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Unsteady sedimentation analysis of spherical particles in Newtonian fluid media using analytical methods

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Abstract Unsteady settling behavior of solid spherical particles falling in water as a Newtonian fluid is investigated using a drag coefficient of the form given by Ferreira et al. Differential transformation method (DTM), Galerkin method (GM), collocation method (CM), and numerical methods are applied to analyze the characteristics of particles motion. The influence of physical parameters on terminal velocity is discussed and moreover, comparing the techniques, it is showed that GM and CM are very efficient for solving the governing equation and DTM with Padé approximation has the best agreement with numerical results. The novelty of this work is introducing three simple and exact analytical method for solving the nonlinear equation of sedimentation and applied it in many industrial and chemical applications.

1. Introduction

Description of the motion of immersed bodies in fluids has long been a subject of great interest due to its wide range of applications in industry e.g. sediment transport, deposition in pipelines, alluvial channels, etc [1–2]. The settling mechanism of solid particle, bubble, or drop, both in Newtonian and non-Newtonian fluids, is reported by Clift et al. [3] and Chhabra [4] and several types of drag coefficients for spherical and non-spherical particles were

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Nomenclature

a, b, c, d	constants
Acc	acceleration (unit: m/s^2)
CM	collocation method
C_D	drag coefficient
D	particle diameter (unit: m)
DTM	differential transformation method
$f(\eta)$	analytic function
g	acceleration due to gravity (unit: m/s^2)
GM	Galerkin method
H	constant value
m	particle mass (unit: kg)
n	number of iteration
Num	numerical method
P	Padé approximation
$R(x)$	residual function
t	time (unit: s)
u	velocity (unit: m/s)
$\tilde{u}$	trial function
W_i	weight function
$x(t)$	analytic function
$X(k)$	DTM transformed function
$[L,M]$	order of accuracy in Padé

Greek letters

μ	dynamic viscosity (unit: kg/(m · s))
ρ	fluid density (unit: kg/m^3)
ρ_s	spherical particle density (unit: kg/m^3)

presented by Haider and Levenspiel [5]. Guo [6] and Mohazzabi [7] have studied the behavior of spheres and objects falling into fluids. It is well known that a particle falling vertically in a stationary fluid under the influence of gravity accelerates until the gravitational body force is balanced by the resistance forces, including buoyancy and drag forces. At the equilibrium, particle reaches a constant velocity so-called “terminal velocity” or “settling velocity”. The Knowledge of the terminal velocity of solids and particles falling in liquids is required in many industrial applications such as mineral processing, solid-liquid mixing, hydraulic transport, slurry systems, abrasive water jets, fluidized bed reactors and so on [3]. Reviewing the technical literature, it is clear that most of the pervious investigations are performed for steady-state conditions (at terminal velocity) and few of them has been studied the unsteady motion of falling objects (accelerating motion).

Recently, several attempts have been made to develop analytical tools to solve the motion equation of falling objects in fluids. Ganji [8] employed Variational Iteration Method (VIM) and derived a semi-exact solution for the instantaneous velocity of the particle over time, setting in incompressible fluid. Yaghoobi and Torabi [9] investigated the acceleration motion of a vertically falling non-spherical particle in incompressible Newtonian media by VIM. They considered water, glycerin and ethylene-glycol as liquid phase also they combined VIM with Padé approximation for increasing the solution accuracy of the non-spherical particle equation of motion [10]. More applications of VIM for solving the nonlinear differential equations are presented by Mohyud-Din et al. [11]. Jalaal et al. [12] used Homotopy Analysis Method (HAM) and obtained the solution of the one-dimensional non-linear particle equation. They demonstrated that using appropriate initial guess and auxiliary parameter, HAM is an accurate and reliable method.

Jalaal et al. [13] applied He’s homotopy perturbation method (HPM) [14,15] to solve the acceleration motion of a vertically falling spherical particle in incompressible Newtonian media. Mohyud-Din et al. [16] used HPM for solving a wide class of nonlinear problems and they suggest that this method is capable to cope with the versatility of the physical problems such as sedimentation process, the system of linear partial differential equations for waves [17] and MHD flow over nonlinear stretching sheeting [18]. Torabi and Yaghoobi [19] combined HPM with Padé approximation for increasing the solution accuracy of the particle equation of motion. The motion of a spherical particle rolling down an inclined plane submerged in a Newtonian environment has been studied by Jalaal et al. [20,21] through HPM.

The main concept of differential transformation method (DTM) was first introduced by Zhou [22] in 1986. This method can be applied directly for linear and nonlinear differential equation without requiring linearization, discretization, or perturbation and this can be the main advantage of this method. Borhanifar et al. [23] employed DTM on some PDEs and their coupled versions. Abdel-Halim Hassan [24] has applied the DTM for different systems of differential equations. Yildirim et al. [25] used DTM for solving the partial and fractional differential Fornberg-Whitham equation and they observed that this method is an effective method for strongly nonlinear partial equations. Rashidi et al. [26] applied Padé approximation to increase the convergence of DTM series in a mixed convection problem an inclined flat plate embedded in a porous medium.

Collocation and Galerkin are two kinds of the methods of weighted residuals (MWR) which are approximation techniques for solving differential equations. Stern and Rasmussen [27] used collocation method and trigonometric series to approximate the solution and coefficients of a third order linear differential equation. Hu and Li [28] and Herrera et al. [29] applied Collocation method for Poisson’s and advection-diffusion equations, respectively. Hendi and Albugami [30] solved Fredholm-Volterra integral equation using collocation and Galerkin methods. Legendre wavelet Galerkin method has been used for solving ordinary differential equations with non analytic solution by Mohammadi et al. [31].

