delta} F; \tag{23}$$

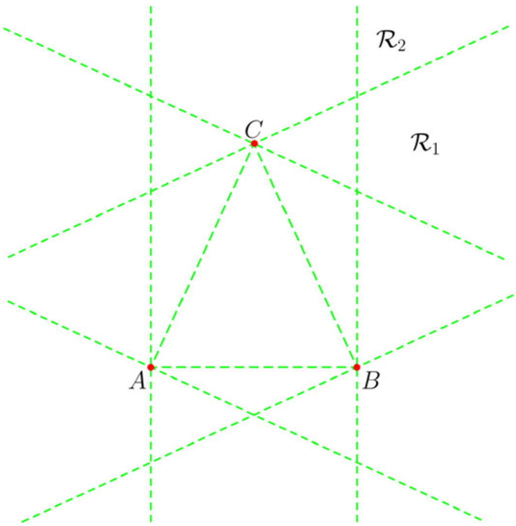

Fig. 2. Paradan regions for a triangle.

$$W_\Delta := \Delta_{\mathrm{reg}} + \Delta^\perp \quad \text{for } \Delta \in \mathcal{B}; \tag{24}$$

$$W := \bigcap_{\Delta \in \mathcal{B}} W_\Delta. \tag{25}$$

The set W is a disjoint union of open sets which we will call *Paradan regions* (see Fig. 2 for an illustration). In fact, W is the complement in $\mathbb{R}^d$ of a finite set of walls of codimension 1, $W^c = E_1 \cup \cdots \cup E_K \cup \{\text{facets of } P\}$, where each wall E_i is contained in a hyperplane perpendicular to an element of $\mathcal{B}$. Note that, in the case of a moment polytope $\mu(M)$ of a toric manifold, the set W is the same as in (7).

4.1. Tangent cones

For each $\Delta \in \mathcal{B}$, we define the tangent cone of P at Δ by

$$\mathbf{C}_\Delta := \bigl\{ y + r(x-y) \bigm| r \geqslant 0,\ y \in \Delta,\ x \in P \bigr\}.$$

It is a full-dimensional cone with apex Δ (i.e. Δ is the maximal affine space contained in $\mathbf{C}_\Delta$). Taking ε in $\mathbb{R}^d$, we denote by $\beta(\varepsilon, \Delta)$ its orthogonal projection onto the affine space Δ, and we take the intersection of $\beta(\varepsilon, \Delta) + \Delta^\perp$ (the orthogonal space of Δ at $\beta(\varepsilon, \Delta)$) with the tangent cone of Δ:

$$\mathbf{C}_{\Delta^\perp,\varepsilon} := \bigl(\beta(\varepsilon, \Delta) + \Delta^\perp\bigr) \cap \mathbf{C}_\Delta.$$

This is now a pointed cone (i.e. a cone with a single point as apex) with vertex $\beta(\varepsilon, \Delta)$. Note that $\mathbf{C}_\Delta$ is the direct product of the affine space Δ and the pointed cone $\mathbf{C}_{\Delta^\perp,\varepsilon}$. Then, considering

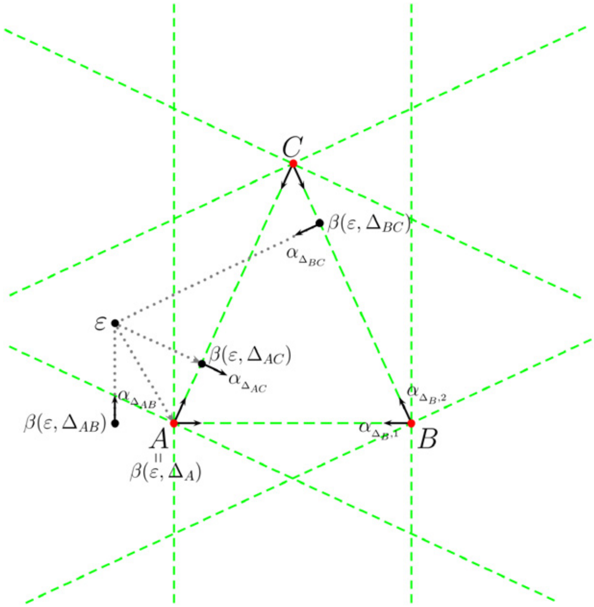

Fig. 3. Projections and generating vectors for some faces of a triangle.

vectors $\alpha_{\Delta,j} \in \mathbb{R}^d$ along the edges of $\mathbf{C}_{\Delta^\perp,\varepsilon}$, pointing away from the vertex ($j = 1, \ldots, r_\Delta$, where $r_\Delta = \dim \mathbf{C}_{\Delta^\perp,\varepsilon} = \operatorname{codim} \Delta$), the tangent cone $\mathbf{C}_\Delta$ can be written as

$$\mathbf{C}_\Delta = \Delta + \mathbf{C}_{\Delta^\perp,\varepsilon} = \Delta + \sum_{j=1}^{r_\Delta} \mathbb{R}^+ \alpha_{\Delta,j}. \tag{26}$$

Hence, $\mathbf{C}_\Delta$ is the cone along Δ which contains P and is bounded by the affine spaces in $\mathcal{B}$ which contain Δ. The vectors $\alpha_{\Delta,j}$, which are only determined up to a positive scalar, will be called the *generators* of $\mathbf{C}_\Delta$ (see Fig. 3 for an illustration).

4.2. Polarization

We will now "polarize" the tangent cones defined in the previous section. For that, let us first fix a point $\varepsilon \in W$ and denote by $\beta(\varepsilon, \Delta)$ the orthogonal projection of ε onto $\Delta \in \mathcal{B}$. By the definition of W, for each $\Delta \in \mathcal{B}$ for which $\beta(\varepsilon, \Delta) \in P$, we have $\langle \beta_\Delta, \alpha_{\Delta,j} \rangle \neq 0$ for all $j = 1, \ldots, r_\Delta$, where $\beta_\Delta := \beta(\varepsilon, \Delta) - \varepsilon$. Indeed, if $\operatorname{codim} \Delta = 1$, then β_Δ is perpendicular to Δ and consequently parallel to $\alpha_{\Delta,1}$, the generator of C_Δ. If, on the other hand, $\operatorname{codim} \Delta \neq 1$, then, taking a generator $\alpha_{\Delta,j}$ of C_Δ, we have that $\langle \beta_\Delta, \alpha_{\Delta,j} \rangle = 0$ implies $\beta_\Delta \perp \tilde{\Delta}$, where $\tilde{\Delta} \in \mathcal{B}$ is the affine space

$$\tilde{\Delta} := \Delta + \operatorname{span} \alpha_{\Delta,j}$$

(note that $\dim \tilde{\Delta} = \dim \Delta + 1$ and that $\beta_\Delta \perp \Delta$ and $\beta_\Delta \perp \alpha_{\Delta,j}$). Then, since $\beta(\varepsilon, \Delta) \in \Delta$, ε would be in the hyperplane $H_{\Delta,\tilde{\Delta}}$ through Δ that is perpendicular to the face $F \in \mathcal{B}'$ that generates $\tilde{\Delta}$, which is impossible by the definition of W ($H_{\Delta,\tilde{\Delta}} \subset W^c_{\tilde{\Delta}}$).

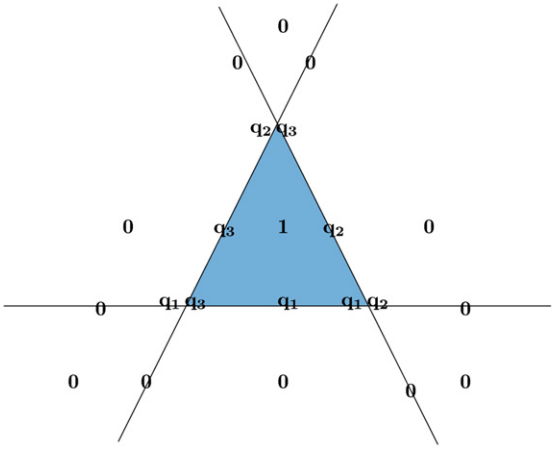

Fig. 4. Weighted characteristic function for a triangle with weights (q_1, q_2, q_3).

Polarizing the vectors $\alpha_{\Delta,j}$ according to β_Δ, that is, taking the vectors $\alpha^+_{\Delta,j} = \epsilon^j_{\beta_\Delta}\alpha_{\Delta,j}$ with $\langle\alpha^+_{\Delta,j}, \beta_\Delta\rangle > 0$ and $\epsilon^j_{\beta_\Delta} = \pm 1$, we define $\mathbf{C}^\sharp_\Delta$, the *polarized tangent cone* of P at Δ as

$$\mathbf{C}^\sharp_\Delta := \Delta + \sum_{j=1}^{r_\Delta} \mathbb{R}^+ \alpha^+_{\Delta,j}. \tag{27}$$

4.3. Weighted characteristic functions

Let us now see how to assign weights to each affine space in $\mathcal{B}$ in order to obtain our polytope decomposition. Take $\Delta_1, \dots, \Delta_N$, the codimension-1 elements of $\mathcal{B}$, that is, the affine subspaces generated by the facets of P. Each $\Delta \in \mathcal{B}$ that is generated by a non-trivial face of P (i.e. $\Delta \neq \emptyset, \mathbb{R}^d$) can be described as an intersection

$$\bigcap_{i \in J_\Delta} \Delta_i,$$

where J_Δ denotes the index set of the hyperplanes Δ_i that contain Δ. Note that, since P is simple, the number of elements of J_Δ is equal to r_Δ, the codimension of Δ. To each Δ_i we assign an arbitrary complex number q_i and to each affine space $\Delta \in \mathcal{B} \setminus \{\mathbb{R}^d\}$ we assign the value $\prod_{i \in J_\Delta} q_i$. Finally, to $\Delta = \mathbb{R}^d$ we assign the value 1. With this, we define a weight function $w : P \to \mathbb{C}$, given by $w(x) = \prod_{i \in J_{\Delta_x}} q_i$, where Δ_x is the smallest-dimensional element of $\mathcal{B}$ that contains x. Using this function, we define the *weighted characteristic function*

$$\mathbf{1}^w_P(x) = \begin{cases} w(x), & \text{if } x \in P, \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

(see Fig. 4 for an illustration).

