Incompressible surfaces in handlebodies and isotopies

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Abstract

It is shown that every properly embedded incompressible surface in a handlebody can be constructed by a canonical gluing process. A simple condition is given which asserts that the result of the gluing process is an incompressible surface. A new notion of isotopy is introduced in order to distinguish surfaces belonging to distinct isotopy classes. Several examples (known and new) are constructed.

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1. Introduction and statements of results

Let \( M \) be a 3-dimensional orientable differentiable manifold with or without boundary and let \( S \subset M \) be a properly embedded surface, i.e., the interior \( \text{Int}(S) \) and the boundary \( \partial S \) of \( S \) satisfy the inclusions

\[ \text{Int}(S) \subset \text{Int}(M) \quad \text{and} \quad \partial S \subset \partial M, \]

\( S \) is transverse to \( \partial M \), and the intersection of \( S \) with a compact subset of \( M \) is compact in \( S \). A \textit{compressible disk} for \( S \subset M \) is an embedded disk \( D \subset M \) such that \( \partial D \subset S \), \( \text{Int}(D) \subset M \setminus S \) and \( \partial D \) is an essential loop in \( S \), i.e., the map \( \partial D \to S \) induces an injection \( \pi_1(\partial D) \to \pi_1(S) \). A properly embedded surface \( S \subset M \) is \textit{incompressible} if there are no compressible disks for \( S \) and no component of \( S \) is a sphere that bounds a ball. If \( S \subset M \) is connected and 2-sided (i.e., if \( S \) is orientable) then \( S \) is incompressible if and only if the induced map \( \pi_1(S) \to \pi_1(M) \) is injective and \( S \) is not a sphere that bounds a ball (see, for example, [4, Chapter 6]). Incompressible surfaces have been a subject of research in the theory of 3-dimensional manifolds for a long time. The majority of the results concerning incompressible surfaces in 3-manifolds describe various properties of incompressible surfaces under the assumption that they exist.

Let \( M \) denote a handlebody of genus \( n \). This paper shows that every properly embedded incompressible surface in \( M \) can be constructed by a simple canonical gluing process where the pieces are disks. This construction (which is

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made precise in Section 3 below), does not always produce an incompressible surface. In Section 4, a simple sufficient condition is given in Theorem 11 which asserts that the result of the gluing process is incompressible.

We further analyze the gluing process in the construction of properly embedded surfaces in $M$. Roughly speaking, we show (see Theorem 23) that the isotopy class of the resulting surface is determined by the components (i.e., disks and stripes) that are glued together to form the surface. This is achieved through the notion of strong isotopy which is made precise in Section 6. Finally, in Section 7 a number of examples is given in order to demonstrate the variety of surfaces obtained by the gluing construction. It is shown, in particular, that some of the well-known examples of incompressible surfaces in handlebodies (see [6,7]) satisfy the property posited in the assumptions of Theorem 11 and using the gluing process new examples of incompressible surfaces are given.

In order to simplify notation and make the exposition more friendly to the reader we present all constructions, examples and proofs in the case the genus of $M$ is equal to 2. The majority of the proofs extend trivially to arbitrary genus hence, we will only comment on the few occasions where the generalization requires explanation.

2. Definitions and preliminary results

Let $M$ denote a 3-dimensional handlebody of genus 2. $M$ can be represented as the union of a handle of index 0 (i.e., a 3-ball) with two handles of index 1 (i.e., two copies of $D^2 \times [0,1]$). We fix these handles along with (following the standard terminology) longitudes $f_\alpha : S^1 \to \partial M$, $f_\beta : S^1 \to \partial M$ and meridians $m_\alpha : S^1 \to \partial M$, $m_\beta : S^1 \to \partial M$. We also fix the so-called base of each handle, denoted by $g_\alpha$, $g_\beta$, given by the commutators $[f_\alpha, m_\alpha]$ and $[f_\beta, m_\beta]$, respectively. As the fundamental group of $M$ is the free group on two letters, we fix two generators of $\pi_1(M)$, denoted by $\alpha$ and $\beta$, so that $\text{Im} f_\alpha$ (respectively $\text{Im} f_\beta$) represents $\alpha$ (respectively $\beta$).

Let $S$ be a properly embedded surface in $M$.

**Definition 1.** Given a properly embedded surface $S$ in $M$, an embedded disk $D_S$ in $M$ will be called a cutting disk for $S$ if it satisfies the following properties:

(a) $\text{Int}(D_S) \subset \text{Int}(M)$ and $\partial D_S \subset \partial M$, i.e., $D_S$ is properly embedded in $M$.
(b) $D_S$ intersects $S$ transversely.
(c) $\partial D_S$ is freely homotopic to $\text{Im} g$ where $g$ is the base of a (fixed) handle in $M$. In particular, $M \setminus D_S$ consists of two components and, since $n = 2$, the closure of each component is homeomorphic to a handlebody of genus 1 (see Remark 6 at the end of this section for $n > 2$).

If $S$ is an incompressible surface then a cutting disk $D_S$ for $S$ can be chosen to satisfy, in addition, the following property:

(d) No component of $D_S \cap S$ is a circle.

To see the latter, assume, on the contrary, that $D_S \cap S$ contains at least one circle. Let $C$ be an innermost such circle. Then $C$ bounds a disk $D_1$ contained in $D_S$ so that $D_1 \cap S$ is exactly $C$. By incompressibility of $S$, $C$ bounds a disk $D_2$ contained in $S$. Then, by irreducibility of $M$, $D_1 \cup D_2$ bounds a 3-ball. Hence, we may isotope $S$ to eliminate $C$ or, equivalently, we may alter $D_2$ so that the intersection $D_2 \cap S$ does not contain $C$. After a finite number of such steps we obtain a cutting disk satisfying property (d) above.

Note that property (d) implies, in particular, that $D_S \cap S$ is a disjoint union of arcs where each arc has endpoints in $\partial D_S$. Denote by $T_1$ and $T_2$ the union of $D_S$ with each of the two components of $M \setminus D_S$. Apparently, $T_1$ and $T_2$ are handlebodies of genus 1.

**Proposition 2.** Let $S$ be incompressible in $M$ and $D_S$ a cutting disk for $S$ in $M$ decomposing $M$ into two handlebodies $T_1$ and $T_2$. Then $S_i = T_i \cap S$, $i = 1, 2$ are incompressible surfaces in $T_i$.

**Proof.** Assuming $S_1$ is compressible, let $D_1$ be a compressing disk for $S_1$ with $\partial D_1 \subset S_1$. As $S$ is incompressible, $\partial D_1$ bounds a disk, say $\Delta$, contained entirely in $S$. By slightly altering the disk $D_1$ we may assume that $\partial \Delta \cap D_S =$
∂D₁ ∩ D = ∅. By transversality, Δ ∩ D is either empty or, a union of circles. However, by property (d), S ∩ D is a union of arcs. As

\[ Δ \cap D \subset S \cap D \]

we conclude that Δ ∩ D = ∅. This means that Δ lies entirely in S₁ which contradicts the assumption that D is a compressing disk for S₁.

