Available online at www.sciencedirect.com
SCIENCE DIRECT®
PHYSICS LETTERS B
Physics Letters B 574 (2003) 276–282
www.elsevier.com/locate/npe

A non-commutative Minkowskian spacetime from a quantum AdS algebra

Angel Ballesteros [a], N. Rossano Bruno [a,b], Francisco J. Herranz [a]

[a] *Departamento de Física, Universidad de Burgos, Pza. Misael Bañuelos s.n., 09001 Burgos, Spain*
[b] *Dipartimento di Fisica, Università di Roma ROMA TRE, and INFN Sez. Roma Tre, Via Vasca Navale 84, 00146 Roma, Italy*

Received 10 June 2003; received in revised form 28 August 2003; accepted 2 September 2003

Editor: M. Cvetič

Abstract

A quantum deformation of the conformal algebra of the Minkowskian spacetime in $(3+1)$ dimensions is identified with a deformation of the $(4+1)$-dimensional AdS algebra. Both Minkowskian and AdS first-order non-commutative spaces are explicitly obtained, and the former coincides with the well-known κ-Minkowski space. Next, by working in the conformal basis, a new non-commutative Minkowskian spacetime is constructed through the full (all orders) dual quantum group spanned by deformed Poincaré and dilation symmetries. Although Lorentz invariance is lost, the resulting non-commutative spacetime is quantum group covariant, preserves space isotropy and, furthermore, can be interpreted as a generalization of the κ-Minkowski space in which a variable fundamental scale (Planck length) appears.

PACS: 02.20.Uw; 11.30.-j; 04.60.-m

Keywords: Quantum algebras; Deformation; Minkowski; Anti-de Sitter; Poincaré; Non-commutative spacetime

1. Introduction

One of the most relevant applications of quantum groups in physics is the construction of deformed spacetime symmetries that generalize classical Poincaré kinematics beyond Lie algebras, such as the well-known κ-Poincaré [1–3] and the quantum null-plane (or light-cone) Poincaré [4,5] algebras. For all these cases, the deformation parameter has been interpreted as a fundamental scale which may be related with the Planck length. In fact, these results can be seen as different attempts to develop new approaches to physics at the Planck scale, an idea that was early presented in [6]. A further physical development of the κ-Poincaré algebra has led to the so-called doubly special relativity (DSR) theories [7–11] that analyze the fundamental role assigned to the deformation parameter/Planck length as an observer-independent length scale to be considered together with the usual observer-independent velocity scale c, in such a manner that Lorentz invariance is preserved [12–14].

From a dual quantum group perspective, when the quantum spacetime coordinates $\hat{x}^\mu$ conjugated to the κ-Poincaré momentum-space P_μ (translations)

E-mail addresses: angelb@ubu.es (A. Ballesteros), rossano@fis.uniroma3.it (N.R. Bruno), fjherranz@ubu.es (F.J. Herranz).

0370-2693
doi:10.1016/j.physletb.2003.09.014

are considered, the non-commutative κ-Minkowski spacetime arises [15–18]. Some field theories on such a space have been proposed (see [19] and references therein) and its role in DSR theories has been analyzed [20]. More general non-commutative Minkowskian spacetimes can be expressed by means of the following Lie algebra commutation rules [21]:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{1}{\kappa}\left(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu\right), \tag{1}$$

where a^μ is a *constant* four-vector in the Minkowskian space.

In the context of κ-deformations, which are understood as quantum algebras with a dimensionful deformation parameter related with the Planck length, Poincaré symmetry should be taken only as a first stage that should be embedded in some way within more general structures such as deformed conformal or AdS/dS symmetries, that is, quantum $so(4,2)/so(5,1)$ algebras. Thus, it is natural to think that a non-commutative Minkowskian space of the form (1) could also be either embedded or generalized. In this respect, by considering the simplest quantum deformation of the Weyl–Poincaré algebra (isometries plus dilations), $U_\tau(\mathcal{WP})$ [22], a new DSR proposal has been presented in [23]. Such a quantum algebra arises as a Hopf subalgebra of a 'mass-like' quantum deformation $U_\tau(so(4,2))$ of the conformal algebra of the $(3+1)$D Minkowskian space.

The aim of this Letter is to analyze the first-order (in both the deformation parameter and non-commutative coordinates) quantum group dual to $U_\tau(so(4,2))$, to construct the complete (all orders) Hopf algebra dual to $U_\tau(\mathcal{WP})$ and, afterwards, to extract some physical implications of the associated non-commutative Minkowskian spacetime.