Similarly, for each $\Delta \in \mathcal{B}$, we define a weighted characteristic function for the tangent cone $\mathbf{C}_\Delta$. For polarized tangent cones $\mathbf{C}^\sharp_\Delta$ the weighted characteristic function is defined as follows. To each hyperplane Δ_i that contains a facet of $\mathbf{C}^\sharp_\Delta$ and whose tangent cone $\mathbf{C}_{\Delta_i}$ (a half-space) intersects $\mathbf{C}^\sharp_\Delta \setminus \Delta_i$, we assign the weight q_i. If, on the other hand, the tangent cone $\mathbf{C}_{\Delta_i}$ does not

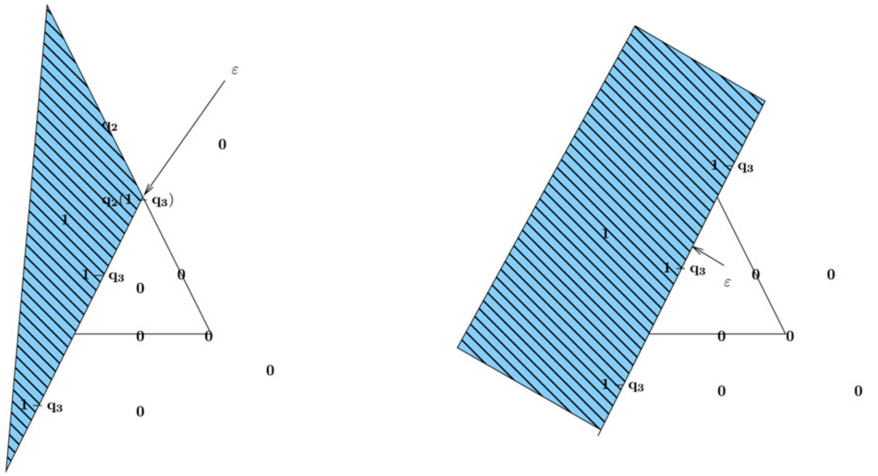

Fig. 5. Weighted characteristic functions for two polarized tangent cones with weights (q_1, q_2, q_3).

intersect $\mathbf{C}^\sharp_\Delta \setminus \Delta_i$ we assign the weight $1 - q_i$ to Δ_i. Then, we define the set $\mathcal{B}^\sharp_\Delta$ of affine spaces generated by faces of $\mathbf{C}^\sharp_\Delta$ and we proceed as we did for polytopes. To each $\tilde{\Delta} \in \mathcal{B}^\sharp_\Delta \setminus \{\mathbb{R}^d\}$ we assign the value

$$\prod_{i \in J^+_{\tilde{\Delta}},\, j \in J^-_{\tilde{\Delta}}} q_i(1 - q_j),$$

where $J^-_{\tilde{\Delta}}$ is the index set of hyperplanes Δ_i containing $\tilde{\Delta}$ whose tangent cone does not intersect $\mathbf{C}^\sharp_\Delta \setminus \Delta_i$, and $J^+_{\tilde{\Delta}} = J_{\tilde{\Delta}} \setminus J^-_{\tilde{\Delta}}$. To $\tilde{\Delta} = \mathbb{R}^d$ we assign the value 1. Finally, we define the weight function and the weighted characteristic function as before: $w : \mathbf{C}^\sharp_\Delta \to \mathbb{C}$ is such that

$$w(x) = \prod_{i \in J^+_{\Delta_x},\, j \in J^-_{\Delta_x}} q_i(1 - q_j),$$

where Δ_x is the smallest-dimensional element of $\mathcal{B}^\sharp_\Delta$ that contains x, and

$$\mathbf{1}^w_{\mathbf{C}^\sharp_\Delta}(x) = \begin{cases} w(x), & \text{if } x \in \mathbf{C}^\sharp_\Delta, \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

(see Fig. 5 for an example).

4.4. Decomposition formulas

Finally, defining $\varphi(\varepsilon, \Delta)$ as

$$\varphi(\varepsilon, \Delta) = \begin{cases} 1, & \text{if } \beta(\varepsilon, \Delta) \cap P \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

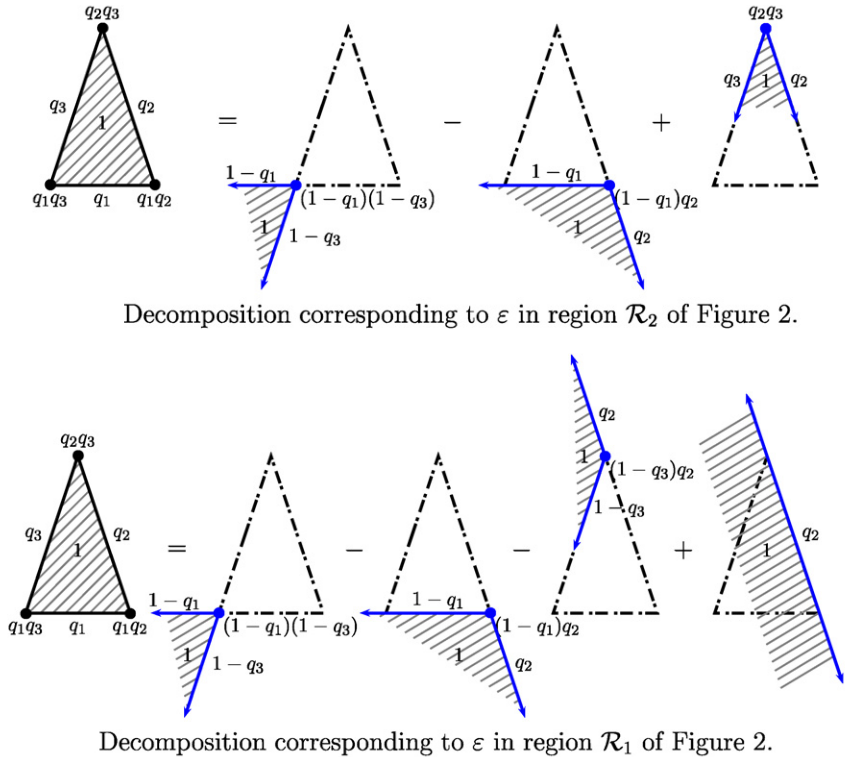

Fig. 6. Polytope decomposition for a triangle.

we obtain, for each $\varepsilon \in W$, the following polytope decomposition formula (see Fig. 6 for an illustration). Note that, by the definition of φ, this formula only takes into account the polarized cones $\mathbf{C}^\sharp_\Delta$ for which $\beta(\varepsilon, \Delta) \cap P \neq \emptyset$, that is, those for which the orthogonal projection of ε onto Δ is in P.

Theorem 4.1. *Let P be a compact convex simple polytope of dimension d in $\mathbb{R}^d$. For any choice of ε in W, we have*

$$\mathbf{1}^w_P = \sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta} \varphi(\varepsilon, \Delta) \mathbf{1}^w_{\mathbf{C}^\sharp_\Delta}, \tag{31}$$

where the sum is taken over the set $\mathcal{B}$ of affine spaces generated by the faces of P, where $\mathbf{C}^\sharp_\Delta$ is the polarized tangent cone of P at $\Delta \in \mathcal{B}$ with respect to the vector $\beta(\varepsilon, \Delta) - \varepsilon$ (where $\beta(\varepsilon, \Delta)$ is the orthogonal projection of ε onto Δ), where m_Δ is the number of generators of the tangent cone $\mathbf{C}_\Delta$ whose signs change by polarization, and where $\mathbf{1}^w_P$ and $\mathbf{1}^w_{\mathbf{C}^\sharp_\Delta}$ are the weighted characteristic functions of the polytope and of the polarized cones, respectively.

Proof. We will prove this formula in two steps. First, we will find an ε in a Paradan region for which formula (31) holds. Then, we will show that the right-hand side is independent of the choice of ε.

Step 1. Let us choose an ε such that $\varphi(\varepsilon, \Delta) = 0$ for every $\Delta \in \mathcal{B}$ with $\dim \Delta > 0$, i.e. we choose

$$\varepsilon \in \bigcap_{F \text{ a facet of } P} \left(F + \Delta_F^{\perp}\right)^c.$$

Then (31) becomes

$$\mathbf{1}_P^w = \sum_{v \text{ a vertex of } P} (-1)^{m_v} \mathbf{1}^w_{\mathbf{C}_v^\sharp}. \tag{32}$$

Choosing a vector $\xi \in \mathbb{R}^d$ such that, for every vertex v of P, $\langle \xi, \alpha_{v,j}\rangle > 0$ whenever $\langle v - \varepsilon, \alpha_{v,j}\rangle > 0$, where the $\alpha_{v,j}$'s are the edge vectors at v, formula (32) becomes a weighted version of the Lawrence–Varchenko polytope decomposition (see [1,23,29,39]), where the tangent cones at vertices are polarized according to ξ, and the result follows. Note that this weighted version of the Lawrence–Varchenko relation is different from the ones in [1,23] because here we may assign different weights $q_i \in \mathbb{C}$ to the faces of the polytope, instead of a fixed complex number. Nevertheless, the proof of this decomposition formula follows easily from the ones in [1,23]. Indeed, if x is in the boundary of P, then the left-hand side of (32) is

$$\mathbf{1}_P^w(x) = \prod_{i \in J_{\Delta_x}} q_i$$

while, taking a vertex in Δ_x and a polarizing vector such that $\mathbf{C}_v^\sharp = \mathbf{C}_v$ (see [23]), the right-hand side becomes equal to

$$\mathbf{1}^w_{\mathbf{C}_v^\sharp}(x) = \prod_{i \in J_{\Delta_x}} q_i.$$

Note that, for every other vertex, v', the cone $\mathbf{C}^\sharp_{v'}$ is disjoint from the relative interior of the face that generates Δ_x and so it does not contain x.

This choice of polarizing vector ξ for which (32) becomes a weighted version of the Lawrence–Varchenko polytope decomposition can be done in the following way. First, we consider the vertex v_0 of P that is closest to ε. Clearly, for v_0 we have $\mathbf{C}_{v_0} = \mathbf{C}^\sharp_{v_0}$ (where this cone is polarized with respect to the vector $v_0 - \varepsilon$). Then, for any other vertex v and for each edge vector $\alpha_{v,j}$ satisfying $\langle v - \varepsilon, \alpha_{v,j}\rangle > 0$, we take the hyperplane $H^\varepsilon_{v,j}$ through ε which is perpendicular to $\alpha_{v,j}$. These hyperplanes intersect at ε and each of them separates the whole space $\mathbb{R}^d$ into two open regions. Let us denote by $(H^\varepsilon_{v,j})^+$ those regions that contain v_0 and take a vector ξ starting at ε and ending somewhere on the intersection

$$\bigcap_{v \text{ a vertex of } P} \bigcap_{\substack{j \text{ s.t.} \\ \langle v-\varepsilon, \alpha_{v,j}\rangle > 0}} \left(H^\varepsilon_{v,j}\right)^+$$

(we can take for instance $\xi = v_0 - \varepsilon$). Then clearly $\langle \xi, \alpha_{v,j}\rangle > 0$ for all edge vectors $\alpha_{v,j}$ satisfying $\langle v - \varepsilon, \alpha_{v,j}\rangle > 0$ (see Fig. 7).

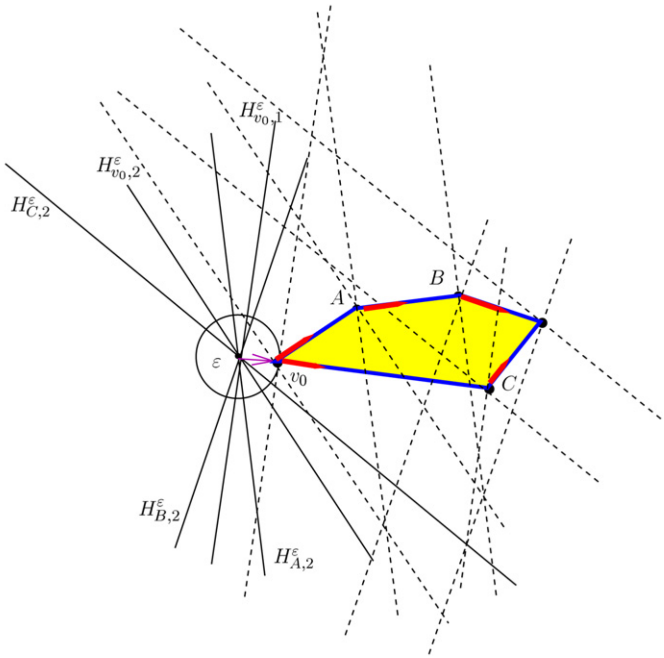

Fig. 7.