In this paper, first DTM and DTM-Padé approximations are used to reproduce the results of Torabi and Yaghoobi

[9,10]. Moreover, Collocation and Galerkin analytical methods and a numerical technique are applied to solve the equation of motion of a spherical particle settling equation. The presented analytical methods are simpler and faster than numerical one.

2. Problem description

For modeling the particle sediment phenomenon, consider a small, rigid particle with a spherical shape of diameter D and mass of m and density of ρ_s, falling in infinite extent filled water as an incompressible Newtonian fluid. Density of water, ρ, and its viscosity, μ, are known. We considered the gravity, buoyancy, Drag forces and added mass (virtual mass) effect on particle. According to the Basset-Boussinesq-Ossen (BBO) equation for the unsteady motion of the particle in a fluid, for a dense particle falling in light fluids and by assuming $\rho \ll \rho_s$ Basset history force is negligible. So by rewriting force balance for the particle, the equation of motion is gained as follows [19],

$$m\frac{\mathrm{d}u}{\mathrm{d}t} = mg(1-\frac{\rho}{\rho_s}) - \frac{1}{8}\pi D^2\rho C_D u^2 - \frac{1}{12}\pi D^3\rho\frac{\mathrm{d}u}{\mathrm{d}t} \quad (1)$$

where C_D is the drag coefficient, in the right hand side of the Eq. (1), the first term represents the buoyancy affect, the second term corresponds to drag resistance, and the last term is due to the added mass effect which is due to acceleration of fluid around the particle. The main difficulty of solving Eq. (1) is non-linear terms due to the non-linearity nature of the drag coefficient C_D. Ferreira et al. [32], in their analytical study, suggested a correlation for C_D of spherical particles which has good agreement with the experimental data in a wide range of Reynolds number, $0 \le Re \le 10^5$. This appropriate Equation is

$$C_D = \frac{24}{Re}\left(1+\frac{1}{48}Re\right) \quad (2)$$

Jalaal et al. [13] have shown that Eq. (2) represents a more accurate resistance of the particle in comparison with the pervious equations presented by others. Substituting Eq. (2) in to Eq. (1) and mass of the spherical particle

$$m = \frac{1}{6}\pi D^3\rho_s \quad (3)$$

Eq. (1) can be rewritten as

$$a\frac{\mathrm{d}u}{\mathrm{d}t} + bu + cu^2 - d = 0,\ u(0) = 0 \quad (4)$$

Where

$$a = \frac{1}{12}\pi D^3(2\rho_s + \rho) \quad (5)$$

$$b = 3\pi D\mu \quad (6)$$

$$c = \frac{1}{16}\pi D^2\rho \quad (7)$$

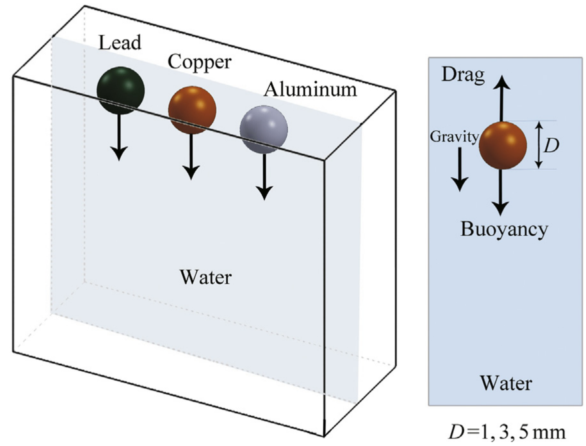

Figure 1 Schematic view of particles settling in water. It is assumed particles are falling freely such that no particle-particle or particle-wall interactions exists.

Table 1 Properties of the selected materials.

Material	Density/(kg/m^3)	Viscosity/(kg/(m · s))
Aluminum	2702	–
Copper	8940	–
Lead	11340	–
Water	996.51	0.001

$$d = \frac{1}{6}\pi D^3 g(\rho_s - \rho) \quad (8)$$

Eq. (4) is a non-linear equation with an initial condition and it can be solved by numerical and analytical methods. In the present study, we choose three different materials for solid particle, Aluminum, Copper and Lead and considered three different diameters (1, 3, and 5 mm) for them. A schematic of described problem is shown in Figure 1. Physical properties of the selected material are shown in Table 1 and the resulted coefficients a, b, c, and d from Eqs. (5)-(8) are listed in Table 2 and Eq. (4) as a non-linear equation, is solved by Numerical method, DTM-Padé approximation, collocation method and Galerkin method. It is necessary to inform that professional version of this problem with more complexity can be solved by CFD methods which is available in the literatures [33–36].

3. Principles of methods and their application in problem

3.1. *Differential transformation method*

The priciple of DTM for the nonlinear differential equations is previously described in details by Torabi and Yaghoobi [9]. Thus, we only present a short summary of needed transformations and the application of this method

Table 2 Coefficient in Eq. (4).