In the next section we identify the deformation $U_\tau(so(4,2))$, formerly obtained in a conformal basis, with a $(4+1)$D quantum AdS algebra at both algebra and dual group levels. By using the Hopf subalgebra spanned by the deformed Poincaré and dilation generators we compute in Section 3 the associated dual quantum group by making use of the quantum $\mathcal{R}$-matrix. The last section is devoted to derive the physical consequences conveyed by the resulting non-commutative Minkowskian spacetime which is covariant under quantum group transformations and does preserve space isotropy, although its Lorentz invariance is lost. Moreover, this new non-commutative spacetime generalizes κ-Minkowski space since it is defined through Lie algebra commutation rules whose structure constants are the quantum group entries associated to the Lorentz sector.

2. Quantum AdS algebra

Let us consider the quantum deformation [22] of the conformal algebra of the $(3+1)$D Minkowskian spacetime, $U_\tau(so(4,2)) \equiv U_\tau(\mathcal{CM}^{3+1})$, which is spanned by the generators of rotations J_i, time and space translations P_μ, boosts K_i, conformal transformations C_μ and dilations D. Hereafter we assume $c = \hbar = 1$ ($\omega = 1$ in [22]), sum over repeated indices, Latin indices $i, j, k = 1, 2, 3$, and Greek indices $\mu, \nu = 0, 1, 2, 3$. The non-vanishing deformed commutation rules and coproduct for $U_\tau(\mathcal{CM}^{3+1})$ are given by

$$\begin{aligned}
&[J_i, J_j] = \varepsilon_{ijk} J_k, \qquad [J_i, K_j] = \varepsilon_{ijk} K_k,\\
&[J_i, P_j] = \varepsilon_{ijk} P_k, \qquad [K_i, K_j] = -\varepsilon_{ijk} J_k,\\
&[K_i, P_0] = \mathrm{e}^{-\tau P_0} P_i, \qquad [D, P_i] = P_i,\\
&[K_i, P_i] = \frac{\mathrm{e}^{\tau P_0} - 1}{\tau}, \qquad [D, P_0] = \frac{1 - \mathrm{e}^{-\tau P_0}}{\tau},
\end{aligned} \tag{2}$$

$$\begin{aligned}
&[J_i, C_j] = \varepsilon_{ijk} C_k, \qquad [D, C_i] = -C_i,\\
&[D, C_0] = -C_0 + \tau D^2, \qquad [P_0, C_0] = -2D,\\
&[K_i, C_0] = C_i, \qquad [K_i, C_i] = C_0 - \tau D^2,\\
&[C_0, C_i] = -\tau(D C_i + C_i D),\\
&[P_0, C_i] = \mathrm{e}^{-\tau P_0} K_i + K_i \mathrm{e}^{-\tau P_0},\\
&[P_i, C_j] = 2(\delta_{ij} D - \varepsilon_{ijk} J_k),\\
&[C_0, P_i] = 2K_i + \tau(D P_i + P_i D),
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1,\\
&\Delta(P_i) = 1 \otimes P_i + P_i \otimes \mathrm{e}^{\tau P_0},\\
&\Delta(J_i) = 1 \otimes J_i + J_i \otimes 1,\\
&\Delta(D) = 1 \otimes D + D \otimes \mathrm{e}^{-\tau P_0},\\
&\Delta(K_i) = 1 \otimes K_i + K_i \otimes 1 - \tau D \otimes \mathrm{e}^{-\tau P_0} P_i,\\
&\Delta(C_0) = 1 \otimes C_0 + C_0 \otimes \mathrm{e}^{-\tau P_0},\\
&\Delta(C_i) = 1 \otimes C_i + C_i \otimes \mathrm{e}^{-\tau P_0} + 2\tau D \otimes \mathrm{e}^{-\tau P_0} K_i\\
&\qquad\quad - \tau^2\left(D^2 + D\right) \otimes \mathrm{e}^{-2\tau P_0} P_i.
\end{aligned} \tag{4}$$

The deformation parameter τ will be identified with the Planck length: $\tau \sim L_p$ [23].