Step 2. Recall that the complement W^c of W is a finite family of walls of codimension 1, $W^c = E_1 \cup \cdots \cup E_K \cup \{\text{facets of } P\}$, where each wall E_i is contained in a hyperplane perpendicular to an element of $\mathcal{B}$. Let ε_1 and ε_2 be in two contiguous Paradan regions $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, and let E be its common "wall" (either one of the E_i's or a facet of P). Let ε_t be any path in $\mathbb{R}^d$ from ε_1 to ε_2 that crosses a single wall (i.e. E) once. When ε_t crosses E, the sign of

$$\left\langle \beta(\varepsilon_t, \Delta) - \varepsilon_t, \alpha_{\Delta,k} \right\rangle \tag{33}$$

(for $\Delta \in \mathcal{B} \setminus \mathbb{R}^d$ and for a generator $\alpha_{\Delta,k}$ of the tangent cone at Δ) flips exactly when $\Delta \cap \partial P$ is contained in E and $\alpha_{\Delta,k}$ is perpendicular to E. Hence, if $\dim \Delta \neq d-1$, this sign flips iff Δ is contained in (exactly) one affine space $\tilde{\Delta}$ perpendicular to E with $\dim \tilde{\Delta} = \dim \Delta + 1$ (see Fig. 8 for an illustration). Indeed, we can take

$$\tilde{\Delta} := \Delta + \operatorname{span} \alpha_{\Delta,k}$$

and unicity follows from dimensional reasons. On the other hand, if $\dim \Delta = d - 1$, the sign of (33) flips iff $\Delta \cap P \subseteq E$. In this case, we define $\tilde{\Delta}$ to be the entire space $\mathbb{R}^d$.

Let us assume without loss of generality that the sign of (33) flips from negative to positive as ε_t crosses E. In this case, the polarized tangent cones at Δ before and after ε_t crosses the wall are

$$\left(\mathbf{C}^{\sharp}_{\Delta}\right)^1 = \Delta + \sum_{j\neq k} \mathbb{R}^+ \alpha^{\sharp}_{\Delta,j} - \mathbb{R}^+ \alpha_{\Delta,k} \quad \text{and} \quad \left(\mathbf{C}^{\sharp}_{\Delta}\right)^2 = \Delta + \sum_{j\neq k} \mathbb{R}^+ \alpha^{\sharp}_{\Delta,j} + \mathbb{R}^+ \alpha_{\Delta,k}.$$

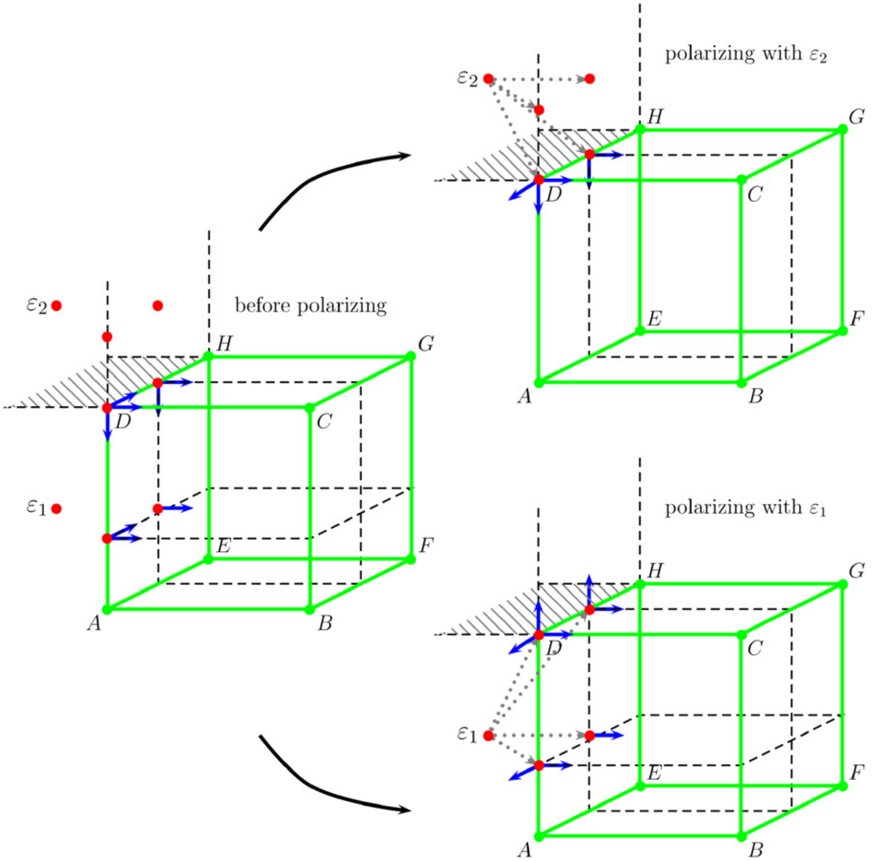

Fig. 8. Generators of some polarized tangent cones $\mathbf{C}^\sharp_\Delta$ for a cube, with ε_1 and ε_2 in two contiguous Paradan regions.

Hence, the corresponding contributions of Δ to the right-hand side of (31) are

$$\pm\mathbf{1}^w_{(\mathbf{C}^\sharp_\Delta)^1} \quad \text{and} \quad \mp\mathbf{1}^w_{(\mathbf{C}^\sharp_\Delta)^2}.$$

Note that the union of the two cones $(\mathbf{C}^\sharp_\Delta)^1$ and $(\mathbf{C}^\sharp_\Delta)^2$ is the polarized tangent cone at $\tilde{\Delta}$, $\mathbf{C}^\sharp_{\tilde{\Delta}}$, for both ε_1 and ε_2 (cf. Fig. 9), and so

$$\mathbf{1}^w_{(\mathbf{C}^\sharp_\Delta)^1} + \mathbf{1}^w_{(\mathbf{C}^\sharp_\Delta)^2} = \mathbf{1}^w_{\mathbf{C}^\sharp_{\tilde{\Delta}}}.$$

On the other hand, we have $\beta(\varepsilon_1, \tilde{\Delta}) \cap P \neq \emptyset$, while $\beta(\varepsilon_2, \tilde{\Delta}) \cap P = \emptyset$. Hence, the corresponding contributions of $\tilde{\Delta}$ to the right-hand side of (31) are

$$\mp\mathbf{1}^w_{\mathbf{C}^\sharp_{\tilde{\Delta}}} \quad \text{and} \quad 0.$$

Indeed, $\beta(E, \Delta) \cap \partial P = \Delta \cap \partial P$ and $\beta(\beta(\varepsilon_i, \tilde{\Delta}), \Delta) - \beta(\varepsilon_i, \tilde{\Delta}) = r_i \alpha_{\Delta,k}$ for $i = 1, 2$, with $r_1 < 0$ and $r_2 > 0$ (cf. Figs. 8 and 9).

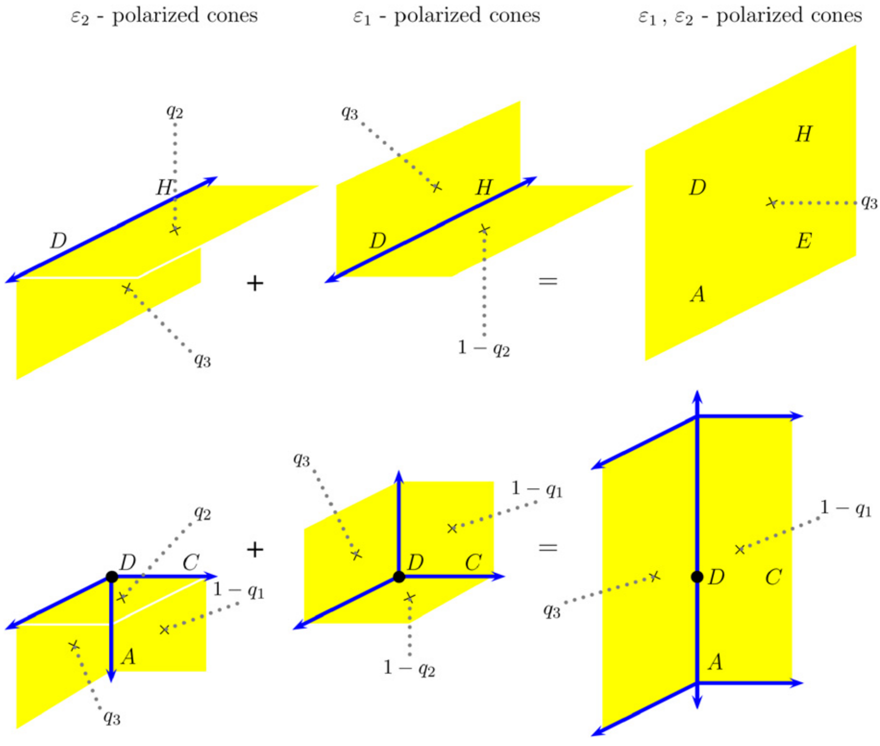

Fig. 9. Polarized tangent cones $\mathbf{C}^{\sharp}_{\Delta}$ at the affine subspaces Δ_D, Δ_{DH}, Δ_{DA} and Δ_{AEHD} for the cube in Fig. 8.

Consequently, the differences of the contributions of Δ to the formula in (31) before and after ε_t crosses the wall, and those of $\tilde{\Delta}$, sum to zero.

Moreover, for a given $\tilde{\Delta} \in \mathcal{B}$, if $\varphi(\varepsilon_t, \tilde{\Delta})$ changes when crossing E, the intersection of $\tilde{\Delta}$ with E contains $\Delta \cap P$ for (exactly) one element Δ of $\mathcal{B}$ with $\dim \Delta = \dim \tilde{\Delta} - 1$ and the result follows. □

Remark 34. This new polytope decompositions (31) generalize the weighted version of the Lawrence–Varchenko relation for a simple polytope presented in [1]. There, the edge vectors emanating from each vertex are flipped in a systematic way using a polarizing vector, and the weighted characteristic function of the polytope is expressed (*only*) in terms of the weighted characteristic functions of the polarized cones supported at the vertices. In (31), not only the polarization is carried out differently, but, for some values of ε, we consider the weighted characteristic functions of polarized tangent cones to faces other than vertices. Indeed, given $\varepsilon \in W$, we obtain a *different* polarizing vector for each face of the polytope by taking ε as starting point, and its projections onto the faces of the polytope as end points, whenever these projections are nonempty. Then we polarize the tangent cones of the corresponding faces accordingly.

4.5. Other decomposition formulas

If we polarize the generators of tangent cones with respect to $\varepsilon - \beta(\varepsilon, \Delta)$ instead of $\beta(\varepsilon, \Delta) - \varepsilon$, and multiply each term on the right-hand side of (31) by a factor $(-1)^{\dim \Delta}$, we obtain new polytope decompositions, under the same hypotheses and statements of Theorem 4.1.