Liquid	Solid (Particle)	Diameter/mm	*a*	*b*	*c*	*d*
Water	**Aluminum**	1	0.0000016756496	0.00000942477796	0.000195664281	0.000008760256
		3	0.0000452425392	0.00002827433389	0.001760978529	0.000236526917
		5	0.0002094562001	0.00004712388981	0.004891607024	0.001095032024
	Copper	1	0.0000049418587	0.00000942477796	0.000195664281	0.000040801768
		3	0.0001334301866	0.00002827433389	0.001760978529	0.001101647738
		5	0.0006177323456	0.00004712388981	0.004891607024	0.005100221010
	Lead	1	0.0000061984958	0.00000942477796	0.000195664281	0.000053129377
		3	0.0001673593872	0.00002827433389	0.001760978529	0.001434493196
		5	0.0007748119783	0.00004712388981	0.004891607024	0.006641172207

Table 3 Fundamental operations of the differential transform method.

Origin function	Transformed function
$x(t)=\alpha f(x)\pm\beta g(t)$	$X(k)=\alpha F(k)\pm\beta G(k)$
$x(t)=\frac{d^m f(t)}{dt^m}$	$X(k)=\frac{(k+m)!F(k+m)}{k!}$
$x(t)=f(t)g(t)$	$X(k)=\sum_{l=0}^{k}F(l)G(k-l)$
$x(t)=t^m$	$X(k)=\delta(k-m)=\begin{cases}1, & \text{if } k=m,\\ 0, & \text{if } k\neq m.\end{cases}$
$x(t)=\exp(t)$	$X(k)=\frac{1}{k!}$
$x(t)=\sin(\omega t+\alpha)$	$X(k)=\frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2}+\alpha\right)$
$x(t)=\cos(\omega t+\alpha)$	$X(k)=\frac{\omega^k}{k!}\cos\left(\frac{k\pi}{2}+\alpha\right)$

for the current problem. For understanding method's concept, suppose that $x(t)$ is an analytic function in domain D, and $t=t_i$ represents any point in the domain. The function $x(t)$ is then represented by one power series whose center is located at t_i. The Taylor series expansion function of $x(t)$ is in form of:

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i}\qquad\forall t\in D\tag{9}$$

The Maclaurin series of $x(t)$ can be obtained by taking $t_i=0$ in Eq. (9) expressed as:

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}\qquad\forall t\in D\tag{10}$$

As explained in [22] the differential transformation of the function $x(t)$ is defined as follows:

$$X(k)=\sum_{k=0}^{\infty}\frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}\tag{11}$$

where $X(k)$ represents the transformed function and $x(t)$ is the original function. The differential spectrum of $X(k)$ is confined within the interval $t\in[0,H]$, where H is a constant value. The differential inverse transform of $X(k)$ is defined as follows:

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k)\tag{12}$$

From Eq. (12), it can be carried out easily that the theory of differential transformation is based upon the Taylor series expansion. The values of function $X(k)$ at values of argument k are referred to as discrete, i.e. $X(0)$ is known as the zero discrete, $X(1)$ as the first discrete, etc. The more discrete available, the more precise it is possible to restore the unknown function. The function $x(t)$ consists of the T-function $X(k)$, and its value is given by the sum of the T-function with $(t/H)k$ as its coefficient. In real applications, at the right choice of constant H, the larger values of argument k the discrete of spectrum reduce rapidly. The function $x(t)$ is expressed by a finite series and Eq. (12) can be written as:

$$x(t)=\sum_{k=0}^{n}\left(\frac{t}{H}\right)^k X(k)\tag{13}$$

Some important mathematical operations performed by differential transform method are listed in Table 3.

3.1.1. Padé approximation

There are some techniques to increase the convergence of series. Among them, the Padé technique is widely applied. Suppose that a function $f(\eta)$ is represented by a power series so that

$$f(\eta)=\sum_{i=0}^{\infty}c_i\eta^i\tag{14}$$

This expansion is the fundamental starting point of any analysis using Padé approximants. The notation c_i, $i=0, 1, 2, \ldots$ is reserved for the given set of coefficients and $f(\eta)$ is the associated function. $[L,M]$ Padé approximant is a rational fraction

$$P[L,M]=\frac{a_0+a_1\eta+\ldots+a_L\eta^L}{b_0+b_1\eta+\ldots+b_M\eta^M}\tag{15}$$

This has a Maclaurin expansion which agrees with Eq. (14) as far as possible.

3.1.2. Application in problem

Applying the DTM transformation from Table 3 into Eq. (4) to find $u(t)$ we have:

$$a(k+1)U(k+1)+bU(k)+c\sum_{l=0}^{k}U(l)U(k-l)-d\delta(k)=0,\ U(0)=0\tag{16}$$

Rearranging Eq. (16), we have

$$U(k+1) = \frac{d\delta(k) - c\sum_{l=0}^{k}[U(l)U(k-l)] - bU(k)}{a(k+1)}, \quad U(0)=0 \tag{17}$$

where

$$\delta(k) = \begin{cases} 1 & \text{if} \quad k=0 \\ 0 & \text{if} \quad k\neq 0 \end{cases} \tag{18}$$

By solving Eq. (17) and using the initial condition, the DTM terms are obtained as

$$\begin{aligned} U(1) &= \frac{d}{a} \\ U(2) &= -\frac{1}{2}\frac{bd}{a^2} \\ U(3) &= \frac{1}{6}\frac{d(b^2-2cd)}{a^3} \\ U(4) &= -\frac{1}{24}\frac{bd(b^2-8cd)}{a^4} \end{aligned} \tag{19}$$