Some properties of $U_\tau(so(4,2))$ can be unveiled by studying its Lie bialgebra structure, that is, the first-order relations in the deformation parameter, generators and dual coordinates. If we write the deformed coproduct Δ as formal power series in τ,

$$\Delta = \sum_{k=0}^{\infty} \Delta_{(k)} = \sum_{k=0}^{\infty} \tau^k \delta_{(k)}, \tag{5}$$

the cocommutator δ is given by the skewsymmetric part of the first-order deformation,

$$\delta = \delta_{(1)} - \sigma \circ \delta_{(1)}, \tag{6}$$

where $\sigma(a \otimes b) = b \otimes a$. In our case, from (4) we find that

$$\begin{aligned}
&\delta(P_0) = 0, \qquad \delta(P_i) = \tau P_i \wedge P_0, \\
&\delta(J_i) = 0, \qquad \delta(D) = -\tau D \wedge P_0, \\
&\delta(K_i) = -\tau D \wedge P_i, \qquad \delta(C_0) = -\tau C_0 \wedge P_0, \\
&\delta(C_i) = -\tau C_i \wedge P_0 + 2\tau D \wedge K_i,
\end{aligned} \tag{7}$$

where $\wedge$ denotes the skewsymmetric tensor product. Next, Lie bialgebra duality [24,25] leads to the dual commutation rules in the form

$$\delta(Y_i) = f_i^{jk} Y_j \wedge Y_k \quad \Rightarrow \quad \left[\hat{y}^j, \hat{y}^k\right] = f_i^{jk} \hat{y}^i, \tag{8}$$

where Y_i is a generic generator and $\hat{y}^i$ its associated dual quantum group coordinate fulfilling $\langle \hat{y}^i | Y_j \rangle = \delta^i_j$. Therefore if we denote by $\{\hat{x}^\mu, \hat{\theta}^i, \hat{\xi}^i, \hat{d}, \hat{c}^\mu\}$ the dual non-commutative coordinates of the generators $\{P_\mu, J_i, K_i, D, C_\mu\}$, respectively, we obtain from (7) the following non-vanishing first-order quantum group commutation rules:

$$\begin{aligned}
&\left[\hat{x}^0, \hat{x}^i\right] = -\tau \hat{x}^i, \qquad \left[\hat{x}^0, \hat{d}\right] = \tau \hat{d}, \\
&\left[\hat{x}^0, \hat{c}^\mu\right] = \tau \hat{c}^\mu, \\
&\left[\hat{d}, \hat{x}^i\right] = -\tau \hat{\xi}^i, \qquad \left[\hat{d}, \hat{\xi}^i\right] = 2\tau \hat{c}^i.
\end{aligned} \tag{9}$$

The non-commutative Minkowskian spacetime is then characterized by

$$\left[\hat{x}^0, \hat{x}^i\right] = -\tau \hat{x}^i, \qquad \left[\hat{x}^i, \hat{x}^j\right] = 0, \tag{10}$$

which coincides with the usual κ-Minkowski space (1) with $a^\mu = (1, 0, 0, 0)$, provided that $\tau = -1/\kappa$. Nevertheless, in the next section we shall compute the full (all orders) dual quantum group to $U_\tau(\mathcal{WP})$, and the associated non-commutative spacetime will generalize the first-order relations (10).

We stress that all the above expressions can be rewritten in terms of deformed symmetries of the $(4+1)$D AdS spacetime, AdS^{4+1}. Let L_{AB} $(A < B)$ and T_A $(A, B = 0, 1, \dots, 4)$ be the Lorentz and translations undeformed generators obeying

$$\begin{aligned}
&[L_{AB}, L_{CD}] = \eta_{AC} L_{BD} - \eta_{AD} L_{BC} - \eta_{BC} L_{AD} \\
&\qquad\qquad + \eta_{BD} L_{AC}, \\
&[L_{AB}, T_C] = \eta_{AC} T_B - \eta_{BC} T_A, \\
&[T_A, T_B] = -\frac{1}{R^2} L_{AB},
\end{aligned} \tag{11}$$

such that $\eta = (\eta_{AB}) = \mathrm{diag}(-1, 1, 1, 1, 1)$ is Lorentz metric associated to $so(4,1)$, L_{0B} are the four boosts in AdS^{4+1} and R is the AdS radius related with the cosmological constant by $\Lambda = 6/R^2$. Then the following change of basis $(i = 1, 2, 3)$:

$$\begin{aligned}
&T_0 = -\frac{1}{2R}(C_0 + P_0), \qquad T_1 = \frac{1}{R} D, \\
&T_{i+1} = \frac{1}{2R}(C_i + P_i), \qquad L_{01} = \frac{1}{2}(C_0 - P_0), \\
&L_{0,i+1} = K_i, \qquad L_{1,i+1} = \frac{1}{2}(C_i - P_i), \\
&L_{23} = J_3, \qquad L_{24} = -J_2, \\
&L_{34} = J_1,
\end{aligned} \tag{12}$$

connects $\mathcal{CM}^{3+1}$ with AdS^{4+1} and can be taken in the deformed case as a way to identify (2)–(4) as the quantum deformation $U_\tau(so(4,2)) \equiv U_\tau(AdS^{4+1})$. The dual Lorentz $\hat{l}^{AB}$ and spacetime $\hat{t}^A$ quantum AdS-coordinates can also be written in terms of the conformal ones as