Theorem 4.2. *Let P be a compact convex simple polytope of dimension d in $\mathbb{R}^d$. For any choice of $\varepsilon \in W$, we have*

$$\mathbf{1}_P^w = \sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta + \dim \Delta} \varphi(\varepsilon, \Delta) \mathbf{1}^w_{\mathbf{C}^\sharp_\Delta}, \tag{35}$$

where the sum is taken over the set $\mathcal{B}$ of affine spaces generated by the faces of P, where $\mathbf{C}^\sharp_\Delta$ is the polarized tangent cone of P at $\Delta \in \mathcal{B}$ with respect to the vector $\varepsilon - \beta(\varepsilon, \Delta)$ (where $\beta(\varepsilon, \Delta)$ is the orthogonal projection of ε onto Δ), where m_Δ is the number of generators of the cone $\mathbf{C}_\Delta$ whose sign changes by polarization, and where $\mathbf{1}_P^w$ and $\mathbf{1}^w_{\mathbf{C}^\sharp_\Delta}$ are the weighted characteristic functions of the polytope P and of the polarized cones, respectively.

Proof. The fact that the right-hand side of (35) does not depend on ε can be proved as in the proof of Theorem 4.1. Hence, we just have to show that we can find an ε in some Paradan region for which (35) holds.

For that, we again choose an ε such that $\varphi(\varepsilon, \Delta) = 0$ for every $\Delta \in \mathcal{B}$ with $\dim \Delta > 0$, i.e. we choose

$$\varepsilon \in \bigcap_{F \text{ a facet of } P} \left(F + \Delta_F^\perp\right)^c.$$

Then (35) becomes

$$\mathbf{1}_P^w = \sum_{v \text{ a vertex of } P} (-1)^{m_v} \mathbf{1}^w_{\mathbf{C}^\sharp_v}. \tag{36}$$

Choosing a vector $\xi \in \mathbb{R}^d$ such that, for every vertex v of P, $\langle \xi, \alpha_{v,j} \rangle > 0$ whenever $\langle \varepsilon - v, \alpha_{v,j} \rangle > 0$, where the $\alpha_{v,j}$'s are the edge vectors at v, formula (36) becomes a weighted version of the Lawrence–Varchenko polytope decomposition (see [1,23,29,39]), where the tangent cones at vertices are polarized according to ξ, and the result follows.

This choice of polarizing vector ξ can be done in the following way. First we consider the vertex v_0 of P that is furthest away from ε. Clearly, for v_0 we have $\mathbf{C}_{v_0} = \mathbf{C}^\sharp_{v_0}$ (where this cone is polarized with respect to the vector $\varepsilon - v_0$). Then, for any other vertex v and for each edge vector $\alpha_{v,j}$ satisfying $\langle \varepsilon - v, \alpha_{v,j} \rangle > 0$, we take the hyperplane $H^0_{v,j}$ through v_0 which is perpendicular to $\alpha_{v,j}$. These hyperplanes intersect at v_0 and each of them separates the whole space $\mathbb{R}^d$ into two open regions. Let us denote by $(H^0_{v,j})^+$ those regions that contain ε and take a vector ξ starting at v_0 and ending somewhere on the intersection

$$\bigcap_{v \text{ a vertex of } P} \bigcap_{\substack{j \text{ s.t.} \\ \langle \varepsilon - v, \alpha_{v,j} \rangle > 0}} \left(H^0_{v,j}\right)^+$$

(we can take for instance $\xi = \varepsilon - v_0$). Then clearly $\langle \xi, \alpha_{v,j} \rangle > 0$ for all edge vectors $\alpha_{v,j}$ satisfying $\langle \varepsilon - v, \alpha_{v,j} \rangle > 0$ (see Fig. 10). □

Remark 37. We have seen in the above proof that, choosing ε in an appropriate region, the polytope decomposition formula (35) becomes the Lawrence–Varchenko relation. In addition, in

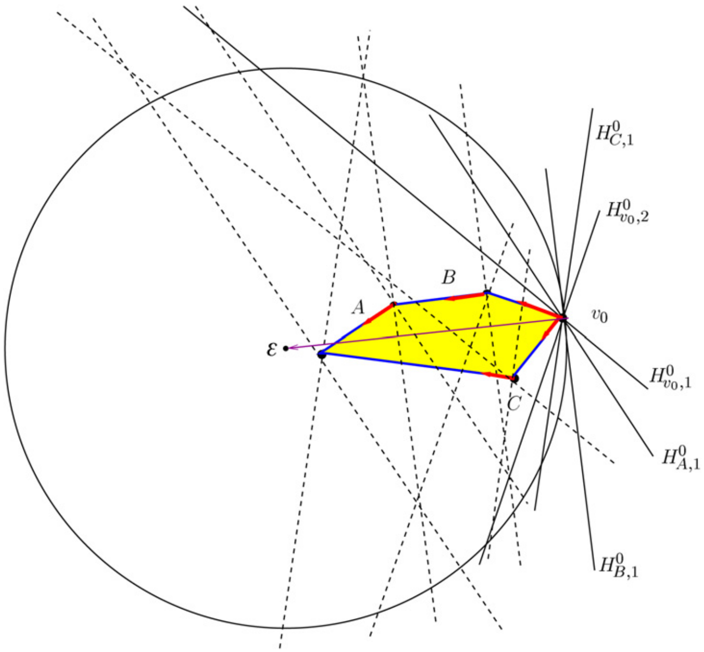

Fig. 10.

some cases, we can also choose ε so that (35) becomes the weighted Brianchon–Gram formula of [2]. Indeed, considering for each vertex v of P, the cone $\mathbf{C}_v^d$ generated by the inward normal vectors to the facets through v, and taking the intersection

$$P_d := \bigcap_{v \text{ vertex of } P} \mathbf{C}_v^d,$$

then, whenever $\operatorname{int}(P_d \cap P) \neq \emptyset$, we can take $\varepsilon \in \operatorname{int}(P_d \cap P)$, and obtain $m_\Delta = 0$ and $\varphi(\varepsilon, \Delta) = 1$ for every Δ in $\mathcal{B}$. Then, with this choice of ε, (35) becomes the weighted Brianchon–Gram formula:

$$\mathbf{1}_P^w = \sum_F (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}^w, \tag{38}$$

where the sum is over all faces F of P.

5. The weighted Euler–Maclaurin formula

As an application of our polytope decompositions, we will give new weighted Euler–Maclaurin formulas with remainder for the sum of the values of a smooth function f on the integral points of a simple polytope P.

5.1. Weighted Euler–Maclaurin for intervals

Let us first recall the weighted Euler–Maclaurin formula for this sum presented in [3] (see also [23,27]). Let q be any complex number and let f be any $\mathcal{C}^m$ function on the real line ($m \geqslant 1$). For integers $a < b$ and $k = \lfloor m/2 \rfloor$, the weighted sum of the values of f on the integral points of the interval $[a, b]$ is defined as

$$\sum_{[a,b]}{}^{q} f := qf(a) + f(a+1) + \cdots + f(b-1) + qf(b)$$
$$= \mathbf{Q}_q^{2k}(D_1)\mathbf{Q}_q^{2k}(D_2) \int\limits_{a-h_1}^{b+h_2} f(x)\,dx \bigg|_{h_1=h_2=0} + R_m(f), \tag{39}$$

with

$$R_m(f) := (-1)^{m-1} \int\limits_a^b P_m(x) f^{(m)}(x)\,dx, \tag{40}$$

for

$$P_m(x) := \frac{1}{m!} B_m\big(\{x\}\big), \tag{41}$$

where B_m is the mth Bernoulli polynomial and $\{x\} := x - \lfloor x \rfloor$ is the fractional part of x, where

$$D_1 := \frac{\partial}{\partial h_1}, \qquad D_2 := \frac{\partial}{\partial h_2},$$

and where $\mathbf{Q}_q^{2k}(S)$ denotes the truncation at the even integer $2k$ of the power series

$$\mathbf{Q}_q(S) = (q-1)S + \mathbf{Td}(S) = 1 + \left(q - \frac{1}{2}\right)S + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} S^{2k}$$
$$= \left(q - \frac{1}{2}\right)S + \frac{S/2}{\tanh(S/2)}. \tag{42}$$

Here $\mathbf{Td}$ is the classical Todd function defined by

$$\mathbf{Td}(S) := S/\big(1 - e^{-S}\big) = 1 - b_1 S + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} S^{2k},$$

with b_k the kth Bernoulli number [8].

Moreover, $P_m(x)$ is given by

$$P_{2k+1}(x) := (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{(2n\pi)^{2k+1}} \tag{43}$$

if $m = 2k + 1$ is odd, and by

$$P_{2k}(x) := (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2\cos(2n\pi x)}{(2n\pi)^{2k}} \tag{44}$$

if $m = 2k$ is even.

Remark 45. The functions $\mathbf{Q}_q^{2k}$ satisfy the following symmetry property

$$\mathbf{Q}_q^{2k}(S) = \mathbf{Q}_{1-q}^{2k}(-S). \tag{46}$$

Indeed, $\mathbf{Q}_q^{2k}(S)$ is a polynomial with constant coefficients, $1 + (q - \frac{1}{2})S +$ terms of even degree independent of q.

Equation (39), when applied to a $\mathcal{C}^m$ function of compact support, gives the weighted Euler–Maclaurin formula for the half ray $[a, \infty)$:

$$\begin{aligned}\sum_{[a,\infty)} {}^q f &:= qf(a) + f(a+1) + f(a+2) + \cdots \\ &= \mathbf{Q}_q^{2k}(D_1) \int_{a-h_1}^{\infty} f(x)\,dx \bigg|_{h_1=0} + R_m(f),\end{aligned} \tag{47}$$

where

$$R_m(f) := (-1)^{m-1} \int_a^{\infty} P_m(x) f^{(m)}(x)\,dx \quad \text{and} \quad k = \left\lfloor \frac{m}{2} \right\rfloor. \tag{48}$$

Moreover, for the half ray $(-\infty, a]$, we have

$$\sum_{(-\infty,a]} {}^q f := qf(a) + f(a-1) + f(a-2) + \cdots \tag{49}$$

and so, considering the function g defined by $g(x) = f(a - x)$, we obtain

$$\begin{aligned}\sum_{(-\infty,a]} {}^q f = \sum_{[0,\infty)} {}^q g &= \mathbf{Q}_q^{2k}(D_1) \int_{-h_1}^{\infty} g(x)\,dx \bigg|_{h_1=0} + (-1)^{m-1} \int_0^{\infty} P_m(x) g^{(m)}(x)\,dx \\ &= \mathbf{Q}_q^{2k}(D_1) \int_{-\infty}^{a+h_1} f(x)\,dx \bigg|_{h_1=0} + R_m(f),\end{aligned} \tag{50}$$

where now

$$R_m(f) := (-1)^{m-1} \int_{-\infty}^{a} P_m(x) f^{(m)}(x)\, dx \tag{51}$$

(here we used the parity and the 2π-periodicity of $\sin(x)$ and $\cos(x)$). From (47) and (50) and symmetry property (46), we obtain the Euler–Maclaurin formula for the whole real line $\mathbb{R}$:

$$\sum_{\mathbb{R}}{}' f := \sum_{x\in\mathbb{Z}} f(x) = \sum_{(-\infty,0]}{}^{q} f + \sum_{[0,\infty)}{}^{(1-q)} f = \int_{\mathbb{R}} f(x)\, dx + (-1)^{m-1} \int_{\mathbb{R}} P_m(x) f^{(m)}(x)\, dx. \tag{52}$$

5.2. Twisted weighted Euler–Maclaurin formulas for intervals

We will now consider the twisted weighted sum for a half ray

$$\sum_{n\geqslant 0}{}^{q} \lambda^n f(n) = q f(0) + \sum_{n=1}^{\infty} \lambda^n f(n), \tag{53}$$

where $\lambda \neq 1$ is a Kth root of unity with K a positive integer. Let $Q_{m,\lambda}$ be the distributions defined recursively in [23] by

$$Q_{0,\lambda}(x) := -\sum_{n\in\mathbb{Z}} \lambda^n \delta(x-n)$$

and

$$\frac{d}{dx} Q_{m,\lambda}(x) = Q_{m-1,\lambda}(x) \quad \text{and} \quad \int_0^K Q_{m,\lambda}(x)\, dx = 0.$$

Moreover, let us consider the polynomials defined in [3] by

$$\mathbf{N}_q^{k,\lambda}(S) := \left(q + \frac{\lambda}{1-\lambda}\right) S + Q_{2,\lambda}(0) S^2 + Q_{3,\lambda}(0) S^3 + \cdots + Q_{k,\lambda}(0) S^k,$$

where $\lambda \neq 1$ is a root of unity.