Now representing Eq. (19) in series form [19], $u(t)$ function will be obtained. After five iterations in DTM series for $a=b=c=d=1$, the $u(t)$ function is obtained as

$$u(t) = t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{7}{24}t^4 - \frac{1}{24}t^5 \tag{20}$$

Subsequently, applying Padé approximation to Eq. (20) (for Padé [4,4] accuracy), we have,

$$\text{Padé}\,[4,4](u(t)) = \frac{\frac{3441}{13480}t^4 - \frac{224}{1011}t^3 - \frac{227}{674}t^2 + t}{1 + \frac{55}{337}t + \frac{9}{337}t^2 + \frac{7}{1685}t^3 + \frac{1}{1685}t^4} \tag{21}$$

3.2. Galerkin method

Suppose a differential operator D, is applied on a function u to produce a function p.

$$D(u(x)) = p(x) \tag{22}$$

u is approximated by a function $\tilde{u}$, which is a linear combination of basic functions chosen from a linearly independent set. That is,

$$u \cong \tilde{u} = \sum_{i=1}^{n} c_i\varphi_i \tag{23}$$

Now, when substituted into the differential operator, D, the result of the operations is not, in general, $p(x)$. Hence an error or residual will exist as,

$$E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0 \tag{24}$$

The main idea of the Galerkin and collocation method is to force the residual to zero in some average sense over the domain. That is

$$\int_X R(x)W_i(x) = 0, \quad i = 1,2,\ldots,n \tag{25}$$

where the number of weight functions W_i is exactly equal the number of unknown constants c_i in $\tilde{u}$ function. The result is a set of n algebraic equations for the unknown constants c_i. In Galerkin method weight functions are:

$$W_i = \frac{\partial\tilde{u}}{\partial c_i}, \quad i = 1,2,\ldots,n \tag{26}$$

3.2.1. Application in problem

Now GM is applied to the Eq. (4) for solving the particle settling equation when $a=b=c=d=1$. First consider the trial function as

$$\tilde{u}(t) = c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 \tag{27}$$

which satisfies the initial condition in Eq. (4). Using Eq. (26), weight functions will be obtained as:

$$W_1 = t,\ W_2 = t^2,\ W_3 = t^3,\ W_4 = t^4,\ W_5 = t^5,\ W_6 = t^6 \tag{28}$$

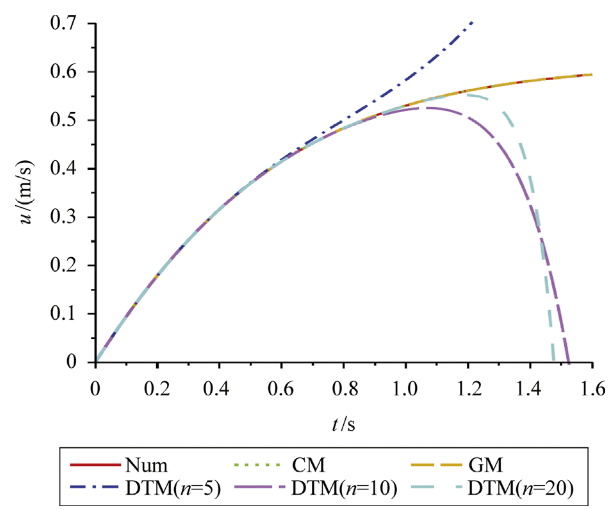

Figure 2 Comparison between numerical, collocation, Galerkin methods and DTM (in three different iterations) for Eq. (4) and $a=b=c=d=1$.

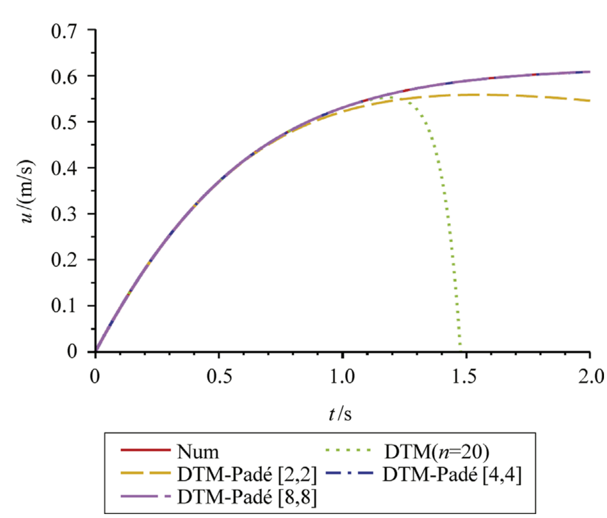

Figure 3 Convergence of the DTM-Padé regarding to the method order, [L,M], for Eq. (4) and $a=b=c=d=1$.

Table 4 Obtained values for u(m/s) from different methods for Eq. (4) and $a=b=c=d=1$.