It is well known that an incompressible surface in a handlebody of genus 1 is homeomorphic to either, a disk or, an annulus whose homotopy generator is noncontractible in the handlebody. Examples of incompressible surfaces intersecting the disk D₁ in the handlebodies T₁ and T₂ are shown in Fig. 1. In fact, only the boundaries which are embedded in ∂T₁ are drawn.

Recall also that a map \( f : X \times I \to Y \) is a proper isotopy if for all \( t \in I \), \( f|_{X \times \{t\}} \) is a proper embedding. In this case we will be saying that \( f(X \times \{0\}) \) and \( f(X \times \{1\}) \) are properly isotopic in Y.

**Lemma 3.** Let S be incompressible in M and D₁ a cutting disk for S in M decomposing M into two handlebodies T₁ and T₂. Then there exists a cutting disk D₁,α \( \subset T₁ \) such that:

1. \( T₁ \setminus D₁,α \) consists of two components, one being, after taking closure, homeomorphic to a 3-ball, denoted by C₀, and the other being, after taking closure, homeomorphic to a handlebody of genus 1.
2. For each component K in \( S \cap (\overline{T₁ \setminus C₀}) \) there exists a proper isotopy \( f : K \times I \to \overline{T₁ \setminus C₀} \) with \( f(K \times \{0\}) = K \) and \( f(K \times \{1\}) \) being a properly embedded surface K’ which is of the form shown in Figs. 1(c) or, 1(d) or, 1(e). Moreover, K ∩ D₁,α is properly isotopic to K’ ∩ D₁,α in D₁,α.
3. Minimality condition: The total number of components of \( S \cap D₁,α \) is minimal amongst all such disks satisfying properties (1) and (2).

**Notation 4.** By property (2) in the above lemma, each component in \( S \cap (\overline{T₁ \setminus C₀}) \) can be of three different types which are demonstrated in Figs. 1(c), 1(d) and 1(e). We will be referring to the first two by the word stripe and, by abuse of language, we will be saying that the disk in Fig. 1(c) is a stripe of type α₀ and the disk in Fig. 1(d) is a stripe of type α¹ (the notation α¹ and α₀ is based on whether the particular stripe represents the generator α or not). A component in \( S \cap (\overline{T₁ \setminus C₀}) \) having exactly one subarc of its boundary in D₁,α will be called an elementary disk (it is shown in Fig. 1(e)) and will be denoted by αe.

**Remark 5.** Note also that we may alter the cutting disk D₁,α or, equivalently, enlarge the 3-ball C₀ so that all stripes of type α₀ are eliminated. In fact, each stripe of type α₀ will give rise to two elementary disks. This procedure is called elementary isotopy of type αe and it is explicitly defined in Section 6 (see Definition 19).
It is clear that a stripe of type $\alpha^1$ is a space homeomorphic to $I \times I$ properly embedded in $T_1$ so that the subarcs $\{0\} \times I$ and $\{1\} \times I$ of its boundary are contained in $D_{\alpha \sigma}$. We will be denoting these arcs by $\sigma_-(\alpha^1)$ and $\sigma_+(\alpha^1)$ respectively. Similar notation will be used for stripes of type $\alpha^0$. For an elementary disk, the subarc of its boundary belonging to $D_{\alpha \sigma}$ will be written $\sigma(\alpha e)$.

**Proof of Lemma 3.** Let $X$ be a component of $S \cap T_1$. By Proposition 2, $X$ is homeomorphic to either a disk or, an annulus. It is easy to see that we may choose a cutting disk $D_{S,X} \subset T_1$ such that

(a) $D_{S,X}$ satisfies property (1), i.e., $\overline{I} \setminus D_{S,X}$ consists of a 3-ball and a handlebody, say $T_X$, of genus 1,

(b) the component $X$ intersects $T_X$ into stripes and elementary disks.

Let $X'$ be an other component of $S \cap T_1$. Find a cutting disk $D_{S,X'} \subset T_X$ satisfying (a) and (b). Observe now that $D_{S,X'}$ can be chosen so, that in addition, the handlebody $T_X'$ determined by $D_{S,X'}$ intersects $X$ into stripes and elementary disks. Hence, $D_{S,X'}$ is a cutting disk satisfying property (2) for both components $X$ and $X'$. We have shown the existence of a cutting disk satisfying properties (1) and (2). The minimality condition in (3) is now an elementary choice. $\square$

The above lemma corresponds to the generator $\alpha$ of $\pi_1(M)$. Apparently, the dual statement with respect to the generator $\beta$ holds. Moreover, all notation introduced for stripes of type $\alpha^1$, $\alpha^0$ and elementary disks $\alpha^\epsilon$ has its analogue for the components of $S \cap (T_2 \setminus C_\beta)$ denoted by $\sigma_-(\beta^1)$, $\sigma_+ (\beta^1)$, $\sigma_-(\beta^0)$, $\sigma_+ (\beta^0)$ and $\sigma (\beta^\epsilon)$. In conclusion, we have shown that for any incompressible surface $S$ in $M$, there exists cutting disks $D_{S,\alpha}$ and $D_{S,\beta}$ decomposing $M$ into two handlebodies of genus 1, denoted by $T_\alpha$ and $T_\beta$ and one 3-ball denoted by $C_{\alpha,\beta}$ so that the components of

$$(s_{\alpha}) \ S \cap T_\alpha \text{ is the disjoint union of stripes of type } \alpha^1, \alpha^0 \text{ or, elementary disks } \alpha^\epsilon,$$

$$(s_{\beta}) \ S \cap T_\beta \text{ is the disjoint union of stripes of type } \beta^1, \beta^0 \text{ or, elementary disks } \beta^\epsilon,$$

$$(s_{\alpha,\beta}) \ S \cap C_{\alpha,\beta} \text{ is the disjoint union of disks intersecting either, } D_{S,\alpha} \text{ or, } D_{S,\beta} \text{ or, both.}$$

From now on, given a (incompressible) surface $S$, it will be always understood that a pair $D_{S,\alpha}, D_{S,\beta}$ of cutting disks for $S$ satisfy properties (1) and (2) of Lemma 3. If the minimality condition of Lemma 3 is required, it will be stated. Moreover, when the surface $S$ is explicitly understood, we will be suppressing the notation and write $D_\alpha, D_\beta$ instead of $D_{S,\alpha}, D_{S,\beta}$

**Remark 6 (On higher genus).** In the case $M$ is a handlebody of genus $n > 2$ all procedures in this section can be carried out by fixing $n$ handlebodies of $M$. The definition of the cutting disk $D_S$ is identical. However, $M \setminus D_S$ consists of two components whose closures are homeomorphic to a handlebody of genus 1 and a handlebody of genus $n - 1$. It is easy to see that for any incompressible surface $S$ in $M$, there exist cutting disks $D_{S,\alpha}, D_{S,\beta}, \ldots, D_{S,\omega}$, with the cardinality of $\{\alpha, \beta, \ldots, \omega\}$ being equal to $n$, decomposing $M$ into $n$ handlebodies of genus 1, denoted by $T_\alpha, T_\beta, \ldots, T_\omega$ and one 3-ball denoted by $C_{\alpha,\beta,\ldots,\omega}$. Properties $(s_{\alpha}), (s_\beta), \ldots, (s_\omega)$ and $(s_{\alpha,\beta,\ldots,\omega})$ analogous to $(s_{\alpha}), (s_\beta)$ and $(s_{\alpha,\beta})$ above hold in an identical way.