Since

$$\frac{d}{dx} \mathbf{1}_{[n,n+1)}(x) = \delta(x-n) - \delta\big(x-(n+1)\big),$$

we have

$$\frac{d}{dx}\left(\sum_{n\in\mathbb{Z}} \lambda^n \mathbf{1}_{[n,n+1)}(x)\right) = \frac{\lambda-1}{\lambda} \sum_{n\in\mathbb{Z}} \lambda^n \delta(x-n) = \frac{1-\lambda}{\lambda} Q_{0,\lambda}(x),$$

implying that

$$Q_{1,\lambda}(x) = \frac{\lambda}{1-\lambda} \sum_{n\in\mathbb{Z}} \lambda^n \mathbf{1}_{[n,n+1)}.$$

Note that

$$\int_0^K Q_{1,\lambda}(x)\,dx = \frac{\lambda}{1-\lambda} \sum_{n=0}^{K-1} \lambda^n = 0.$$

On the other hand, integrating by parts, we have

$$\int_0^\infty Q_{1,\lambda}(x) f'(x)\,dx = \frac{\lambda}{1-\lambda} \sum_{n=0}^{\infty} \int_n^{n+1} \lambda^n f'(x)\,dx = -\frac{\lambda}{1-\lambda} f(0) + \lambda f(1) + \lambda^2 f(2) + \cdots$$

and so,

$$\begin{aligned} qf(0) + \sum_{n\geqslant 1} \lambda^n f(n) &= \left(q + \frac{\lambda}{1-\lambda}\right) f(0) + \int_0^\infty Q_{1,\lambda}(x) f'(x)\,dx \\ &= \left(q + \frac{\lambda}{1-\lambda}\right) f(0) - Q_{2,\lambda}(0) f'(0) + Q_{3,\lambda}(0) f''(0) - \cdots + (-1)^{k-1} Q_{k,\lambda}(0) f^{(k-1)}(0) \\ &\quad + (-1)^{k-1} \int_0^\infty Q_{k,\lambda}(x) f^{(k)}(x)\,dx. \end{aligned}$$

Then, since

$$(-1)^{m-1} f^{(m-1)}(0) = \left(\frac{\partial}{\partial h}\right)^m \int_{-h}^\infty f(x)\,dx \bigg|_{h=0},$$

we obtain the following twisted Euler–Maclaurin formula:

Proposition 54. *(See [3,23].) Let $k > 1$ and let $f \in \mathcal{C}^k(\mathbb{R})$ be compactly supported. Then*

$$\sum_{n\geqslant 0}{}^{q} \lambda^n f(n) = \mathbf{N}_q^{k,\lambda}\left(\frac{\partial}{\partial h}\right) \int_{-h}^\infty f(x)\,dx \bigg|_{h=0} + (-1)^{k-1} \int_0^\infty Q_{k,\lambda}(x) f^{(k)}(x)\,dx. \tag{55}$$

Remark 56. If, for $\lambda \neq 1$, we write $\lambda = e^{2\pi i j/K}$, then, by the Poisson formula, we have

$$Q_{0,\lambda}(x) = -\sum_{n\in\mathbb{Z}} \lambda^n \delta(x-n) = -\sum_{r\in\mathbb{Z}} e^{2\pi i (r + \frac{j}{K})x}.$$

Hence, for $m > 1$, we obtain

$$Q_{m,\lambda}(x) = -\frac{1}{(2\pi i)^m} \sum_{r\in\mathbb{Z}} \frac{e^{2\pi i(r+\frac{j}{K})x}}{(r+\frac{j}{K})^m}, \tag{57}$$

and so

$$Q_{m,\lambda}(0) = -\frac{1}{(2\pi i)^m} \sum_{r\in\mathbb{Z}} \frac{1}{(r+\frac{j}{K})^m}$$

is the $(m-1)$th coefficient of the Taylor series expansion of

$$\frac{1}{1-e^{2\pi i\frac{j}{K}-s}} = \frac{1}{1-\lambda e^{-s}}$$

at $s = 0$. Note that the derivative of $\frac{1}{1-e^{2\pi i\frac{j}{K}-s}}$ with respect to s is equal to

$$\frac{1}{4\sin^2(\frac{\pi j}{K} - \frac{s}{2i})} = \frac{1}{4\pi^2} \sum_{r\in\mathbb{Z}} \frac{1}{(r+\frac{j}{K}-\frac{s}{2\pi i})^2}$$

(since $\frac{\pi^2}{\sin^2 \pi z} = \sum_{r\in\mathbb{Z}} \frac{1}{(r+z)^2}$) and higher order derivatives are obtained differentiating this series expansion. Consequently, considering the operators

$$\mathbf{T}(\lambda, S) := \frac{S}{1-\lambda e^{-S}}$$

defined in [10], we have that $\mathbf{N}_q^{k,\lambda}(S)$ is the truncation at the integer k of the power series

$$\mathbf{N}_q^{\lambda}(S) := \left(q + \frac{\lambda}{1-\lambda}\right)S - \frac{S}{1-\lambda} + \mathbf{T}(\lambda, S) = (q-1)S + \mathbf{T}(\lambda, S).$$

From (57) it is clear that the operators $\mathbf{N}_q^{m,\lambda}$ satisfy the following symmetry property

$$\mathbf{N}_{1-q}^{m,\lambda^{-1}}(S) = \mathbf{N}_q^{m,\lambda}(-S). \tag{58}$$

Remark 59. If, for $\lambda = 1$, we define

$$\mathbf{N}_q^{k,1}(S) := \mathbf{Q}_q^{2\lfloor k/2\rfloor}(S) \quad \text{and} \quad Q_{k,1} := P_k,$$

then formula (55) becomes formula (47) and so it is still valid. Note that, if $\lambda \neq 1$, $\mathbf{N}_q^{k,\lambda}(S)$ is a multiple of S and that, if $\lambda = 1$, then $\mathbf{N}_q^{k,\lambda}(S) = 1 +$ a multiple of S. Moreover, still when $\lambda = 1$, symmetry property (58) becomes property (46).

5.3. Weighted Euler–Maclaurin formulas for cones

For a subset $J \subset \{1,\dots,d\}$, let $\mathbf{S}_J$ be the standard J-sector $\mathbf{S}_J := \{x \in \mathbb{R}^d \mid x_j \geqslant 0 \text{ for } j \in J\}$. Iterating Eqs. (47) and (52), we obtain an Euler–Maclaurin formula for $\mathbf{S}_J$ ($J \neq \emptyset$) and a $\mathcal{C}^m$ function of compact support f. Indeed, considering $J = \{j_1,\dots,j_n\}$ and $\mathbf{1}^w_{\mathbf{S}_J}$, the weighted characteristic function for the J-sector defined in Section 4.3,

$$\sum_{\mathbf{S}_J\cap\mathbb{Z}^d}{}^{w} f := \sum_{\substack{x_j\in\mathbb{Z}^+,\, j\in J\\ x_j\in\mathbb{Z},\, j\notin J}} \big(\mathbf{1}^w_{\mathbf{S}_J} f\big)(x_1,\dots,x_d)$$
$$= \prod_{j\in J} \mathbf{Q}^{2k}_{q_j}(D_j) \int_{\mathbf{S}_J(h_J)} f(x)\,dx \bigg|_{h_J=0} + R^{J_{\mathrm{st}}}_m(f), \tag{60}$$

where $D_j = \partial/\partial h_j$, where $h_J = (h_{j_1},\dots,h_{j_n})$, where

$$\mathbf{S}_J(h_J) = \big\{x \in \mathbb{R}^d \;\big|\; x_j \geqslant -h_j,\ \text{for } j \in J\big\}$$

is the shifted J-sector, and where the remainder $R^{J_{\mathrm{st}}}_m(f)$ is given by

$$\begin{aligned} &R^{J_{\mathrm{st}}}_m(f) \\ &\quad := \sum_{I\subseteq J} \sum_{\substack{R\supseteq J\\ R\subseteq\{1,\dots,d\}\\ R\neq I}} (-1)^{(m-1)(|R|-|I|)} \prod_{i\in I} \mathbf{Q}^{2k}_{q_i}(D_i) \\ &\qquad \times \int_{\mathbf{S}_J(h_J)} \prod_{i\in R\setminus I} P_m(x_i) \prod_{j\in R\setminus I} \left(\frac{\partial}{\partial x_j}\right)^m f(x)\,dx \bigg|_{h_J=0}. \end{aligned}$$

If $J = \emptyset$ then $\mathbf{S}_J$ is the whole space $\mathbb{R}^d$ and so

$$\sum_{\mathbf{S}_J\cap\mathbb{Z}^d}{}^{w} f = \int_{\mathbb{R}^d} f(x)\,dx + R^{\emptyset}_m(f), \tag{61}$$

with

$$R^{\emptyset}_m(f) := \sum_{\substack{R\neq\emptyset\\ R\subseteq\{1,\dots,d\}}} (-1)^{(m-1)|R|} \int_{\mathbb{R}^d} \prod_{i\in R} P_m(x_i) \prod_{j\in R} \left(\frac{\partial}{\partial x_j}\right)^m f(x)\,dx. \tag{62}$$

Let us now consider a regular integral J-sector $\mathbf{C}_J$, the image of the standard J-sector by an affine transformation

$$x \mapsto A_J x := Mx + b, \quad \text{with } M \in SL(d,\mathbb{Z}) \text{ and } b \in \mathbb{R}^d.$$

Moreover, let us denote by $\mathbf{C}_J(h)$ the expanded sector, image of $\mathbf{S}_J(h)$ under this affine transformation. For a $\mathcal{C}^m$ function of compact support f, let us consider $g := A^*_J f = f \circ A_J$. Then,

$$\sum_{\mathbf{C}_J\cap\mathbb{Z}^d} {}^w f := \sum_{\mathbf{S}_J\cap\mathbb{Z}^d} {}^w g = \prod_{j\in J} \mathbf{Q}_{q_j}^{2k}(D_j) \int\limits_{\mathbf{S}_J(h_J)} g(x)\,dx\bigg|_{h_J=0} + R_m^{J_{\mathrm{st}}}(g),$$

and we obtain the Euler–Maclaurin formula for a regular J-sector

$$\sum_{\mathbf{C}_J\cap\mathbb{Z}^d} {}^w f = \prod_{j\in J} \mathbf{Q}_{q_j}^{2k}(D_j) \int\limits_{\mathbf{C}_J(h_J)} f(x)\,dx\bigg|_{h_J=0} + R_m^{\mathbf{C}_J}(f), \tag{63}$$

where $R_m^{\mathbf{C}_J}(f) = R_m^{J_{\mathrm{st}}}(g)$.