Time/s	Numerical method	Collocation method	Galerkin method	DTM ($n=20$)	DTM-Padé [2,2]	DTM-Padé [4,4]	DTM-Padé [8,8]
0	0	0	0	0	0	0	0
0.2	0.17911335972	0.17916766	0.17911334	0.1791044	0.1791133	0.1791133	0.179113346
0.4	0.31600713125	0.31602345	0.31600709	0.3157894	0.3160070	0.3160070	0.316007092
0.6	0.41502836932	0.41506767	0.41502829	0.4137931	0.4150281	0.4150281	0.415028297
0.8	0.48383755960	0.48385852	0.48383508	0.4799999	0.4838364	0.4838364	0.483837459
1.0	0.53032985719	0.53032540	0.53010503	0.5217391	0.5303244	0.5303244	0.530329756
1.2	0.56115087915	0.56116612	0.55212889	0.5454545	0.5611313	0.5611313	0.561150740
1.4	0.58132582197	0.58132775	0.38037826	0.5562913	0.5812713	0.5812713	0.581325694
1.6	0.59442282455	0.59457135	–	0.5581395	0.5942956	0.5942956	0.594422665
1.8	0.60287916150	0.60512051	–	0.5538461	0.6026196	0.6026196	0.602879031
2	0.60832017092	0.62039362	–	0.5454545	0.6078431	0.6078431	0.608320058

Table 5 Calculated errors (%) for various methods solving Eq. (4) and $a=b=c=d=1$.

Time/s	Collocation method	Galerkin method	DTM($n=20$)	DTM-Padé [2,2]	DTM-Padé [4,4]	DTM-Padé [8,8]
0	0	0	0	0	0	0
0.2	0.000303	0.000303	7.27454E-08	4.95894E-05	7.33037E-08	7.44203E-08
0.4	0.000274	5.17E-05	1.24524E-07	0.000688774	1.36866E-07	1.23258E-07
0.6	0.000155	9.47E-05	1.88481E-07	0.002976341	4.32802E-07	1.73302E-07
0.8	8.6E-05	4.33E-05	5.1164E-06	0.007931504	2.31877E-06	2.06488E-07
1.0	5.75E-05	8.39E-06	0.000423933	0.016198837	1.02894E-05	1.89122E-07
1.2	2.36E-05	2.72E-05	0.016077644	0.027971681	3.4736E-05	2.46742E-07
1.4	0.000102	3.32E-06	0.345671144	0.043064372	9.36325E-05	2.18938E-07
1.6	0.002018	0.00025	4.902019398	0.06103953	0.000213981	2.66898E-07
1.8	0.012315	0.003718	-	0.081331402	0.000430478	2.14978E-07
2	0.047649	0.019847	-	0.10334299	0.000784182	1.85468E-07

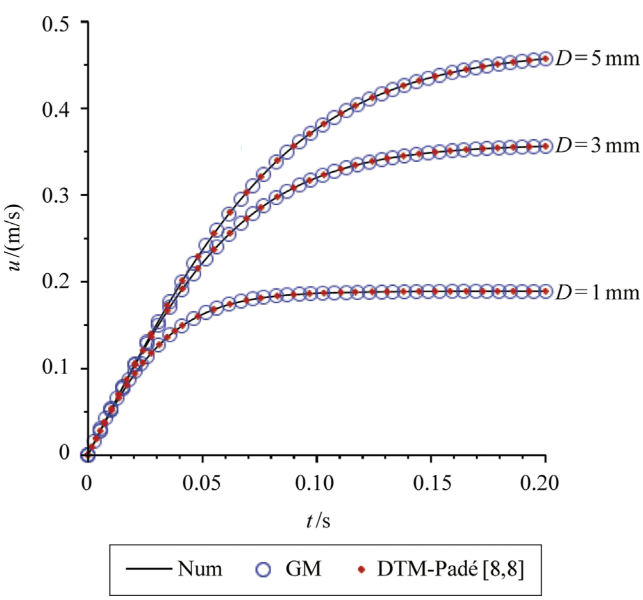

Figure 4 Velocity variation for different particle diameters (Aluminum).

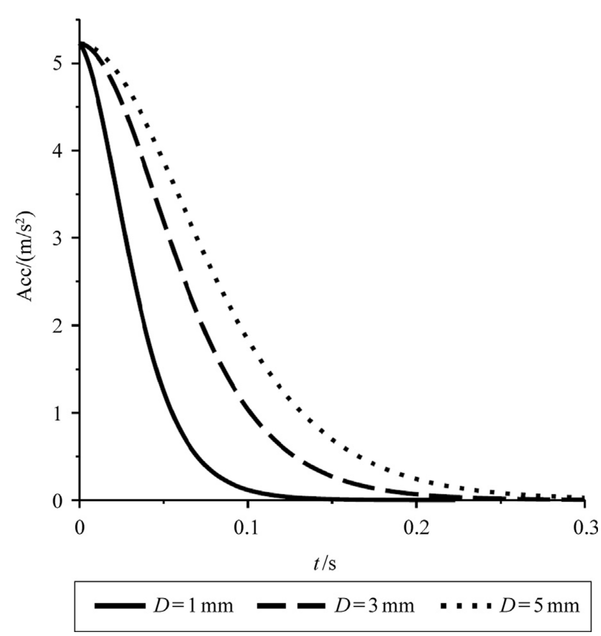

Figure 5 Acceleration variation for different particle diameters (Aluminum).