3. Construction of incompressible surfaces

In this section we will describe a process of constructing properly embedded surfaces in $M$. Moreover, we will show that all incompressible surfaces can be obtained (up to proper isotopy) via this construction.

**Main Construction 7.** Start with a handlebody $M$ of genus 2 and let $D_\alpha$ and $D_\beta$ be two disjoint properly embedded disks satisfying property (c) of Definition 1 so that $D_\alpha$ and $D_\beta$ bound a 3-ball $C_{\alpha,\beta} \subset M$. Note that we do not say that $D_\alpha$ and $D_\beta$ are cutting disks because a cutting disk refers to an embedded surface in $M$. Consider finitely many embedded pairwise disjoint disks $K_j$, $j = 1, \ldots, k$ in $C_{\alpha,\beta}$ (typical examples of such disks are drawn in Fig. 2. Let $\sigma_1, \ldots, \sigma_m$ (respectively $t_1, \ldots, t_q$) be the subarcs of the boundaries of the disks $K_j$, $j = 1, \ldots, k$ contained in $D_\alpha$ (respectively $D_\beta$). Consider the following attaching procedures:
\(\sigma\) can be attached to all of the subarcs \(\sigma_{m_1}, \sigma_{m_2} \in \{\sigma_1, \ldots, \sigma_m\}\) attaching a stripe \(\alpha^1\) having \(\sigma_- (\alpha^1) = \sigma_{m_1}\) and \(\sigma_+ (\alpha^1) = \sigma_{m_2}\).

\((A^0)\) To distinct subarcs \(\sigma_{m_3}, \sigma_{m_4} \in \{\sigma_1, \ldots, \sigma_m\}\) attach a stripe \(\alpha^0\) having \(\sigma_- (\alpha^0) = \sigma_{m_3}\) and \(\sigma_+ (\alpha^0) = \sigma_{m_4}\).

\((A')\) To a single subarc \(\sigma_{m_5} \in \{\sigma_1, \ldots, \sigma_m\}\), attach an elementary disk \(\alpha^e\) having \(\sigma (\alpha^e) = \sigma_{m_5}\).

The above procedures can take place in any order with repetitions allowed. We impose the restriction that each subarc in \(\{\sigma_1, \ldots, \sigma_m\}\) is used once. We perform analogous attaching procedures \((B^1), (B^0)\) and \((B^e)\) to the subarcs \(\{\tau_1, \ldots, \tau_q\}\) contained in \(D_\beta\). It is not clear that a repeated application of the above procedures \((A^1), (A^0), (A^e), (B^1), (B^0), (B^e)\) can always be terminated so that a stripe or an elementary disk is attached to all of the subarcs \(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_q\). For such a termination, a symmetrical distribution of the subarcs \(\sigma_1, \ldots, \sigma_m\) (respectively \(\tau_1, \ldots, \tau_q\)) on the disk \(D_\alpha\) (respectively \(D_\beta\)) is needed. A precise necessary and sufficient condition for terminating the construction is given in the proposition below.

**Proposition 8.** Let \(D_\alpha\) be a disk with arcs \(\sigma_1, \ldots, \sigma_m\) properly embedded. Stripes of type \(\alpha^1, \alpha^0\) or elementary disks \(\alpha^e\) can be attached to all of the subarcs \(\sigma_1, \ldots, \sigma_m\) if and only if there exists (after relabeling \(\sigma_1, \ldots, \sigma_m\), if necessary) an index \(k\), \(0 \leq k \leq m\) and two disjoint properly embedded arcs \(d, d'\) splitting \(D_\alpha\) into three regions \(D_1, D_2\) and \(D_3\) (see Fig. 3), where \(D_2\) is the region having both \(d, d'\) as boundary curves, with the following properties:

1. \(\sigma_1, \ldots, \sigma_k \subset D_2\).
2. \(\sigma_{k+1}, \ldots, \sigma_m \subset D_1 \cup D_3\).
3. Considering \(D_2\) as an axis of reflection mapping \(D_1\) onto \(D_3\), the image under such reflection of the arcs in \(D_1\) is properly isotopic with the arcs in \(D_3\) (in particular, \(m - k\) is even).
4. Denote by \(D_2^+\) and \(D_2^-\) the two components of \(\partial D_2\) in \(\partial D_\alpha\). Let \(\ell\) be an index, \(0 \leq \ell \leq k\), such that (after relabeling if necessary)
   - each of the arcs \(\sigma_i\) for \(i \leq \ell\) has one endpoint in \(D_2^+\) and the other in \(D_2^-\),
   - each of the arcs \(\sigma_i\) for \(i > \ell\) has both its endpoints either in \(D_2^+\) or in \(D_2^-\) and
   - the region between \(\sigma_j, \sigma_{j+1}\) (for all \(j = 1, \ldots, \ell - 1\)) contains, if any, arcs \(\sigma_i\) with \(i\) satisfying \(\ell < i \leq k\).

To obtain \(\ell\) is just a matter of relabeling to the above requirements. Consider an embedded arc \(d''\) with one endpoint in \(d\) and the other in \(d'\) intersecting each \(\sigma_i\) for \(i \leq \ell\) at a single point. Consider \(d''\) as an axis of reflection of \(D_2\) onto itself keeping \(\sigma_1\) invariant for all \(i \leq \ell\). It is then required that for each \(j = 1, \ldots, \ell - 1\) the image (under the reflection with respect to the axis \(d''\)) of the arcs contained in the region of \(D_2\) bounded by \(\sigma_j, \sigma_{j+1}\), \(d''\) and \(D_2^+\) are properly isotopic to the arcs contained in the region of \(D_2\) bounded by \(\sigma_j, \sigma_{j+1}\), \(d''\) and \(D_2^-\).

**Proof.** The indices \(k, \ell\) in properties (1)–(4) above are intended to separate the arcs \(\sigma_1, \ldots, \sigma_m\) into three groups, namely, those arcs to which stripes of type \(\alpha^1\) can be attached (these are indexed \(\sigma_{k+1}, \ldots, \sigma_m\)), those arcs to which stripes of type \(\alpha^0\) can be attached (these are indexed \(\sigma_{\ell+1}, \ldots, \sigma_k\)) and those arcs to which elementary disks \(\alpha^e\) can be attached (these are indexed \(\sigma_1, \ldots, \sigma_\ell\)). Fig. 3 is drawn with \(\ell = 3\) and \(k = 8\).
Let $\sigma_1, \sigma_2$ be two arcs in $D_\alpha$ and suppose a stripe of type $\alpha^1$ is attached to $\sigma_1, \sigma_2$ with $\sigma_-(\alpha^1) = \sigma_1$ and $\sigma_+(\alpha^1) = \sigma_2$. Let $\sigma_3, \sigma_4$ be two (pairwise distinct from $\sigma_1, \sigma_2$) subarcs in $D_\alpha$. By elementary topological considerations it can be seen that

(a) A stripe of type $\alpha^1$ can be attached to $\sigma_3, \sigma_4$ with $\sigma_-(\alpha^1) = \sigma_3$ and $\sigma_+(\alpha^1) = \sigma_4$ if and only if the arcs $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are as shown in Figs. 4(a), 4(a'). This establishes the necessity of condition (3).