5.4. Weighted Euler–Maclaurin formula for regular simple integral polytopes

From (63) we can write an Euler–Maclaurin formula for a regular integral polytope P with N facets, by using a polytope decomposition from Theorem 4.1. First, we write P as an intersection of N half spaces

$$P = \bigcap_{j=1}^{N} H_i,$$

where $H_i = \{x \in \mathbb{R}^d\colon\ \langle x, \eta_i\rangle + \lambda_i \geqslant 0\}$, for $i = 1, \ldots, N$. The vector η_i is an inward normal vector to the ith facet of P. Since P is a regular integral polytope the polarized tangent cones $\mathbf{C}_\Delta^\sharp$ are regular integral sectors. Moreover, we can write each tangent cone as

$$\mathbf{C}_\Delta = \bigcap_{j\in J_\Delta} H_j,$$

where $J_\Delta = \{j_1, \ldots, j_n\}$ is the index set of half spaces that contain $\mathbf{C}_\Delta$ ($n = \operatorname{codim}\Delta$). Then, using Theorem 4.1, we have

$$\begin{aligned}
\sum_{P\cap\mathbb{Z}^d} {}^w f &:= \sum_{P\cap\mathbb{Z}^d} \mathbf{1}_P^w f = \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta}\varphi(\varepsilon,\Delta) \sum_{\mathbf{C}_\Delta^\sharp\cap\mathbb{Z}^d} {}^w f \\
&= \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta}\varphi(\varepsilon,\Delta)\left(\prod_{j=1}^{N} \mathbf{Q}_{q_j}^{2k}(D_j) \int\limits_{\mathbf{C}_\Delta^\sharp(h_\Delta)} f(x)\,dx\bigg|_{h_\Delta=0} + R_m^{\mathbf{C}_\Delta^\sharp}(f)\right) \\
&= \prod_{j=1}^{N} \mathbf{Q}_{q_j}^{2k}(D_j) \int\limits_{P(h_1,\ldots,h_N)} f(x)\,dx\bigg|_{h=0} + S_m^P(f),
\end{aligned} \tag{64}$$

where

$$S_m^P(f) := \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta}\varphi(\varepsilon,\Delta) R_m^{\mathbf{C}_\Delta^\sharp}(f), \tag{65}$$

where $h_\Delta = (h_{j_1}, \dots, h_{j_n})$, and where the dilated polytope $P(h_1, \dots, h_N)$ is obtained by shifting the ith facet outward by a "distance" h_i. Here we used the fact that, when multiplying the differential operator in the first term of the right-hand side of (63) by any operator of the form $\mathbf{Q}^{2k}_{q_j}(D_j)$, with $j \notin J$, all that will remain of $\mathbf{Q}^{2k}_{q_j}(D_j)$ is the constant term 1, not affecting the final result. Note that both $\sum_{P\cap\mathbb{Z}^d} {}^w f$ and

$$\prod_{j=1}^{N} \mathbf{Q}^{2k}_{q_j}(D_i) \int\limits_{P(h_1,\dots,h_N)} f(x)\,dx \bigg|_{h=0}$$

do not depend on the choice of ε (that is, they do not depend on the Paradan region used). Consequently, the remainder is also independent of this choice.

Remark 66. Alternatively, using the polytope decomposition of Theorem 4.2, we obtain a different expression for the remainder in (65). Indeed, we get

$$S^P_m(f) := \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta+\dim\Delta} \varphi(\varepsilon, \Delta) R_m^{\mathbf{C}^\sharp_\Delta}(f), \tag{67}$$

where now the tangent cones $\mathbf{C}_\Delta$ are polarized with respect to the vectors $\varepsilon - \beta(\varepsilon, \Delta)$.

5.5. *Weighted Euler–Maclaurin formula for simple integral polytopes*

To extend formula (64) to simple integral polytopes we need to obtain an Euler–Maclaurin formula for simple J-sectors. We can describe a simple J-sector, $\mathbf{C}_J$, with $J = \{j_1, \dots, j_n\}$ as the intersection of n half spaces H_j in general position

$$\mathbf{C}_J = \bigcap_{j\in J} H_j, \tag{68}$$

where $H_j := \{x \in \mathbb{R}^d \mid \langle x, \eta_j\rangle + \lambda_j \geqslant 0\}$ for rational vectors η_j. Clearly, $\mathbf{C}_J$ is a polarized tangent cone $\mathbf{C}^\sharp_{\Delta_J}$ along the affine space Δ_J defined by

$$\Delta_J = \bigcap_{j\in J} \partial H_j$$

(see (26)). Clearing denominators we can assume the η_j's to be integral and we impose the normalizing condition that they are primitive elements of the dual lattice $\mathbb{Z}^{d*}$. Let $T_J \subseteq \mathbb{R}^{d*}$ be the subspace generated by the vectors η_i and let Ξ_J be the sublattice of $\mathbb{Z}^{d*} \cap T_J$ generated by these vectors. Then, to $\mathbf{C}_J$ we associate the finite abelian group

$$\Gamma_J := \left(\mathbb{Z}^{d*} \cap T_J\right)/\Xi_J. \tag{69}$$

Alternatively, we can consider the projection $\pi_J : \mathbb{R}^d \to \mathbb{R}^d/\operatorname{lin}(\Delta_J)$, where $\operatorname{lin}(\Delta_J)$ is the vector subspace of $\mathbb{R}^d$ parallel to Δ_J. Then we take the cone $\widetilde{\mathbf{C}}_J := \pi_J(\mathbf{C}_J)$ as well as the vectors $\tilde\eta_j := \pi_J(\eta_j)$. Note that these vectors are inward normals to the facets of $\widetilde{\mathbf{C}}_J$. Let $\{\tilde\alpha_{j_1}, \dots, \tilde\alpha_{j_n}\}$

be the dual basis in $\mathbb{R}^n$ (that is, such that $\langle\tilde{\alpha}_k, \tilde{\eta}_l\rangle = \delta_{kl}$ for $k, l \in J$). The $\tilde{\alpha}_j$'s are the projections on $\mathbb{R}^d/\operatorname{lin}(\Delta_J)$ of the vectors α_j defined in Section 4 (the generators of $\mathbf{C}_{\Delta_J}$) and generate a lattice $\hat{\ell}$ in $\mathbb{R}^n$ which is a finite extension of $\mathbb{Z}^n$ (this extension is trivial exactly when $\mathbf{C}_J$ is regular). Then Γ_J is equal to

$$\Gamma_J = \mathbb{Z}^{n*}/\hat{\ell}^*. \tag{70}$$

This group is trivial exactly when $\mathbf{C}_J$ is regular, and its order is $|\Gamma_J| = |\det \hat{\ell}|$.

Moreover, as it is shown in [23], $\gamma \mapsto e^{2\pi i\langle\gamma,\tilde{x}\rangle}$ defines a character of Γ_J, whenever $\tilde{x} \in \hat{\ell}$, which is trivial iff $\tilde{x} \in \mathbb{Z}^n$. Since, by a theorem of Frobenius, the average value of a character on a finite group is equal to zero if the character is non-trivial, and equal to one otherwise, we have

$$\frac{1}{|\Gamma_J|}\sum_{\gamma\in\Gamma_J} e^{2\pi i\langle\gamma,\tilde{x}\rangle} = \begin{cases} 1, & \text{if } \tilde{x} \in \mathbb{Z}^n, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\tilde{x} \in \hat{\ell}$. Consequently, for any compactly supported function f on $\mathbb{R}^d$,

$$\sum_{\mathbf{C}_J\cap\mathbb{Z}^d}{}^{w} f = \frac{1}{|\Gamma_J|}\sum_{\gamma\in\Gamma_J}\sum_{x}{}^{w} e^{2\pi i\langle\gamma,\tilde{x}\rangle} f(x), \tag{71}$$

where we sum over all

$$x = y + \sum_{j\in J} m_j\alpha_j \tag{72}$$

with $y \in \Delta_J \cap \mathbb{Z}^d$, $j \in J$ and $m_j \in \mathbb{Z}^+$, and where $\tilde{x} := \pi_J(x)$. Moreover, the cone $\mathbf{C}_J$ is the image of the standard J-sector $\mathbf{S}_J$ under an affine map

$$t \mapsto A_J t := U_J t + b, \quad \text{with } b \in \mathbb{R}^d, \tag{73}$$

where, for $j \in J$, $U_J \in GL(d, \mathbb{Z})$ carries the vectors e_j of the standard basis of $\mathbb{R}^d$ into the basis $\{\alpha_j\}_{j\in J}$, and the remaining basis vectors to an orthonormal basis of the space $(\operatorname{span}(\{\alpha_j\}_{j\in J}))^{\perp}$. Thus, $|\det U_J| = 1/|\det\hat{\ell}| = 1/|\Gamma_J|$. On the other hand, since in (72) we have $y \in \mathbb{Z}^d$, we get

$$e^{2\pi i\langle\gamma,\tilde{x}\rangle} = \prod_{j\in J}\lambda_j^{m_j}, \quad \text{with } \lambda_j = e^{2\pi i\langle\gamma,\tilde{\alpha}_j\rangle},$$

and so the inner sum in (71) becomes

$$\begin{aligned} &\sum_{j\in J}\sum_{y\in\Delta_J\cap\mathbb{Z}^n}\sum_{m_j\geqslant 0}{}^{q_j}\left(\prod_{l\in J}\lambda_l^{m_l}\right) f\left(y + \sum_{l\in J} m_l\alpha_l\right) \\ &\quad = \sum_{j\in J}\sum_{\substack{1\leqslant i\leqslant d\\ i\notin J}}\sum_{m_i\in\mathbb{Z}}\sum_{m_j\geqslant 0}{}^{q_j}\left(\prod_{l\in J}\lambda_l^{m_l}\right) g(m_1, \ldots, m_d), \end{aligned} \tag{74}$$

where $g = f \circ A_J$. Iterating the twisted remainder formula for the half ray (55) and the Euler–Maclaurin formula (52) for the whole real line, the sum in (74) can be written as

$$\prod_{j\in J} N_{q_j}^{k,\lambda_j}\left(\frac{\partial}{\partial h_j}\right) \int\limits_{\mathbf{S}_J(h_{j_1},\dots,h_{j_n})} g_J(t)\,dt\bigg|_{h=0} + R^{\mathrm{std}}_{\mathbf{q}_J,k}(\lambda_{j_1},\dots,\lambda_{j_n};g), \tag{75}$$

where again $\mathbf{S}_J(h_{j_1},\dots,h_{j_n}) = \{(t_1,\dots,t_d) \mid t_j \geqslant -h_j \text{ for } j \in J\}$ denotes the dilated standard J-sector, and where, for $\mathbf{q}_J := (q_{j_1},\dots,q_{j_n})$, the remainder is given by