$$\begin{aligned}
&\hat{t}^0 = -R\left(\hat{c}^0 + \hat{x}^0\right), \qquad \hat{t}^1 = R\hat{d}, \\
&\hat{t}^{i+1} = R\left(\hat{c}^i + \hat{x}^i\right), \qquad \hat{l}^{01} = \hat{c}^0 - \hat{x}^0, \\
&\hat{l}^{0,i+1} = \hat{\xi}^i, \qquad \hat{l}^{1,i+1} = \hat{c}^i - \hat{x}^i, \\
&\hat{l}^{23} = \hat{\theta}^3, \qquad \hat{l}^{24} = -\hat{\theta}^2, \\
&\hat{l}^{34} = \hat{\theta}^1.
\end{aligned} \tag{13}$$

Hence the (first-order) non-vanishing commutation rules for the non-commutative AdS^{4+1} spacetime turn out to be

$$\begin{aligned}
&\left[\hat{t}^0, \hat{t}^1\right] = -\tau R \hat{t}^1, \qquad \left[\hat{t}^0, \hat{t}^{i+1}\right] = -\tau R^2 \hat{l}^{1,i+1}, \\
&\left[\hat{t}^1, \hat{t}^{i+1}\right] = -\tau R^2 \hat{l}^{0,i+1},
\end{aligned} \tag{14}$$

which involve both the boost $\hat{l}^{0,i+1}$ and the rotation $\hat{l}^{1,i+1}$ quantum coordinates. A Lie bialgebra contraction analysis [26] shows that the contraction $R \to \infty$ from $U_\tau(AdS^{4+1})$ and its dual to a $(4+1)$D quantum Poincaré algebra/group is well defined whenever the deformation parameter is transformed as $\tau \to \tau R^2$.

Therefore, the maps (12) and (13) can be used to express the same quantum deformation of $so(4,2)$ within two physically different frameworks: $U_\tau(\mathcal{CM}^{3+1}) \leftrightarrow U_\tau(AdS^{4+1})$. In fact, such a quantum group relationship might further be applied in order to analyze the role that quantum deformations of $so(4,2)$ could play in relation with the "AdS-CFT correspondence" that relates local QFT on $AdS^{(d-1)+1}$ with a conformal QFT on the (compactified) Minkowskian spacetime $\mathcal{CM}^{(d-2)+1}$ [27–29] (in our case up to $d=5$). We also remark that a more general (three-parameter) quantum deformation of $o(3,2)$ can be found in [30], where the connection between the corresponding quantum $\mathcal{CM}^{2+1}$ and AdS^{3+1} algebras is explicitly described.

3. Quantum Weyl–Poincaré group

The classical r-matrix associated to $U_\tau(so(4,2))$ reads

$$r = -\tau D \wedge P_0 \equiv \tau R^2 T_1 \wedge T_0 + \tau R\, T_1 \wedge L_{01}, \tag{15}$$

which satisfies the classical Yang–Baxter equation [31]. Therefore, $U_\tau(so(4,2))$ is a triangular (or twisting) quantum deformation, different to the Drinfeld–Jimbo type, which is supported by the Hopf subalgebra spanned by $\{D, P_0\}$. The universal $\mathcal{T}$-matrix for the latter can be written as [32]

$$\mathcal{T} = \mathrm{e}^{\hat{d} D}\mathrm{e}^{\hat{x}^0 P_0}, \tag{16}$$

while the $\mathcal{R}$-matrix reads

$$\mathcal{R} = \exp(\tau P_0 \otimes D)\exp(-\tau D \otimes P_0). \tag{17}$$

Since this element is also a universal $\mathcal{R}$-matrix for both $U_\tau(\mathcal{WP}) \subset U_\tau(so(4,2))$ [22], the corresponding dual quantum groups can be deduced explicitly by applying the FRT procedure [33]. This requires a matrix representation R for (17) as well as to choose a matrix element T of the quantum group with non-commutative entries. However, the consideration of the complete $U_\tau(so(4,2))$ structure (in both conformal and AdS bases) precludes a clear identification of the non-commutative spacetime coordinates as these would appear as arguments of functions that also depend on many other coordinates. In this respect see, for instance, [34] for the construction of a quantum AdS space from a q-$SO(3,2)$ of Drinfeld–Jimbo type.

Since we are mainly interested in the structure and physical consequences of the associated non-commutative spacetime coordinates (dual to the translation generators), we shall consider here the FRT construction for the Weyl–Poincaré Hopf subalgebra in the conformal basis.