Applying Eq. (25), a set of algebraic equations is defined. Solving this set of equations, c_1 to c_6 coefficients will be calculated as follows:

$$\begin{aligned} c_1 &= 1.001706321,\ c_2 = -0.5118166769, \\ c_3 &= -0.1420897415,\ c_4 = 0.2995302815, \\ c_5 &= -0.1405245851, \\ c_6 &= 0.02351981005 \end{aligned} \tag{28}$$

Consequently the velocity equation for particle is:

$$u(t) = 1.001706321t - 0.5118166769t^2 - 0.1420897415t^3 + 0.2995302815t^4 - 0.1405245851t^5 + 0.02351981005t^6 \quad (29)$$

3.3. Collocation method

Governing equations in Collocation method are similar to the GM one (i.e. Eqs. (22) to (25)), however, the weighting functions are taken from the family of Dirac δ functions in the domain. That is, $W_i(x) = \delta(x - x_i)$. The Dirac δ function has the property that:

$$\delta(x - x_i) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{Otherwise} \end{cases} \quad (30)$$

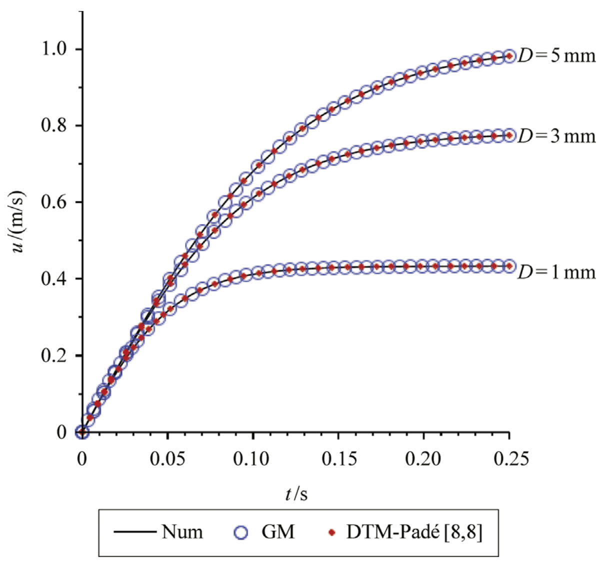

Figure 6 Velocity variation for different particle diametersb (Copper).

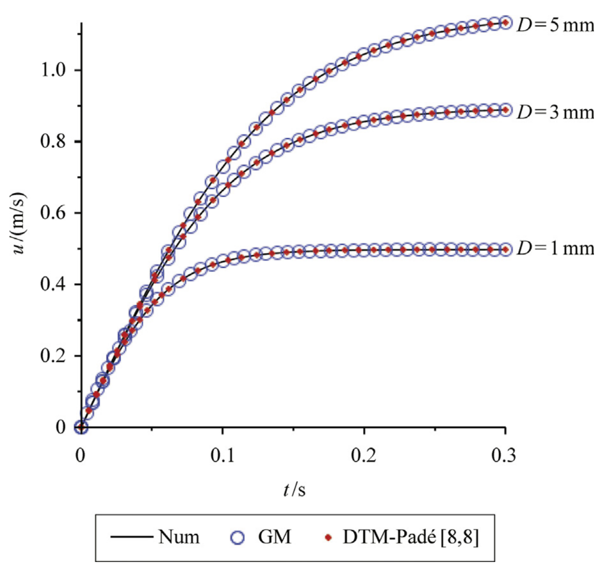

Figure 7 Velocity variation for different particle diameters (Lead).

And residual function in Eq. (24) must force to be zero at specific points.

3.3.1. Application in problem

Consider the pervious assumptions ($a = b = c = d = 1$) for solving the Eq. (4). With the trial function as Eq. (27) and Eq. (24), residual function, $R(c_1, c_2, c_3, c_4, c_5, c_6, t)$, is found. This function must be close to zero and for reaching this importance. For reaching this purpose, six specific points should be chosen in the domain $t \in [0, 1.6]$. These

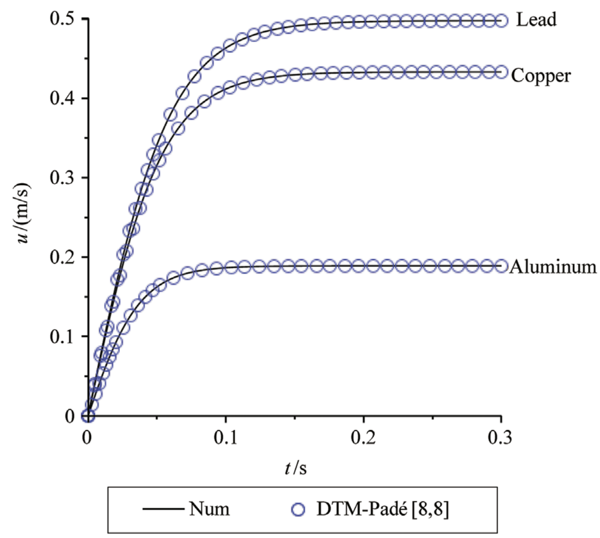

Figure 8 Comparison of velocity variation over time for different particle materials when $D = 1$ mm.