(b) A stripe of type $\alpha^0$ can be attached to $\sigma_3, \sigma_4$ with $\sigma_-(\alpha^1) = \sigma_3$ and $\sigma_+(\alpha^1) = \sigma_4$ if and only if the arcs $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are as shown in Fig. 4(b).

(c) A stripe of type $\alpha^e$ can be attached to $\sigma_3$, with $\sigma(\alpha^e) = \sigma_3$ if and only if the arcs $\sigma_1, \sigma_2, \sigma_3$ are as shown in Fig. 4(c).

The analogous necessary and sufficient condition concerning

(d) Two stripes of type $\alpha^0$ is shown in Figs. 4(d) and 4(d').

(e) Two elementary disks $\alpha^e$ is shown in Fig. 4(e).

(f) A stripe of type $\alpha^0$ combined with an elementary disk $\alpha^e$ is shown in Fig. 4(f).

Condition (4) above asserts that all subarcs with endpoints in $D_2^+$ can be mutually used with the subarcs in $D_2^-$ in order to attach stripes of type $\alpha^0$. Then, there is no obstruction to attaching elementary discs $\alpha^e$ to the remaining subarcs $\sigma_i$ for $i \leq \ell$. In Fig. 3 the region bounded by $\sigma_1$ and $\sigma_2$ satisfies the third bullet of condition (4) whereas the region bounded by $\sigma_2$ and $\sigma_3$ does not.

**Remark (On higher genus).** In the case of a handlebody of genus $n > 2$ the symmetrical distribution of the arcs $\sigma_1, \ldots, \sigma_m \subset D_{S,\alpha}$ posited in the above proposition must be required for all cutting disks $D_{S,\alpha}, D_{S,\beta}, \ldots, D_{S,\omega}$ (where cardinality of $\{\alpha, \beta, \ldots, \omega\}$ is equal to $n$).
Let $S$ be the collection of connected surfaces obtained by the Main Construction with respect to fixed cutting disks $D_{\alpha}$ and $D_{\beta}$. Apparently, each such surface is properly embedded in $M$. Let $S$ be an arbitrary incompressible surface in $M$. We are about to show that $S$ can be obtained, up to proper isotopy, by the above Main Construction with respect to the fixed cutting disks $D_{\alpha}$ and $D_{\beta}$. Lemma 3 above asserts the existence of cutting disks $D_{S,\alpha}$ and $D_{S,\beta}$ which decompose $S$ into disks and stripes. There exists an ambient isotopy in $\partial M$ taking $\partial D_{S,\alpha}$ onto $\partial D_{\alpha}$.
and $\partial D_{S,\beta}$ onto $\partial D_{\beta}$. By standard isotopy extension theory (see for example [8, Theorem 4.24]) this isotopy extends to an ambient isotopy
\[ F: M \times I \to M. \]
The boundary of $F(D_{S,\alpha} \times \{1\})$ is $\partial D_{\alpha}$ and the boundary of $F(D_{S,\beta} \times \{1\})$ is $\partial D_{\beta}$. Hence, $F$ can be altered so that
\[ F(D_{S,\alpha} \times \{1\}) = D_{\alpha} \quad \text{and} \quad F(D_{S,\beta} \times \{1\}) = D_{\beta}. \]
It is clear that $F(S \times \{1\})$ is an element of $S$ and $S$ is properly isotopic to $F(S \times \{1\})$. Hence, we have the following theorem.

**Theorem 9.** The class $S$ of surfaces obtained by the Main Construction contains a representative from all isotopy classes of incompressible surfaces in $M$.

### 4. Conditions for incompressibility

In this section we describe a criterion which asserts that the result of the Main Construction is an incompressible surface. This criterion is used in Section 7 in order to establish incompressibility of well-known examples (given by W. Jaco [6] and R. Qiu [7]) as well as new examples of incompressible surfaces.

Let $S$ be a surface obtained by the Main Construction. We may assume (see Remark 5) that no stripes of type $\alpha^0$, $\beta^0$ exist, in other words, the cutting disks $D_{\alpha}$, $D_{\beta}$ decompose $S$ into
- components $K_1, K_2, \ldots, K_k$ in $S \cap C_{\alpha\beta}$,
- stripes of type $\alpha^1$, $\beta^1$ and
- elementary disks $\alpha^e$, $\beta^e$.

Observe that the original components of $S \cap C_{\alpha\beta}$ used for constructing $S$ are properly embedded disks in $C_{\alpha\beta}$. However, after altering the cutting disks $D_{\alpha}$, $D_{\beta}$ in order to eliminate the stripes of type $\alpha^0$, $\beta^0$ it may happen that the components of $S \cap C_{\alpha\beta}$ are no longer incompressible (in particular, they may not be embedded disks). It is imperative in the theorem below to assume that the components $K_1, K_2, \ldots, K_k$ in $S \cap C_{\alpha\beta}$ remain disks after the alteration of the cutting disks $D_{\alpha}$, $D_{\beta}$.

**Definition 10.** Consider two distinct stripes $(\alpha^1)_x$, $(\alpha^1)_y$ of type $\alpha^1$ and let $\sigma_1 = \sigma_-(\alpha^1)_x$, $\sigma_2 = \sigma_+(\alpha^1)_x$ and $\sigma_3 = \sigma_-(\alpha^1)_y$, $\sigma_4 = \sigma_+(\alpha^1)_y$ be their intersection with $D_{\alpha}$ (see Figs. 4(a) and 4(a')). In particular, $\sigma_1, \sigma_2 \in D_1$. We will say that $(\alpha^1)_x$, $(\alpha^1)_y$ form a pair of parallel stripes of type $\alpha^1$ in $S$ if either, both $\sigma_1, \sigma_3$ or, both $\sigma_2, \sigma_4$ belong to the same component $K_i \subset C_{\alpha\beta}$ for some $i$.

Note that if all four subarcs $\sigma_1, \sigma_3, \sigma_2, \sigma_4$ belong to a single component $K_i \subset C_{\alpha\beta}$, then $S$ must be compressible. In an identical way, a pair of parallel stripes of type $\beta^1$ in $S$ is defined.

**Theorem 11.** Let $S$ be a properly embedded surface in $M$ obtained by the Main Construction. Let $D_{\alpha}$, $D_{\beta}$ be cutting disks decomposing $S$ into properly embedded components $K_1, K_2, \ldots, K_k$ in $S \cap C_{\alpha\beta}$, stripes of type $\alpha^1$, $\beta^1$ and elementary disks $\alpha^e$, $\beta^e$ (i.e., $D_{\alpha}$, $D_{\beta}$ are chosen so that all stripes of type $\alpha^0$, $\beta^0$ are eliminated). Assume that the components $K_1, K_2, \ldots, K_k$ in $S \cap C_{\alpha\beta}$ are embedded disks. Then, if $S$ does not contain any pair of parallel stripes of type $\alpha^1$ nor of type $\beta^1$ then $S$ is incompressible.