A deformed matrix representation for the defining relations of $U_\tau(\mathcal{WP})$ (2) can be obtained from the 6×6 matrix representation of AdS^{4+1}; namely,

$$\begin{aligned} P_0 &= \frac{\tau}{2}(e_{00} - e_{01} + e_{10} - e_{11}) - e_{02} - e_{12} \\ &\quad + e_{20} - e_{21}, \\ P_i &= e_{0,i+2} + e_{1,i+2} + e_{i+2,0} - e_{i+2,1}, \\ D &= e_{01} + e_{10}, \\ J_i &= -\varepsilon_{ijk}\, e_{j+2,k+2}, \qquad K_i = e_{2,i+2} + e_{i+2,2}, \end{aligned} \tag{18}$$

where e_{ab} $(a,b = 0,\dots,5)$ is the matrix with entries δ_{ab}. We construct the quantum group element T in such a representation by considering the following matrix product, which is consistent with the exponential form of the universal $\mathcal{T}$-matrix (16) for the carrier subalgebra $\{D, P_0\}$:

$$\begin{aligned} T &= \mathrm{e}^{\hat{d} D}\mathrm{e}^{\hat{x}^0 P_0}\mathrm{e}^{\hat{x}^1 P_1}\mathrm{e}^{\hat{x}^2 P_2}\mathrm{e}^{\hat{x}^3 P_3}\mathrm{e}^{\hat{\theta}^1 J_1}\mathrm{e}^{\hat{\theta}^2 J_2}\mathrm{e}^{\hat{\theta}^3 J_3} \\ &\quad \times \mathrm{e}^{\hat{\xi}^1 K_1}\mathrm{e}^{\hat{\xi}^2 K_2}\mathrm{e}^{\hat{\xi}^3 K_3} \\ &= \begin{pmatrix} \hat{\alpha}_+ & \hat{\beta}_- & \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 \\ \hat{\beta}_+ & \hat{\alpha}_- & \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 \\ \hat{x}^0 & -\hat{x}^0 & \hat{\Lambda}^0_0 & \hat{\Lambda}^0_1 & \hat{\Lambda}^0_2 & \hat{\Lambda}^0_3 \\ \hat{x}^1 & -\hat{x}^1 & \hat{\Lambda}^1_0 & \hat{\Lambda}^1_1 & \hat{\Lambda}^1_2 & \hat{\Lambda}^1_3 \\ \hat{x}^2 & -\hat{x}^2 & \hat{\Lambda}^2_0 & \hat{\Lambda}^2_1 & \hat{\Lambda}^2_2 & \hat{\Lambda}^2_3 \\ \hat{x}^3 & -\hat{x}^3 & \hat{\Lambda}^3_0 & \hat{\Lambda}^3_1 & \hat{\Lambda}^3_2 & \hat{\Lambda}^3_3 \end{pmatrix}, \end{aligned} \tag{19}$$

where the non-commutative entries are the quantum Minkowskian coordinates $\hat{x}^\mu$ and

$$\begin{aligned} \hat{\alpha}_\pm &= \cosh\hat{d} \pm \frac{1}{2}\mathrm{e}^{\hat{d}}\big(\hat{x}_\mu\hat{x}^\mu + \tau\hat{x}^0\big), \\ \hat{\gamma}_\nu &= \mathrm{e}^{\hat{d}}\hat{x}_\mu\hat{\Lambda}^\mu_\nu, \end{aligned}$$

$$\hat\beta_\pm = \sinh\hat d \pm \frac{1}{2}e^{\hat d}\big(\hat x_\mu \hat x^\mu + \tau \hat x^0\big),$$
$$\hat\Lambda^\mu_\nu = \hat\Lambda^\mu_\nu\big(\hat\theta^i, \hat\xi^i\big),$$
$$\hat\Lambda^\mu_\nu \hat\Lambda^\rho_\sigma g^{\nu\sigma} = g^{\mu\rho}, \qquad \hat x_\mu = g_{\mu\nu}\hat x^\nu,$$
$$\big(g^{\mu\rho}\big) = \mathrm{diag}(-1,1,1,1). \tag{20}$$

Note that quantum rotation and boost coordinates are jointly expressed through the formal Lorentz entries $\hat\Lambda^\mu_\nu$.