$$\begin{aligned} &R^{\mathrm{std}}_{\mathbf{q}_J,k}(\lambda_{\gamma,j_1},\dots,\lambda_{\gamma,j_n};g) \\ &\quad := \sum_{I\subset J} \sum_{\substack{R\supseteq J\\ R\subseteq\{1,\dots,d\}\\ R\neq I}} (-1)^{(k-1)(|R|-|I|)} \\ &\qquad \times \prod_{i\in I} N_{q_i}^{k,\lambda_{\gamma,i}}\left(\frac{\partial}{\partial h_i}\right) \int\limits_{\mathbf{S}_J(h_{j_1},\dots,h_{j_n})} \prod_{j\in R\setminus I} Q_{k,\lambda_{\gamma,j}}(t_j) \prod_{j\in R\setminus I}\left(\frac{\partial}{\partial t_j}\right)^k g(t)\,dt\bigg|_{h=0}, \end{aligned} \tag{76}$$

with $g = f \circ A$. Let us now change variables by the inverse transformation of (73). Then, the Euler–Maclaurin formula in (71) becomes

$$\sum_{\mathbf{C}^J\cap\mathbb{Z}^d} {}^w f = \sum_{\gamma\in\Gamma_J}\prod_{j\in J} N_{q_j}^{k,\lambda_{\gamma,j}}\left(\frac{\partial}{\partial h_j}\right) \int\limits_{\mathbf{C}_J(h_J)} f(x)\,dx\bigg|_{h_J=0} + R^{\mathbf{C}_J}_{\mathbf{q}_J,k}(f), \tag{77}$$

where $\lambda_{\gamma,j} := e^{2\pi i\langle\gamma,\tilde{\alpha}_j\rangle}$, where, for $h_J := (h_{j_1},\dots,h_{j_n})$, $\mathbf{C}_J(h_J)$ denotes the image of the dilated standard J-sector $\mathbf{S}_J(h_J)$ under the affine transformation A_J of (73), and where the remainder $R^{\mathbf{C}_J}_{\mathbf{q}_J,k}(f)$ is given by

$$\begin{aligned} R^{\mathbf{C}_J}_{\mathbf{q}_J,k}(f) &:= \sum_{\gamma\in\Gamma_J}\sum_{I\subset J} \sum_{\substack{R\supseteq J\\ R\subseteq\{1,\dots,d\}\\ R\neq I}} (-1)^{(k-1)(|R|-|I|)} \prod_{i\in I} N_{q_i}^{k,\lambda_{\gamma,i}}\left(\frac{\partial}{\partial h_i}\right) \\ &\quad \times \int\limits_{\mathbf{C}_J(h_J)} \prod_{j\in R\setminus I} Q_{k,\lambda_{\gamma,j}}\big((U_{jk}^{-1})_k(x-b)\big) \prod_{j\in R\setminus I} D_j^k f(x)\,dx\bigg|_{h=0}, \end{aligned} \tag{78}$$

with $(U_{jk}^{-1})_k$ the jth row of U_J^{-1} and with D_j the directional derivative along the jth column vector of U_J^{-1}. Note that, when $j \in J$, this is the directional derivative along α_j.

Let now P be a simple polytope and choose an ε on some Paradan region. Again we write P as an intersection of half spaces

$$P = \bigcap_{i=1}^N H_i.$$

For each affine space Δ generated by a face of P there is a J_Δ-sector, $\mathbf{C}_{J_\Delta}$ ($J_\Delta \subset \{1, \dots, N\}$), equal to the polarized tangent cone of Δ, $\mathbf{C}^\sharp_\Delta$ (cf. (27)), and so we can associate a finite group Γ_Δ to Δ by simply taking the corresponding group Γ_{J_Δ}. Let $P(h)$ denote the dilated polytope obtained by shifting the ith facet by a distance h_i. Our decompositions of $P(h)$ involve dilated sectors but now, dilating the facets of P outward results in dilating some of the facets of $\mathbf{C}^\sharp_\Delta$ inward and some outward. Explicitly, taking $J_\Delta = \{j_1, \dots, j_{n_\Delta}\}$ such that $\mathbf{C}^\sharp_\Delta = \mathbf{C}_{J_\Delta}$ (see (68)), the inward normal vector to the jth facet of $\mathbf{C}^\sharp_\Delta$ ($j \in J_\Delta$) is

$$\eta^\sharp_{\Delta,j} = \begin{cases} \eta_j, & \text{if } \alpha^\sharp_{\Delta,j} = \alpha_{\Delta,j}, \\ -\eta_j, & \text{if } \alpha^\sharp_{\Delta,j} = -\alpha_{\Delta,j}, \end{cases}$$

where η_j is the inward pointing primitive normal vector to the jth facet of P. Note that, considering the projection $\pi_\Delta : \mathbb{R}^d \to \mathbb{R}^d/\mathrm{lin}(\Delta)$, the vectors $\{\pi_\Delta(\alpha_{\Delta,j})\}_{j \in J_\Delta}$, are the dual basis in $\mathbb{R}^{n_\Delta}$ to $\{\pi_\Delta(\eta_j)\}_{j \in J_\Delta}$. The dilated sectors that appear on the right side of the polytope decompositions of $P(h)$ are then $\mathbf{C}^\sharp_\Delta(h^\sharp_{\Delta,j_1}, \dots, h^\sharp_{\Delta,j_{n_\Delta}})$, where

$$h^\sharp_{\Delta,j_i} = \begin{cases} h_{j_i}, & \text{if } \alpha^\sharp_{\Delta,j_i} = \alpha_{\Delta,j_i}, \\ -h_{j_i}, & \text{if } \alpha^\sharp_{\Delta,j_i} = -\alpha_{\Delta,j_i}. \end{cases}$$

Moreover, the roots of unity that appear in the Euler–Maclaurin formula for $\mathbf{C}^\sharp_\Delta$ are

$$\lambda^\sharp_{\gamma,j,\Delta} = e^{2\pi i \langle \gamma, \tilde{\alpha}^\sharp_{\Delta,j} \rangle} = \begin{cases} \lambda_{\gamma,j,\Delta}, & \text{if } \alpha^\sharp_{\Delta,j} = \alpha_{\Delta,j}, \\ \lambda^{-1}_{\gamma,j,\Delta}, & \text{if } \alpha^\sharp_{\Delta,j} = -\alpha_{\Delta,j}, \end{cases}$$

where $\tilde{\alpha}^\sharp_{\Delta,j} = \pi_\Delta(\alpha^\sharp_{\Delta,j})$.

Hence, for any compactly supported function in $\mathbb{R}^d$ of type $\mathcal{C}^{dk}$ (for an integer $k \geqslant 1$), the decomposition formula of Theorem 4.1 applied to $P(h)$ along with formula (77) give

$$\begin{aligned} \sum_{P \cap \mathbb{Z}^d} {}^w f &= \sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta} \varphi(\varepsilon, \Delta) \sum_{\mathbf{C}^\sharp_\Delta \cap \mathbb{Z}^n} {}^w f \\ &= \sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta} \varphi(\varepsilon, \Delta) \sum_{\gamma \in \Gamma_\Delta} \prod_{j \in J_\Delta} \mathbf{N}^{k, \lambda^\sharp_{\gamma,j,\Delta}}_{q^\sharp_j} \left(\frac{\partial}{\partial h^\sharp_{\Delta,j}} \right) \int_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\, dx \bigg|_{h^\sharp_{J_\Delta} = 0} + R^P_{w,k}(f), \end{aligned} \tag{79}$$

where $h^\sharp_{J_\Delta} = (h^\sharp_{j_1}, \dots, h^\sharp_{j_{n_\Delta}})$ and where

$$R^P_{w,k}(f) := \sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta} \varphi(\varepsilon, \Delta) R^{\mathbf{C}^\sharp_\Delta}_{q^\sharp_{J_\Delta}, k}(f). \tag{80}$$

Remark 81. Using the polytope decomposition of Theorem 4.2 we obtain

$$\sum_{P\cap\mathbb{Z}^d} {}^w f = \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta+\dim\Delta}\varphi(\varepsilon,\Delta) \sum_{\mathbf{C}^\sharp_\Delta\cap\mathbb{Z}^n} {}^w f$$
$$= \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta+\dim\Delta}\varphi(\varepsilon,\Delta) \sum_{\gamma\in\Gamma_\Delta}\prod_{j\in J_\Delta} \mathbf{N}^{k,\lambda^\sharp_{\gamma,j,\Delta}}_{q^\sharp_j}\left(\frac{\partial}{\partial h^\sharp_{\Delta,j}}\right)\int_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\,dx\bigg|_{h^\sharp_{J_\Delta}=0}$$
$$+ R^P_{w,k}(f), \tag{82}$$

with $h^\sharp_J = (h^\sharp_{j_1},\dots,h^\sharp_{j_{n_\Delta}})$ and

$$R^P_{w,k}(f) := \sum_{\Delta\in\mathcal{B}} (-1)^{m_\Delta+\dim\Delta}\varphi(\varepsilon,\Delta) R^{\mathbf{C}^\sharp_\Delta}_{q^\sharp_{J_\Delta},k}(f), \tag{83}$$

where now the tangent cones $\mathbf{C}_\Delta$ are polarized by the vectors $\varepsilon-\beta(\varepsilon,\Delta)$.

Let us now analyze some properties of the groups Γ_Δ. These generalize Claims 61, 62 and 65 in [23] to spaces $\Delta\in\mathcal{B}$ of arbitrary dimensions. Their proofs follow easily from the ones in [23] but we will include them for completeness. For that, we will first introduce some necessary notation. If Δ and $\tilde\Delta$ are two elements of $\mathcal{B}$ with $\Delta\subseteq\tilde\Delta$, then $J_{\tilde\Delta}\subseteq J_\Delta$ and we have an inclusion $T_{J_{\tilde\Delta}}\subseteq T_{J_\Delta}$. Moreover, if Ξ_Δ and $\Xi_{\tilde\Delta}$ are the lattices in $\mathbb{R}^{d*}$ generated by the vectors η_j with $j\in J_\Delta$ and $J_{\tilde\Delta}$, respectively, we have $T_{J_{\tilde\Delta}}\cap\Xi_\Delta=\Xi_{\tilde\Delta}$. Hence, the natural map from $\Gamma_{\tilde\Delta}$ to Γ_Δ is one-to-one and provides a natural inclusion map $\Gamma_{\tilde\Delta}\subseteq\Gamma_\Delta$. Therefore, we can define a subset $\Gamma^\flat_\Delta$ of Γ_Δ by

$$\Gamma^\flat_\Delta := \Gamma_\Delta \Big\backslash \bigcup_{\tilde\Delta\in\mathcal{B}\mid\Delta\subsetneq\tilde\Delta}\Gamma_{\tilde\Delta}$$

and then

$$\Gamma_\Delta = \bigsqcup_{\tilde\Delta\in\mathcal{B}\mid\Delta\subseteq\tilde\Delta}\Gamma^\flat_{\tilde\Delta}. \tag{84}$$

Claim 85. *If $\gamma\in\Gamma_\Delta$ and $j\in J_\Delta$, then $\lambda_{\gamma,j,\Delta'}$ is the same for all $\Delta'\subset\Delta$.*