**Proof.** Suppose $D$ is a compressing disk for $S$. We may assume that $D$ is transverse to $D_{\alpha}$ and $D_{\beta}$ and the number of components of $D \cap D_{\alpha}$ and $D \cap D_{\beta}$ are both minimal. $\partial D$ is freely homotopic to the identity element in $\pi_1(M) = \langle \alpha, \beta \rangle$. Choose a base point $x$ in the 3-ball $C_{\alpha,\beta}$ for the loop $\partial D$. Starting from $x$ and tracing (in any direction) the loop $\partial D$ we may write a word in the letters $\alpha$ and $\beta$ as follows: let $x = x_0, x_1, \ldots, x_{m-1}, x_m = x$ be the ordered sequence of points on the loop $\partial D$ so that each $x_i$, $i = 1, 2, \ldots, m - 1$ belongs to either $D_{\alpha}$ or $D_{\beta}$. By minimality of $D \cap (D_{\alpha} \cup D_{\beta})$ it is clear that, for all $i = 0, 1, \ldots, m - 1$, the subarc $\overline{x_i x_{i+1}}$ of $\partial D$ is contained in either a stripe of type $\alpha^1$ or a stripe of type $\beta^1$ or, the 3-ball $C_{\alpha,\beta}$. If the subarc $\overline{x_i x_{i+1}}$ of $\partial D$ is contained in
- A stripe \( \alpha^1 \) (respectively \( \beta^1 \)) with \( x_i \in \sigma_-^{(\alpha^1)} \) (respectively \( x_j \in \sigma_-^{(\beta^1)} \)) and \( x_{i+1} \in \sigma_+^{(\alpha^1)} \) (respectively \( x_{j+1} \in \sigma_+^{(\beta^1)} \)) we write the letter \( \alpha \) (respectively \( \beta \)).
- A stripe \( \alpha^{-1} \) (respectively \( \beta^{-1} \)) with \( x_i \in \sigma_-^{(\alpha)} \) (respectively \( x_j \in \sigma_-^{(\beta)} \)) and \( x_{i+1} \in \sigma_-^{(\beta)} \) (respectively \( x_{j+1} \in \sigma_-^{(\alpha)} \)) we write the letter \( \alpha^{-1} \) (respectively \( \beta^{-1} \)).
- The 3-ball \( C_{\alpha,\beta} \) we ignore this arc.

In this way we obtain a word \( w(D) \) in the letters \( \alpha \) and \( \beta \) which, as \( D \) is compressing, represents the trivial element in \( \pi_1(M) = \langle \alpha, \beta \rangle \). Therefore, the word \( w(D) \) is either the empty word, which means that \( D \) is entirely contained in the 3-ball \( C_{\alpha,\beta} \) or, contains a sub-word \( \epsilon \) of the form \( \alpha \alpha^{-1} \) or, \( \alpha^{-1} \alpha \) or, \( \beta \beta^{-1} \) or, \( \beta^{-1} \beta \). The former case is impossible since the components of \( S \) in the 3-ball \( C_{\alpha,\beta} \) are disks so \( D \) cannot be a compressing disk contained entirely in \( C_{\alpha,\beta} \). Hence we may assume that the word \( w(D) \) contains a sub-word of the form, say, \( \alpha \alpha^{-1} \).

In other words, \( \partial D \) intersects \( D_\alpha \) into consecutive points \( x_i, x_{i+1}, x_{i+2} \) and \( x_{i+3} \) so that the subarc \( x_i x_{i+1} \) traces a stripe \( \alpha^1 \), the subarc \( x_{i+1} x_{i+2} \) is contained in a single component, say \( K_{01} \), of \( S \cap C_{\alpha,\beta} \) and the subarc \( x_{i+1} x_{i+2} \) traces a stripe \( \alpha^1 \) in the opposite direction. Moreover, the endpoints \( x_i, x_{i+1}, x_{i+2} \) of \( x_i x_{i+1} x_{i+2} \) are contained, by minimality assumption, in distinct subarcs of \( K_{01} \cap D_\alpha \). In particular, the subarcs \( x_i x_{i+1}, x_{i+1} x_{i+2} \) are tracing distinct stripes. This implies that \( S \) contains a pair of parallel stripes, a contradiction. \( \square \)

The condition of parallel stripes posited in the above theorem does not characterize incompressibility. In the last section it is given an example of an incompressible surface containing parallel stripes. Moreover, the above theorem can be used to show incompressibility for well-known examples of incompressible surfaces in handlebodies (see [6,7]) as well as prove incompressibility in some new examples given in Section 7.

5. The minimality condition

In this section we further analyze the minimality condition of Lemma 3. We will describe a geometrical condition which is, in fact, equivalent to the minimality condition of Lemma 3. Moreover, we will show that any pair of cutting disks \( D_\alpha, D_\beta \) can be altered in order to satisfy the geometrical, hence the minimality, condition.

Definition 12. Let \( S \) be a given incompressible surface and \( D_\alpha, D_\beta \) a pair of cutting disks for \( S \). We will be saying that the cutting disks \( D_\alpha, D_\beta \) are geometrically minimal for \( S \) if there does not exists

\begin{enumerate}
\item a (connected) subarc \( \rho \) of \( \partial S \) where \( \rho \) is of the form \( \rho = \rho_1 \cup \rho_{\alpha\beta} \cup \rho_2 \) with
  - each of \( \rho_1, \rho_2 \) is in the boundary of a stripe \( \alpha^0 \) (respectively \( \beta^0 \)) or an elementary disk \( \alpha^e \) (respectively \( \beta^e \)),
  - \( (\partial \rho_1 \cup \partial \rho_{\alpha\beta} \cup \partial \rho_2) \subset \partial D_\alpha \) (respectively \( \subset \partial D_\beta \))
  - \( \rho_{\alpha\beta} \subset C_{\alpha,\beta} \)
  (such an arc \( \rho \) is drawn in Fig. 5(a)),
\item a (connected) subarc \( \sigma \) of \( \partial S \) where \( \sigma \) is of the form \( \sigma = \sigma_1 \cup \sigma_{\alpha\beta} \cup \sigma_2 \) with
  - each of \( \sigma_1, \sigma_2 \) is in the boundary of a stripe of type \( \alpha^1 \) (respectively \( \beta^1 \)),
  - \( (\partial \sigma_1 \cup \partial \sigma_{\alpha\beta} \cup \partial \sigma_2) \subset \partial D_\alpha \) (respectively \( \subset \partial D_\beta \))
  - \( \sigma_{\alpha\beta} \subset C_{\alpha,\beta} \)
  (such an arc \( \sigma \) is drawn in Fig. 5(a)),
\item a subarc \( \sigma \) of the form \( \sigma = \sigma_1 \cup \sigma_{\alpha\beta} \cup \sigma_3 \cup \chi_{\alpha\beta} \cup \sigma_2 \) with
  - each of \( \sigma_1, \sigma_2 \) is in the boundary of a stripe of type \( \alpha^1 \) (respectively \( \beta^1 \)) and \( \sigma_3 \) is in the boundary of a stripe \( \alpha^0 \) (respectively \( \beta^0 \)) or an elementary disk \( \alpha^e \) (respectively \( \beta^e \)),
  - \( (\partial \sigma_1 \cup \partial \sigma_{\alpha\beta} \cup \partial \sigma_3 \cup \partial \chi_{\alpha\beta} \cup \partial \sigma_2) \subset \partial D_\alpha \) (respectively \( \subset \partial D_\beta \)),
  - \( \sigma_{\alpha\beta}, \chi_{\alpha\beta} \subset C_{\alpha,\beta} \)
  (such an arc \( \sigma \) is drawn in Fig. 5(a)).
\end{enumerate}