The representation (18) gives rise to a quantum R-matrix (17) with dimension 36×36. Since P_0^3 vanishes, R reads

$$R = \left(\mathbf{1}\otimes\mathbf{1} + \tau P_0 \otimes D + \frac{1}{2}\tau^2 P_0^2 \otimes D^2\right) \times \left(\mathbf{1}\otimes\mathbf{1} - \tau D \otimes P_0 + \frac{1}{2}\tau^2 D^2 \otimes P_0^2\right), \tag{21}$$

where $\mathbf{1}$ is the 6×6 unit matrix. Next in pursuing the FRT program we impose that

$$RT_1T_2 = T_2T_1R, \tag{22}$$

where $T_1 = T \otimes \mathbf{1}$ and $T_2 = \mathbf{1} \otimes T$. This matrix equation provides the commutation rules among all the entries in (19), which by taking into account (20) can then be reduced to

$$\big[\hat d, \hat\Lambda^\mu_\nu\big] = 0, \qquad \big[\hat x^\alpha, \hat\Lambda^\mu_\nu\big] = 0, \qquad \big[\hat\Lambda^\alpha_\beta, \hat\Lambda^\mu_\nu\big] = 0,$$
$$\big[\hat d, \hat x^\mu\big] = \tau\big(\delta^\mu_0 e^{-\hat d} - \hat\Lambda^\mu_0\big),$$
$$\big[\hat x^\mu, \hat x^\nu\big] = \tau\big(\hat\Lambda^\nu_0 \hat x^\mu - \hat\Lambda^\mu_0 \hat x^\nu\big), \tag{23}$$

where the quantum Lorentz entries $\hat\Lambda^\mu_0$ are given by

$$\hat\Lambda^0_0 = \cosh\hat\xi^1\cosh\hat\xi^2\cosh\hat\xi^3,$$
$$\hat\Lambda^1_0 = \sinh\hat\xi^1\cosh\hat\xi^2\cosh\hat\xi^3,$$
$$\hat\Lambda^2_0 = \sinh\hat\xi^2\cosh\hat\xi^3,$$
$$\hat\Lambda^3_0 = \sinh\hat\xi^3. \tag{24}$$

The coproduct for all the entries in T is just $\Delta(T) = T \dot\otimes T$. By using again (20) the coproduct for $\{\hat d, \hat x^\mu, \hat\Lambda^\mu_\nu\}$ can consistently be found:

$$\Delta(\hat d) = \hat d \otimes 1 + 1 \otimes \hat d,$$
$$\Delta\big(\hat x^\mu\big) = \hat x^\mu \otimes e^{-\hat d} + \hat\Lambda^\mu_\eta \otimes \hat x^\eta,$$
$$\Delta\big(\hat\Lambda^\mu_\nu\big) = \hat\Lambda^\mu_\eta \otimes \hat\Lambda^\eta_\nu, \tag{25}$$

which is a homomorphism of (23). Thus the expressions (23)–(25) together with the counit $\epsilon(T) = \mathbf{1}$ and antipode $S(T) = T^{-1}$ determine the Hopf algebra structure of the quantum Weyl–Poincaré group dual to $U_\tau(\mathcal{WP})$, which is the restriction $\hat c^\mu \equiv 0$ of that dual to $U_\tau(\mathcal{CM}^{3+1})$.

We remark that the commutation relations (9) (with $\hat c^\mu \equiv 0$) can be recovered from (23) by only taking the first-order in all the quantum coordinates (notice that in this case, $\hat\Lambda^0_0 \to 1$ and $\hat\Lambda^i_0 \to \hat\xi^i$).

Recall that the FRT approach was also used in the construction of the null-plane quantum Poincaré group [5]. However, as κ-Poincaré has no universal $\mathcal{R}$-matrix, the associated quantum group was obtained [15–18] through a direct quantization of the semiclassical Poisson–Lie algebra coming from the κ-Poincaré classical r-matrix.

4. A new non-commutative Minkowskian spacetime

Now we focus our attention on some structural physical consequences of the new non-commutative spacetime that comes out from (23)

$$\big[\hat x^\mu, \hat x^\nu\big] = \tau\big(\hat\Lambda^\nu_0(\hat\xi)\hat x^\mu - \hat\Lambda^\mu_0(\hat\xi)\hat x^\nu\big), \tag{26}$$

which can be seen as a generalization of (1) through $a^\mu \to \hat\Lambda^\mu_0(\hat\xi)$.

Firstly, we stress that since $\hat\Lambda^\mu_0$ (24) only depend on the quantum boost parameters and the quantum rotation coordinates $\hat\theta^i$ do not play any role in the spacetime non-commutativity, the isotropy of the space is thus preserved. Furthermore, by taking into account the commutation rules (23), $\hat\Lambda^\mu_0$ can be considered to play the role of the structure constants within the quantum space (26). In fact, the quantum boost coordinates $\hat\xi^i$ can be regarded as scalars (usual commutative parameters) within the quantum Weyl–Poincaré group. From this viewpoint, relations (26) would define a Lie algebraic non-commutative spacetime of the type (1). However, this situation changes in the full quantum conformal group since, as shown in (9), $\hat\xi^i$ and $\hat\Lambda^\mu_0$ no longer commute with the dilation parameter $\hat d$.