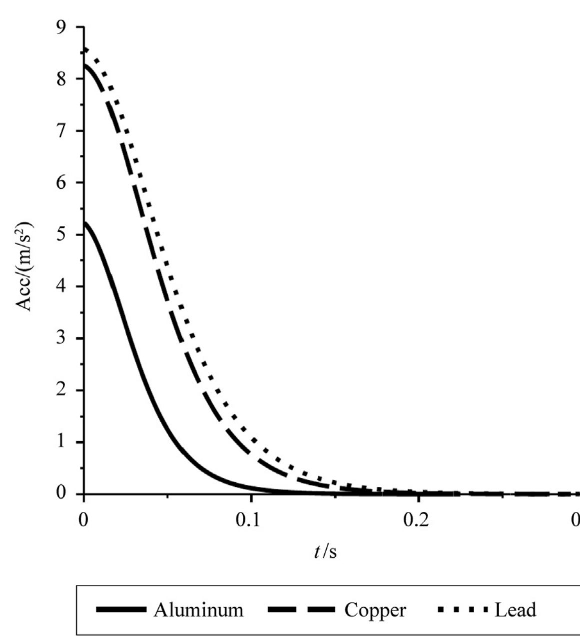

Figure 9 Comparison of acceleration variation for different particle materials ($D = 1$ mm).

points are,

$$t_1 = \frac{8}{35},\ t_2 = \frac{16}{35},\ t_3 = \frac{24}{35},\ t_4 = \frac{32}{35},\ t_5 = \frac{40}{35},\ t_6 = \frac{48}{35} \tag{31}$$

Finally by setting all of these points in to residual function, $R(c_1, c_2, c_3, c_4, c_5, c_6, t)$, a set of equations with six equations and six unknown coefficients will be obtained. Solving this set of equations and introducing these coefficients to trial function, the velocity equation for particle will be found as,

$$u(t) = 1.003795836t - 0.5187232698t^2 - 0.1347023045t^3 + 0.2986629632t^4 - 0.1433338394t^5 + 0.02468713536t^6 \tag{32}$$

4. Results and discussion

At first, a comparison between the three analytical methods and Runge-Kutta methods is provided to select the best and reliable analytical method for present problem. Eq. (4) is considered with all constants are equal to unity (i.e. $a=b=c=d=1$). Results of the solutions are presented in Eqs. (20), (29) and (32) and Runge-Kutta method and they are depicted in Figure 2. From Figure 2, it is revealed that GM and CM have a good agreement with Numerical result, but DTM when time tends to infinity, even in high iterations (20 iterates), cannot estimate a constant velocity as "terminal velocity" and its value suddenly reaches to zero. For eliminating this problem, Padé approximation in different orders such as [2,2], [4,4], [8,8] (e.g. Eq. (21)) is

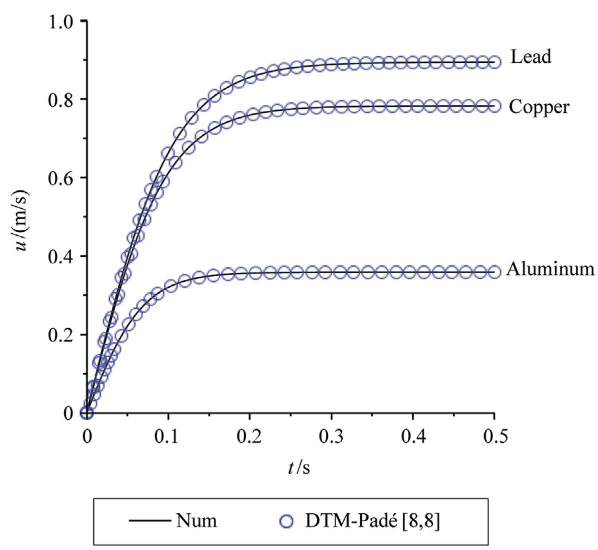

Figure 10 velocity variation for different particle materials when $D=3$ mm.

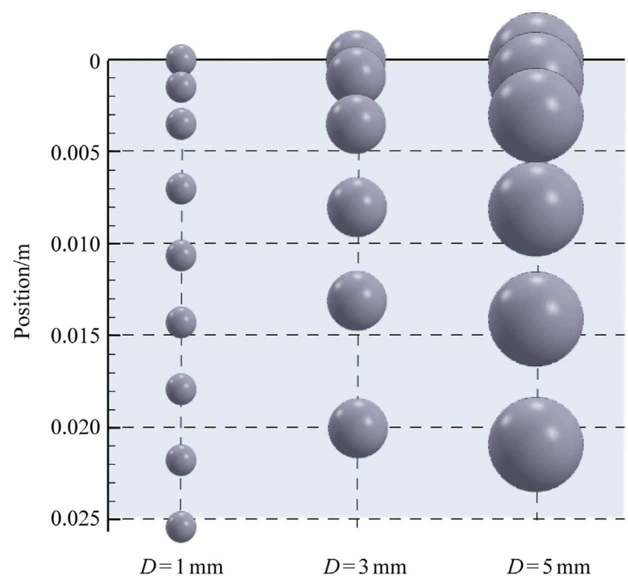

Figure 12 Positions of falling particles for different size of Aluminum particles, time interval = 0.02 s.

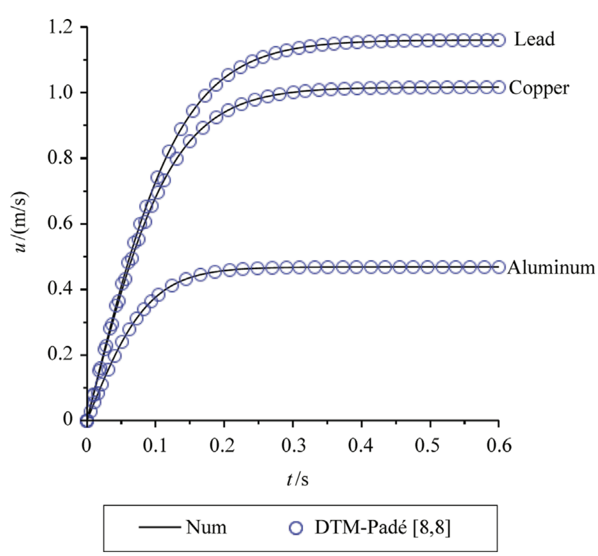

Figure 11 velocity variation for different particle materials when $D=5$ mm.