Notation. In the event of the existence of an arc of type (2) of the above definition we will say that the stripes containing \( \sigma_1, \sigma_2 \) form a reducible pair of (parallel) stripes of type \( \alpha^1 \). Similarly, for stripes of type \( \beta_1 \). In the event of the existence of an arc of type (1) of the above definition we will say that the stripes containing \( \rho_1, \rho_2 \) form a reducible pair of stripes of type \( \alpha^0 \) (or \( \alpha^e \)). Similarly, for stripes of type \( \beta^0 \) (or \( \beta^e \)). The word reducible fits well in
this notation because if such a pair exists then we may isotope $S$ and reduce the number of stripes (see property (E4) of elementary isotopies in Section 6 below) by one.

We will also use another pair of properly embedded disks in $M$ which are defined as follows: let $S$ be a given incompressible surface and let $D_\alpha, D_\beta$ be a pair of cutting disks for $S$. Consider two properly embedded disks $\Delta_\alpha, \Delta_\beta$ in $M$ with the following properties (see Fig. 5b):

1. \( \Delta_1 \cap (D_\alpha \cup D_\beta) = \emptyset \) and \( \Delta_2 \cap (D_\alpha \cup D_\beta) = \emptyset \).
2. \( M \setminus (\Delta_\alpha \cup \Delta_\beta) \) is contractible.
3. Both $\Delta_\alpha, \Delta_\beta$ intersect $S$ transversely.
4. $\Delta_\alpha$ does not intersect any stripe of type $a^0$ or an elementary disk $a^e$ and $\Delta_\alpha$ intersects each boundary component of each stripe of type $a^1$ at exactly one point.
   $\Delta_\beta$ does not intersect any stripe of type $b^0$ or an elementary disk $b^e$ and $\Delta_\beta$ intersects each boundary component of each stripe of type $b^1$ at exactly one point.

In particular, the intersection of $\partial \Delta_\alpha$ with $S$ consists of points entirely contained in the boundary of stripes of type $\alpha^1$ and similarly for the intersection of $\Delta_\beta$ with $S$.

To obtain the disks $\Delta_\alpha, \Delta_\beta$ we may start with two meridians (one for each handle) and choose, accordingly, $\Delta_\alpha, \Delta_\beta$ to be the disks bounded by them.

**Definition 13.** We will be saying that two simple curves $\sigma, \tau$ on $\partial M$ form a back and forth if there exist subarcs $\sigma_1 \subset \sigma$ and $\tau_1 \subset \tau$ such that $\partial \sigma_1 = \partial \tau_1$ and $\sigma_1 \cup \tau_1$ bounds a disk $D \subset \partial M$.

We will be saying that $\partial S$ and $\partial \Delta_\alpha$ form a back and forth if some component of $\partial S$ along with $\partial \Delta_\alpha$ form a back and forth.

We next define a specific type of back and forth: let $\sigma$ be a subarc of $\partial S$ with $\partial \sigma \subset \partial \Delta_\alpha$ and $\text{Int} \sigma \cap \partial \Delta_\alpha = \emptyset$. Apparently, $\partial \sigma$ decomposes $\partial \Delta_\alpha$ into two subarcs, say, $\tau_\alpha$ and $\tau'_\alpha$. If either, $\sigma$ and $\tau_\alpha$ or, $\sigma$ and $\tau'_\alpha$ form a back and forth then we will say that $\partial S$ has a back and forth with respect to $\Delta_\alpha$ (the arc $\sigma$ in Fig. 5(b) has a back and forth with respect to $\Delta_\alpha$ since $\sigma \cup \tau_\alpha$ bounds a disk on $\partial M$). A back and forth of $\partial S$ with respect to $\Delta_\beta$ is defined similarly.
Proposition 14. Let $S$ be an incompressible surface in $M$. Any pair of $D_α$, $D_β$ of cutting disks for $S$ can be altered in order to be geometrically minimal for $S$.

Proof. We will show that if $D_α$, $D_β$ are cutting disks and a subarc of type (1) or (2) or (3) of Definition 12 exists then there exists a pair of cutting disks $D_α'$, $D_β'$ so that the number of components of $(D_α' \cup D_β') \cap S$ is strictly smaller than the number of components of $(D_α \cup D_β) \cap S$. Since this number (of components) is bounded below, we will reach, after a finite number of steps, a pair of cutting disks with no subarcs of type (1)–(3) of Definition 12, hence, geometrically minimal pair of cutting disks for $S$.

We first deal with arcs of type (1) of Definition 12. We will use the following notation: let $a, b, c$ be three (or more than three) points on a simple closed curve. We will denote by $ab$ the subarc of the closed curve with end points $a, b$ which does not contain the third (or any of the remaining) point $c$.

Let $ρ$ be a subarc of $∂S$ of type (1) of Definition 12 being of the form $ρ = ρ₁ ∪ ρ_{αβ} ∪ ρ₂$. Assume $∂ρ₁ = \{x, y\}$, $∂ρ_{αβ} = \{y, z\}$ and $∂ρ₂ = \{z, w\}$. Choose embedded arcs $ρ₁', ρ₂'$ on $∂M$ arbitrarily close to and disjoint from $ρ₁, ρ₂$ respectively so that $∂ρ₁' = \{x', y'\}$, $∂ρ₂' = \{z', w'\}$ with $x', y' \in \overline{xy}$ and $z', w' \in \overline{zw}$. Then choose an embedded arc $ρ'_{αβ}$ on $∂M$ arbitrarily close to and disjoint from $ρ_{αβ}$ with endpoints $y', z'$. Finally, choose an embedded arc $ρ''_{αβ}$ on $∂M$ arbitrarily close to and disjoint from $ρ'_{αβ}$ with endpoints $y'', z''$ so that $y'' \in \overline{y'y'}$ and $z'' \in \overline{z'w'}$. Now form the closed curve

$$ρ' = ρ₁' ∪ ρ'_{αβ} ∪ ρ₂' ∪ w'z'' ∪ ρ''_{αβ} ∪ y''x''.$$

This closed curve is going to be the boundary $∂D_α'$ of the new cutting disk $D_α'$. In order to describe the interior of the disk $D_α$ we distinguish two cases

- $y, z$ belong to a single component of $D_α \cap S$, which will be denoted by $C_{y,z}$.
- $y, z$ belong to distinct components of $D_α \cap S$, which will be denoted by $C_y, C_z$.