Claim 86. *If $\gamma\in\Gamma_\Delta$, $\Delta'\subset\Delta$ and $j\in J_{\Delta'}\setminus J_\Delta$, then $\lambda_{\gamma,j,\Delta'}=1$.*

Claim 87. *If $\gamma\in\Gamma^\flat_\Delta$ and $j\in\Gamma_\Delta$, then $\lambda_{\gamma,j,\Delta}\neq 1$.*

Proof. Let $\gamma\in\Gamma_\Delta$ be represented by

$$\tilde\gamma = \sum_{i\in J_\Delta} b_i\tilde\eta_i \in \mathbb{Z}^{n_\Delta *}$$

for some $b_i \in \mathbb{R}$ (see (70)). Let $\Delta' \subset \Delta$ and let us identify $\mathbb{R}^d/\operatorname{lin}(\Delta)$ with the orthogonal complement $(\operatorname{lin}(\Delta))^\perp$ in $\mathbb{R}^d/\operatorname{lin}(\Delta')$. Since the $\tilde{\alpha}_{\Delta',j}$'s are dual to the $\tilde{\eta}_j$'s for $j \in J_{\Delta'}$, we have

$$\langle \tilde{\gamma}, \tilde{\alpha}_{\Delta',j} \rangle = \begin{cases} b_j, & \text{if } j \in J_\Delta, \\ 0, & \text{if } j \in J_{\Delta'} \setminus J_\Delta. \end{cases}$$

Consequently,

$$\lambda_{\gamma,j,\Delta'} = \begin{cases} e^{2\pi i b_j}, & \text{if } j \in J_\Delta, \\ 1, & \text{if } j \in J_{\Delta'} \setminus J_\Delta \end{cases}$$

is independent of Δ' and is equal to 1 if $j \in J_{\Delta'} \setminus J_\Delta$, and so Claims 85 and 86 follow.

Let $j \in J_\Delta$. If $\lambda_{\gamma,j,\Delta} := e^{2\pi i b_j} = 1$, then $b_j \in \mathbb{Z}$ and so

$$\tilde{\gamma} = \sum_{i \in J_\Delta \setminus \{j\}} b_i \tilde{\eta}_i \tag{88}$$

also represents γ. Let $\tilde{\Delta} \supset \Delta$ be the element of $\mathcal{B}$ such that $J_{\tilde{\Delta}} = J_\Delta \setminus \{j\}$. Then, by (88), $\gamma \in \Gamma_{\tilde{\Delta}}$, and Claim 87 follows. □

With these properties we can further simplify formula (79). First, note that either $h^\sharp_{\Delta,j} = h_j$, $\lambda^\sharp_{\gamma,j,\Delta} = \lambda_{\gamma,j,\Delta}$ and $q^\sharp_j = q_j$, or $h^\sharp_{\Delta,j} = -h_j$, $\lambda^\sharp_{\gamma,j,\Delta} = \lambda^{-1}_{\gamma,j,\Delta}$ and $q^\sharp_j = 1 - q_j$, and so, by symmetry property (58), this gives

$$\mathbf{N}^{k,\lambda^\sharp_{\gamma,j,\Delta}}_{q^\sharp_{\Delta,j}}\left(\frac{\partial}{\partial h^\sharp_{\Delta,j}}\right) = \mathbf{N}^{k,\lambda_{\gamma,j,\Delta}}_{q_j}\left(\frac{\partial}{\partial h_j}\right). \tag{89}$$

Moreover, from Claim 86, we have $\lambda_{\gamma,j,\Delta} = 1$ for $j \notin J_\Delta$, implying that

$$\mathbf{N}^{k,\Delta_{\gamma,j,\Delta}}_{q_j}\left(\frac{\partial}{\partial h_j}\right) = 1 + \text{powers of } \frac{\partial}{\partial h_j}.$$

Since, still for $j \notin J_\Delta$, the cone $\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})$ is independent of h_j, (79) is equal to

$$\sum_{\Delta \in \mathcal{B}} (-1)^{m_\Delta} \varphi(\varepsilon, \Delta) \sum_{\gamma \in \Gamma_\Delta} \prod_{j=1}^N \mathbf{N}^{k,\lambda_{\gamma,j,\Delta}}_{q_j}\left(\frac{\partial}{\partial h_j}\right) \int\limits_{\mathbf{C}^\sharp_\Delta(h^\sharp_J)} f(x)\,dx \Big|_{h=0} + R^P_{w,k}(f), \tag{90}$$

where N is the number of facets of P. Defining

$$\mathbf{N}^k_{\gamma,\Delta} := \prod_{j=1}^N \mathbf{N}^{k,\lambda_{\gamma,j,\Delta}}_{q_j}\left(\frac{\partial}{\partial h_j}\right), \quad \text{for } \gamma \in \Gamma_\Delta \tag{91}$$

we have, from Claim 85, that

$$\mathbf{N}^k_{\gamma,\Delta} = \mathbf{N}^k_{\gamma,\tilde{\Delta}} \quad \text{whenever } \gamma \in \Gamma_{\tilde{\Delta}} \text{ and } \Delta \subset \tilde{\Delta}. \tag{92}$$

Consequently, using (84), formula (90) can be written as

$$\sum_{\Delta\in\mathcal{B}}(-1)^{m_\Delta}\varphi(\varepsilon,\Delta)\sum_{\gamma\in\Gamma_\Delta}\mathbf{N}^k_{\gamma,\Delta}\int\limits_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f)$$

$$=\sum_{\Delta\in\mathcal{B}}(-1)^{m_\Delta}\varphi(\varepsilon,\Delta)\sum_{\tilde\Delta\in\mathcal{B}}\sum_{\gamma\in\Gamma^\flat_{\tilde\Delta}}\mathbf{N}^k_{\gamma,\tilde\Delta}\int\limits_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f)$$

$$=\sum_{\tilde\Delta\in\mathcal{B}}\sum_{\gamma\in\Gamma^\flat_{\tilde\Delta}}\mathbf{N}^k_{\gamma,\tilde\Delta}\sum_{\Delta\subset\tilde\Delta}(-1)^{m_\Delta}\varphi(\varepsilon,\Delta)\int\limits_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f). \tag{93}$$

In the interior summation on the left we can add similar terms that correspond to spaces Δ not included in $\tilde\Delta$. Indeed, these make a zero contribution for the following reason. If Δ is not a subset of $\tilde\Delta$ there exists a $j\in J_{\tilde\Delta}\setminus J_\Delta$. Then, since $j\notin J_\Delta$, the cone $\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})$ does not depend on h_j. On the other hand, since $\gamma\in\Gamma^\flat_{\tilde\Delta}$ and $j\in J_{\tilde\Delta}$, we know, from Claim 87, that $\lambda_{\gamma,j,\tilde\Delta}\neq 1$ and so, by Remark 59, we have that $\mathbf{N}^{k,\lambda_{\gamma,j,\tilde\Delta}}_{q_j}(\frac{\partial}{\partial h_j})$ (one of the factors of $\mathbf{N}^k_{\gamma,\tilde\Delta}$) is a multiple of $(\frac{\partial}{\partial h_j})$.

Therefore, (93) is equal to

$$\sum_{\tilde\Delta\in\mathcal{B}}\sum_{\gamma\in\Gamma^\flat_{\tilde\Delta}}\mathbf{N}^k_{\gamma,\tilde\Delta}\sum_{\Delta\in\mathcal{B}}(-1)^{m_\Delta}\varphi(\varepsilon,\Delta)\int\limits_{\mathbf{C}^\sharp_\Delta(h^\sharp_{J_\Delta})} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f)$$

$$=\sum_{\tilde\Delta\in\mathcal{B}}\sum_{\gamma\in\Gamma^\flat_{\tilde\Delta}}\mathbf{N}^k_{\gamma,\tilde\Delta}\int\limits_{P(h)} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f), \tag{94}$$

and we have our result:

Theorem 5.1. *Let P be a simple polytope in $\mathbb{R}^d$ with N facets and let $f\in\mathcal{C}^{dk}_c(\mathbb{R}^d)$ be a compactly supported function on $\mathbb{R}^d$ for $k\geqslant 1$. Choosing an ε on a Paradan region determined by P, we obtain*

$$\sum_{P\cap\mathbb{Z}^d}{}^w f=\sum_{\Delta\in\mathcal{B}}\sum_{\gamma\in\Gamma^\flat_\Delta}\mathbf{N}^k_{\gamma,\Delta}\int\limits_{P(h)} f(x)\,dx\bigg|_{h=0}+R^P_{w,k}(f), \tag{95}$$

where $\mathbf{N}^k_{\gamma,\Delta}$ is the differential operator described in (91) *and* (92)*, and where the remainder is given by* (80)*. The operator $\mathbf{N}^k_{\gamma,\Delta}$ is of order $\leqslant k$ in each of the variables $h_1,\dots,h_N$ with N the number of facets of P. The remainder is a sum of integrals over sectors, of bounded periodic functions times several partial derivatives of f of order no less than k and no more than kd. Moreover, this remainder is independent of the choice of Paradan region of ε, and is a distribution supported on the polytope P.*

Note that, even though the expression for the remainder (80) depends on ε, its final value is independent of the choice of ε since both the left-hand side and the first term on the right-hand side of (95) are independent of ε.

Remark 96. If we instead use the polytope decompositions of Theorem 4.2 the remainder in (95) will be given by (83).

The Euler–Maclaurin formula (95) obtained in Theorem 5.1 is similar to the one presented in [3]. However, in our formula, we allow the operators $\mathbf{N}_{q_j}^{k,\lambda_{\gamma,j,\Delta}}$ that define $\mathbf{N}_{\gamma,\Delta}^k$ and $\mathbf{N}_0^k$ to have different weights $q_j \in \mathbb{C}$ while, in [3], the q_j's are all equal to some fixed complex number (in [23] this fixed weight is $1/2$). Moreover, we obtain a different expression for the remainder $R_{w,k}^P(f)$ which is now given as a sum over the affine spaces generated by all the faces of the polytope (not only over the vertices). In addition, the intermediate formulas that we obtain in (93) (before adding terms with zero contribution in order to get an integral over the dilated polytope) also involve sums of integrals over the polarized tangent cones to the polytope at the different faces and not only at vertices.

Just as the Euler–Maclaurin formulas in [3,23] our formulas generalize to symbols that is, to smooth functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ for which there is a positive integer N (called the *order* of the symbol) such that, for every d-tuple of non-negative integers $a := (a_1, \dots, a_d)$ there is a constant C_a satisfying $|\partial_1^{a_1} \cdots \partial_d^{a_d} f(x)| \leqslant C_a(1 + |x|)^{N-a}$. In particular, for a polynomial function p in $\mathbb{R}^d$, we obtain the following exact formula

$$\sum_{P\cap\mathbb{Z}^d}{}^{w} p = \sum_{\Delta\in\mathcal{B}} \sum_{\gamma\in\Gamma_\Delta^\flat} \mathbf{N}_{\gamma,\Delta}^k \int_{P(h)} p(x)\,dx \bigg|_{h=0} \tag{97}$$

(where we choose $k \geqslant \deg p + d + 1$). From Remark 56 we see that this is a weighted version of the exact Euler–Maclaurin formula obtained in [10], which is obtained from (97) by making all the weights in w equal to 1.

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