Secondly, relations (26) show that different observers in relative motion with respect to quantum group transformations have a different perception of the spacetime non-commutativity, i.e., Lorentz invariance is lost. Nevertheless, we remark that, in this context, covariance under quantum group transformations is ensured by construction. Explicitly, in the commutative case the T-matrix (19) is just a matrix representation of the transformation group of the spacetime, and the coproduct (25) represents the multiplication of two different elements of the group. A similar interpretation holds in the quantum group case, for which (25) provides the transformation law for the non-commutative spacetime coordinates that can be rewritten as

$$\hat{x}''^{\mu} = \hat{x}^{\mu} \mathrm{e}^{-\hat{d}'} + \hat{\Lambda}^{\mu}_{\eta} \hat{x}'^{\eta}, \tag{27}$$

where the tensor product notation has been replaced by two different copies of the non-commutative coordinates ($\hat{x} \otimes 1 \equiv \hat{x}$, $1 \otimes \hat{x} \equiv \hat{x}'$). The fact that the spacetime (26) is quantum group covariant is a direct consequence of the Hopf algebra structure which implies that

$$\left[\hat{x}''^{\mu}, \hat{x}''^{\nu}\right] = \tau\left(\hat{\Lambda}''^{\nu}_{0}(\hat{\xi}'')\hat{x}''^{\mu} - \hat{\Lambda}''^{\mu}_{0}(\hat{\xi}'')\hat{x}''^{\mu}\right), \tag{28}$$

where the new Lorentz entries are also given by (25):

$$\hat{\Lambda}''^{\mu}_{\nu}(\hat{\xi}'') = \hat{\Lambda}^{\mu}_{\eta}(\hat{\xi})\hat{\Lambda}'^{\eta}_{\nu}(\hat{\xi}'). \tag{29}$$

Therefore, if we assume that two "observers" are actually related through a quantum group transformation (27), they will be "affected" by different structure constants for the spacetime commutation rule (26), yet the latter is manifestly quantum group covariant.

In order to illustrate these results, let us consider the $(1+1)$D case where the Lorentz sector is expressed in terms of a single quantum boost parameter $\hat{\xi} \equiv \hat{\xi}^{1}$ as

$$\left(\hat{\Lambda}^{\mu}_{\nu}\right) = \begin{pmatrix} \cosh\hat{\xi} & \sinh\hat{\xi} \\ \sinh\hat{\xi} & \cosh\hat{\xi} \end{pmatrix}. \tag{30}$$

The transformation (29) shows directly the additivity of the quantum boost parameter (along the same direction):

$$\begin{aligned} \hat{\Lambda}^{0}_{0}(\hat{\xi}'') &= \cosh\hat{\xi}'' = \Delta\left(\hat{\Lambda}^{0}_{0}\right) \\ &= \cosh\hat{\xi} \otimes \cosh\hat{\xi} + \sinh\hat{\xi} \otimes \sinh\hat{\xi} \\ &= \cosh(\hat{\xi} + \hat{\xi}'), \end{aligned} \tag{31}$$

and similarly for the remaining $\hat{\Lambda}^{\mu}_{\nu}$. The non-commutative $(1+1)$D spacetime reads

$$\left[\hat{x}^{1}, \hat{x}^{0}\right] = \tau\left(\hat{x}^{1}\cosh\hat{\xi} - \hat{x}^{0}\sinh\hat{\xi}\right), \tag{32}$$

which implies the following uncertainty relation:

$$\delta\hat{x}^{1}\delta\hat{x}^{0} \geqslant \frac{|\tau|}{2}\left|\langle\hat{x}^{1}\rangle\cosh\hat{\xi} - \langle\hat{x}^{0}\rangle\sinh\hat{\xi}\right|, \tag{33}$$

where δ denotes the root-mean-square deviation and $\langle\hat{x}^{1}\rangle$, $\langle\hat{x}^{0}\rangle$ are the expectation values of the space and time operators.