In the first case observe that, if $y, z$ belong to a single component, so do $x, w$ and, thus, we can find a properly embedded disk $D'_α$ with $∂D'_α = ρ'$ intersecting $S$ transversely so that

$$D'_α \cap S = (D_α \cap S) \setminus (C_{x,w} \cup C_{y,z}).$$

It is easy to see that the arc $ρ$ does not intersect $D'_α$ and, hence, $ρ$ is not of type (1) of Definition 12 with respect to $D'_α$ and no new arc of this type with respect to $D'_α$ is introduced. Clearly, the number of components of $D'_α \cap S$ is strictly smaller than the number of components of $D_α \cap S$.

In the second case, as $ρ'$ bounds a disk on $∂M$ we may find a properly embedded disk $D'_α$ with $∂D'_α = ρ'$ which is a cutting disk and intersects $S$ transversely. By construction of $ρ'$, $∂D'_α$ does not contain the points $x, y, z, w$. Since the number of components of $D'_α \cap S$ is twice the number of components of $∂D'_α \cap S$, it is clear that of components of $D'_α \cap S$ is two less than of components of $D_α \cap S$.

We work similarly in order to eliminate all arcs of type (1) of Definition 12 with end points on $D_β$.

In an identical way as above we may show that if a subarc of type (2) of Definition 12 exists then there exists a pair of cutting disks $D'_α, D'_β$ so that the number of components of $(D'_α \cup D'_β) \cap S$ is strictly smaller than the number of components of $(D_α \cup D_β) \cap S$.

We now deal with arcs of type (3) of Definition 12. By the discussion above we may assume that no arcs of type (1) or type (2) of Definition 12 exist. We proceed to show that if $σ$ is of type (3) of Definition 12, then there exist cutting disks $D'_α, D'_β$ so that the number of components of $(D'_α \cup D'_β) \cap S$ is strictly smaller than the number of components of $(D_α \cup D_β) \cap S$. Recall that $σ$ is of the form

$$σ = σ₁ ∪ σ_{αβ} \cup σ_ɛ \cup χ_{αβ} \cup σ₂$$

with

- $σ₁$ (respectively $σ₂$) is contained in the boundary of a stripe of type $α^1$ and $∂σ₁ = \{x, y\} \subset ∂D_α$ (respectively $∂σ₂ = \{z, w\} \subset ∂D_α$),
- $σ_{αβ}, χ_{αβ}$ are properly embedded in $C_{α, β}$ and $∂σ_{α, β} = \{y, z\}, ∂χ_{αβ} = \{z_2, z\}$ with $z_1, z_2 \in ∂D_α$. 

We distinguish two cases (in the case $D_\sigma$ and disjoint from $\delta \Delta$):  

- $y, z$ belong to a single component of $D_a \cap S$, which will be denoted by $C_{y,z}$.
- $y, z$ belong to distinct components of $D_a \cap S$, which will be denoted by $C_y, C_z$.

In the first case observe that if $y, z$ belong to a single component of $D_a \cap S$ so do $x$, $w$ and we can find a properly embedded disk $D'_a$ as follows: choose embedded arcs $\sigma'_{a_\beta}, \chi'_{a_\beta}$ on $\partial M \cap C_{a, \beta}$ arbitrarily close to and disjoint from $\sigma_{a_\beta}, \chi_{a_\beta}$ respectively so that $\partial \sigma'_{a_\beta} = \{y', z'_1\}$, $\partial \chi'_{a_\beta} = \{z'_2, z'_3\}$ with $z'_1 \in \overline{z_2z_3}$, $z'_1 \in \overline{z_2w}$ and $z'_2 \in \overline{z_2w}$. Now choose a properly embedded cutting disk $D'_a$ with 

$$\partial D'_a = \overline{z'_1w} \cup \overline{wz'_3} \cup \overline{xz'_1} \cup \overline{\sigma'_{a_\beta}} \cup \overline{y'z'} \cup \overline{\chi'_{a_\beta}}.$$ 

It is clear that $\sigma$ is in the boundary of an elementary disk with respect to $D'_a$. In particular, 

$$D'_a \cap S = (D_a \cap S) \setminus (C_{yz} \cup C_{ziy})$$

and no arcs of type (1) and (2) of Definition 12 are introduced with respect to the new cutting disk $D'_a$. Clearly, the number of components of $D'_a \cap S$ is strictly smaller than the number of components of $D_a \cap S$.

We are left to deal with the second case, namely, where $y, z$ belong to distinct components of $D_a \cap S$. Choose properly embedded disks $\Delta_\alpha$, $\Delta_\beta$ satisfying properties (A1)--(A4) above. Apparently, $\partial \Delta_\alpha$ intersects $\sigma$ at exactly two points, say $x_\Delta = \sigma_1 \cap \partial \Delta_\alpha$ and $w_\Delta = \sigma_2 \cap \partial \Delta_\alpha$. The subarc of $\sigma$ with endpoints $x_\Delta, w_\Delta \in \partial \Delta_\alpha$ forms a back and forth with respect to $\Delta_\alpha$ and we will be denoting this subarc by $\sigma_\Delta$. Let $\tau$ be the subarc of $\partial \Delta_\alpha$ which forms a back and forth with $\sigma_\Delta$ and $\tau_\Delta$ its complement in $\partial \Delta$. Choose an embedded arc $\sigma'_{\Delta}$ on $\partial M$ arbitrarily close to and disjoint from $\sigma_\Delta$ with endpoints $x'_\Delta, w'_\Delta \in \tau'_\Delta$. Observe that $\overline{x'_\Delta w'_\Delta} \cup \overline{\sigma'_\Delta}$ is freely homotopic to $\partial \Delta_\alpha$. Choose a properly embedded disk $\Delta'_\alpha$ with 

$$\partial \Delta'_\alpha = \overline{x'_\Delta w'_\Delta} \cup \overline{\sigma'_\Delta}$$

so that $\Delta'_\alpha$ intersects $S$ transversely. Let $\Sigma'$ be a simple closed curve on $\partial M$ which

- intersects $\tau$ at exactly one point,
- intersects $\sigma$ and, hence, $\sigma'_{\Delta}$ at exactly one point,
- is a generator of $\pi_1(T_\alpha)$.

Choose now a tubular neighborhood $\mathcal{N}$ of $\Delta'_\alpha$ with boundary components $\Delta'_1$ and $\Delta'_2$. Let $q_1$ and $q_2$ be the points in the singletons $\partial \Delta'_1 \cap \Sigma$ and $\partial \Delta'_2 \cap \Sigma$ respectively. Consider embedded arcs $\Sigma', \Sigma''$ on $\partial M$ arbitrarily close to and disjoint from $\Sigma$ with boundary points $q_1, q_2$ and $q'_1, q'_2$ respectively. The union of arcs 

$$\overline{q'_1q''_1} \cup \overline{\Sigma''} \cup \overline{q'_2q''_2} \cup \overline{\Sigma'}$$

is a simple closed curve on $\partial M$ freely homotopic to $\partial D_a$. We may choose a properly embedded disk $D'_a$ whose boundary is the above simple curve so that

- $D'_a$ intersects $S$ transversely and
- $D'_a$ is arbitrarily close to the union of the disks $\Delta'_2$ and $\Delta'_2$ along with the stripe bounded by $\Sigma', \Sigma''$.