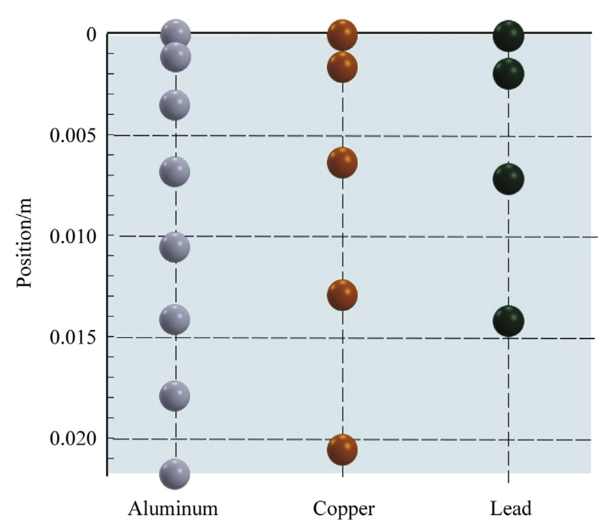

Figure 13 Positions of falling particle for different materials, $D=1$ mm and time interval = 0.02 s.

Table 6 Terminal velocity (m/s) for particles calculated by DTM-Padé and compared by HPM [13] and numerical method.

Particle material	Diameter/mm	Numerical method	DTM-Padé [8,8]	HPM
Aluminum	1	0.1888759074	0.1771953458	0.1888743212
	3	0.3585504311	0.3581483507	0.3585425621
	5	0.4683308257	0.4682662708	0.4683267958
Copper	1	0.4332008767	0.4336398188	0.4332014752
	3	0.7829089418	0.7828929076	0.7828921531
	5	1.0156731422	1.015681926	1.0156721043
Lead	1	0.4975607949	0.4837738933	0.4965829147
	3	0.8944267047	0.8944040156	0.8944159856
	5	1.1589079584	1.158902486	1.1589059863

used and convergence results are depicted in Figure 3. As Figure 3 shows, higher order of Padé approximation leads to obtain results closer to the numerical solution. Table 4 shows the velocity values versus time for applied analytical methods and are compared with numerical procedure. The errors (%) of these methods, with respect to the numerical method, were listed in Table 5. As seen in Table 5, DTM-Padé [8,8] is the best method for this equation, also GM has a good agreement and acceptable accuracy, so it can be considered as an efficient method. In the following section, these two methods are used for analysis the practical settling motion of some spherical particles in water.

Afterwards, to present some practical examples, Aluminum, Copper and Lead are selected in various sizes submerged in water. Physical properties of the materials and calculated coefficients for Eq. (4) in these practical examples are listed in Tables 1 and 2, respectively. Figures 4 and 5 depict the variations of velocity and acceleration for Aluminum in different diameters. Velocity variations for copper and lead are showed in Figures 6 and 7. The effect of the particle material on velocity and acceleration for these three metals, Aluminum, Copper and Lead, are investigated in Figures 8 and 9 for a constant diameter of 1 mm. these effects for larger diameters, $D=3$ mm and $D=5$ mm are shown in Figures 10 and 11 respectively. Figures 12 and 13 presented the position of the particle in sedimentation process for each time step equal to 0.02 s. Figure 12 shows the effect of particle size and Figure 13 explains the material of particle's effect on settling position.

Generally speaking, the physical behavior of the settling particles is well captured. For all the selected particles, terminal velocity is calculated and presented in Table 6. Results also are compared with HPM which already was done by Jalal et al. [13]. It can be concluded from Table 6 that DTM-Padé [8,8] estimated the terminal velocity excellently, although it's errors might increase compared with HPM when t tends to infinity. Outcomes reveal that the value of the terminal velocity increases significantly with particle diameter and density. Moreover, the smaller particle reaches to its terminal velocity earlier. Thus, the acceleration period is shorter for smaller and lighter particles. From a physical point of view, it can be concluded that larger particles reach zero acceleration (terminal velocity) more slowly. Results also explain that when particle is massive (because of greater density or diameter) it has lower position in the same time steps because of its greater terminal velocity.

5. Conclusion

Many problems in environment such as two-phase solid-liquid flows include particle equation of motion. The majority of the pervious investigations in this field is related to numerical studies and is limited to steady state situations. In the current study, for the first time, three analytical methods are introduced for the unsteady settling behavior of solid spherical particles falling in water which has many applications in chemical sciences. DTM, DTM-Padé approximations, Galerkin method (GM) and collocation method (CM) and numerical method were presented to particles sedimentation and different particles material (Aluminum, Copper, and Lead) behavior are investigated. The following main conclusions can be drawn from the current study:

I. Padé approximation is needed for convergence of DTM at terminal velocity.
II. DTM-Padé had an excellent accuracy for the current problem.
III. Galerkin method is the simplest method among applied methods that has a good agreement with numerical results. Accuracy of this method and reduction in the size of computational domain make it as a wider applicable method.
IV. The presented semi-exact solutions are very simple and can be extended for much harder problems including high nonlinear terms and/or integral terms that, in contrast to the current problem [6,7], do not have any exact solutions

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