To end with, we also stress that if the following new space variables $\hat{X}^{i}$ in the $(3+1)$D spacetime (26) are considered

$$\hat{x}^{0} \to \hat{x}^{0}, \qquad \hat{x}^{i} \to \hat{X}^{i} = \hat{x}^{i}\hat{\Lambda}^{0}_{0} - \hat{x}^{0}\hat{\Lambda}^{i}_{0}, \tag{34}$$

the transformed commutation rules for the quantum spacetime are given by

$$\left[\hat{X}^{i}, \hat{x}^{0}\right] = \tau\hat{\Lambda}^{0}_{0}(\hat{\xi})\hat{X}^{i}, \qquad \left[\hat{X}^{i}, \hat{X}^{j}\right] = 0, \tag{35}$$

which, in turn, can be interpreted as a generalization of the κ-Minkowski space (10) with a "variable" Planck length $\tau' = \tau\hat{\Lambda}^{0}_{0}(\hat{\xi})$ that does depend on *all* the quantum boost parameters (in the $(1+1)$D case, this yields $\tau' = \tau\cosh\hat{\xi}$). This result is a direct consequence of imposing a larger quantum group symmetry than Poincaré. Moreover, if the quantum conformal transformations and parameters are taken into account and the corresponding quantum group is constructed, then $\hat{\Lambda}^{0}_{0}$ becomes a non-central operator in such a manner that (35) (and also (26)) defines a quadratic non-commutative spacetime. This suggests that a further study of non-Lie spacetime algebras derived from conformal or AdS quantum symmetries could be meaningful.

Acknowledgements

This work was partially supported by MCyT and JCyL, Spain (Projects BFM2000-1055 and BU04/03), and by INFN-CICyT (Italy–Spain).

References

[1] J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, Phys. Lett. B 264 (1991) 331.
[2] S. Giller, P. Kosinski, J. Kunz, M. Majewski, P. Maslanka, Phys. Lett. B 286 (1992) 57.
[3] J. Lukierski, H. Ruegg, A. Nowicky, Phys. Lett. B 293 (1992) 344.
[4] A. Ballesteros, F.J. Herranz, M.A. del Olmo, M. Santander, Phys. Lett. B 351 (1995) 137.
[5] A. Ballesteros, F.J. Herranz, C.M. Pereña, Phys. Lett. B 391 (1997) 71.
[6] S. Majid, Class. Quantum Grav. 5 (1988) 1587.

[7] G. Amelino-Camelia, Phys. Lett. B 510 (2001) 255.
[8] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35.
[9] J. Magueijo, L. Smolin, Phys. Rev. Lett. 88 (2002) 190403.
[10] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 1643.
[11] J. Kowalski-Glikman, S. Nowak, Phys. Lett. B 539 (2002) 126.
[12] N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B 522 (2001) 133.
[13] N.R. Bruno, Phys. Lett. B 547 (2002) 109.
[14] J. Lukierski, A. Nowicki, Int. J. Mod. Phys. A 18 (2003) 7.
[15] P. Maslanka, J. Phys. A 26 (1993) L1251.
[16] S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348.
[17] S. Zakrzewski, J. Phys. A 27 (1994) 2075.
[18] J. Lukierski, H. Ruegg, Phys. Lett. B 329 (1994) 189.
[19] G. Amelino-Camelia, M. Arzano, L. Doplicher, hep-th/0205047.
[20] J. Kowalski-Glikman, S. Nowak, Int. J. Mod. Phys. D 12 (2003) 299.
[21] J. Lukierski, V.D. Lyakhovsky, M. Mozrzymas, Phys. Lett. B 538 (2002) 375.
[22] F.J. Herranz, Phys. Lett. B 543 (2002) 89.
[23] A. Ballesteros, N.R. Bruno, F.J. Herranz, hep-th/0305033, J. Phys. A 36 (2003), in press.
[24] S. Majid, Foundations of Quantum Group Theory, Cambridge Univ. Press, Cambridge, 1995.
[25] V. Chari, A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994.
[26] A. Ballesteros, N.A. Gromov, F.J. Herranz, M.A. del Olmo, M. Santander, J. Math. Phys. 36 (1995) 5916.
[27] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.
[28] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253.
[29] K.-H. Rehren, A proof of the AdS-CFT correspondence, in: H.-D. Doebner, et al. (Eds.), Quantum Theory and Symmetries, World Scientific, Singapore, 2000, p. 278.
[30] J. Lukierski, V.D. Lyakhovsky, M. Mozrzymas, Mod. Phys. Lett. A 18 (2003) 753.
[31] V.G. Drinfeld, Sov. Math. Dokl. 27 (1983) 68.
[32] A. Ballesteros, F.J. Herranz, M.A. del Olmo, C.M. Pereña, M. Santander, J. Phys. A 28 (1995) 7113.
[33] L.D. Faddeev, N.Y. Reshetikhin, L.A. Takhtajan, Lengingrad Math. J. 1 (1990) 193.
[34] Z. Chang, Eur. Phys. J. C 17 (2000) 527.