Denote by $(x_\alpha^1)_{x_\Delta}$ and $(w_\alpha^1)_{w_\Delta}$ the (distinct) stripes of type $\alpha^1$ with respect to $D_a$ containing $x_\Delta$ and $w_\Delta$ respectively. Since all stripes of type $\alpha^1$ with respect to $D_a$ distinct from $(x_\alpha^1)_{x_\Delta}$ and $(w_\alpha^1)_{w_\Delta}$ intersect $\Delta_\alpha$ at points inside $\tau'_\alpha$, it is clear that all stripes of type $\alpha^1$ with respect to $D_a$ remain stripes of type $\alpha^1$ with respect to $D'_a$. Since, in addition, $\Sigma$ intersects each elementary disk at exactly one boundary point, it follows that $D'_a$ is a cutting disk. Moreover, the points $x_\Delta, w_\Delta$ belong to the same component in $D'_a \cap S$, hence, the number of components of $D'_a \cap S$ is one less than
the number of components of $D_α \cap S$. This concludes the reduction in the case of an arc of type (3) of Definition 12 and the proof of the proposition is complete. □

The above proposition shows that if $S$ is an incompressible surface in $M$ then there exists a pair of cutting disks $D_{S,α}, D_{S,β}$ satisfying properties (1) and (2) of Lemma 3 and, in addition, $D_{S,α}, D_{S,β}$ are geometrically minimal for $S$. In view of this, we may impose in the Main Construction the additional requirement that the attaching procedures $(A_1^γ), (A_0^β), (A^γ)$ and $(B_1^β), (B^β)$ are performed so that no arcs of the form (1), (2) and (3) of Definition 12 are created. Let $S_{gm}$ be the following subclass of $S$:

$$S_{gm} = \{ S \in S: D_α, D_β \text{ are geometrically minimal for } S \}.$$

Using the above proposition (instead of Lemma 3) we have the following corollary whose proof is identical to the proof of Theorem 9.

**Corollary 15.** The class $S_{gm}$ of surfaces obtained by the Main Construction with the additional requirements imposed by Definition 12 contains a representative from all isotopy classes of incompressible surfaces in $M$.

**Proposition 16.** Let $S$ be an incompressible surface in $M$ and $D_α, D_β$ a pair of cutting disks for $S$ satisfying properties (1) and (2) of Lemma 3. Then $D_α, D_β$ are geometrically minimal for $S$ if and only if $D_α, D_β$ satisfy the minimality condition of Lemma 3.

**Proof.** It is clear from the construction in the above proposition that if $D_α, D_β$ satisfy the minimality condition of Lemma 3 then they are geometrically minimal for $S$. The converse will be shown as a corollary to Theorem 23 in Section 6. □

We now show that geometrically minimal cutting disks carry a property which will be crucial in the proof of Theorem 23 below.

**Proposition 17.** Let $S$ be an incompressible surface, $D_α, D_β$ a pair of geometrically minimal cutting disks for $S$ and $Δ_α, Δ_β$ properly embedded disks satisfying properties $(Δ1)$–$(Δ4)$. Then there does not exist a subarc of $∂S$ forming a back and forth with respect to $∂Δ_α$ nor with respect to $∂Δ_β$.

**Proof.** First observe that it suffices to show the conclusion for a single choice of properly embedded disks $Δ_α, Δ_β$ satisfying properties $(Δ1)$–$(Δ4)$. For, if $Δ'_α, Δ'_β$ is an other such pair then there exists an ambient isotopy from $Δ_α \cup Δ_β$ onto $Δ'_α \cup Δ'_β$ which fixes point-wise the ball $C_{α,β}$.

Assume, on the contrary, that $σ$ is a subarc of $∂S$ which has a back and forth with respect to $Δ_α$. We may assume that $σ \cap ∂Δ_β = ∅$ because if $σ$ forms a back and forth with respect to $Δ_α$ and $σ \cap ∂Δ_β \neq ∅$ then, by property $(Δ2)$ above, there must exist a subarc $σ_β$ of $σ$ with endpoints on $Δ_β$ which forms a back and forth with respect to $Δ_β$. Now either, $σ_β$ enjoys the property $σ_β \cap ∂Δ_α = ∅$ (in which case we proceed with the subarc $σ_β$) or, there exists a subarc $σ_{β,α}$ of $σ_β$ with endpoints on $Δ_α$ which forms a back and forth with respect to $Δ_α$. After a finite number of steps we obtain a subarc $σ'$ of $σ$ with endpoints on $∂Δ_α$ (respectively on $∂Δ_β$) which forms a back and forth with respect to $Δ_α$ (respectively $Δ_β$) and $σ' \cap ∂Δ_β = ∅$ (respectively $σ' \cap ∂Δ_α = ∅$).

Apparently, the subarc $σ$ uniquely determines a subarc $σ_D$ of $∂S$ with endpoints on $∂D_α$ so that $σ_D \supset σ$ and $σ_D$ is minimal with respect to the latter property. In particular, $σ_D$ does not intersect $∂Δ_β$. We will show that, since $D_α, D_β$ is a pair of geometrically minimal cutting disks for $S$, such an arc $σ_D$ cannot exist.

Observe that as $σ$ is a subarc of $∂S$ it must be a union of arcs with endpoints on $(∂D_α \cup ∂D_β)$. In particular, without loss of generality, we may assume that $σ_D$ is of the form

$$σ_D = σ_1 \cup σ_{αβ} \cup τ_1 \cup ⋯ \cup τ_m \cup χ_{αβ} \cup σ_2,$$

where

- $σ_1$ (respectively $σ_2$) is contained in the boundary of a stripe of type $α^1$ and $∂σ_1 = \{ x, y \} \subset ∂D_α$ (respectively $∂σ_2 = \{ z, w \} \subset ∂D_α$).
− \( \sigma_{a\beta}, \chi_{a\beta} \) are properly embedded in \( C_{a,\beta} \) and \( \partial \sigma_{a\beta} = \{ y, z_1 \}, \partial \chi_{a\beta} = \{ z_2, z \} \) with \( z_1, z_2 \in ( \partial D_a \cup \partial D_\beta ) \),
− each \( \tau_i, i = 1, \ldots, m \) is either properly embedded in \( C_{a,\beta} \) with \( \partial \tau_i \in ( \partial D_a \cup \partial D_\beta ) \) or, is contained in the boundary of a stripe of type \( \alpha^0 \) (respectively \( \beta^0 \)) or, is contained in the boundary of an elementary disk \( \alpha^e \) (respectively \( \beta^e \)) with \( \partial \tau_i \in \partial D_\alpha \) (respectively \( \partial D_\beta \)).

We will first show that the arc \( \sigma_D \), equivalently the arc \( \sigma \), does not intersect \( \partial D_\beta \). Recall that \( \sigma \) forms a back and forth with a subarc of \( \partial D_\alpha \) and let \( D \) be the disk bounded by these two subarcs. For, if \( \sigma_D \cap \partial D_\beta \neq \emptyset \) (which means that the boundary curve \( \partial D_\beta \) intersects the disk \( D \)) we may choose a subarc \( \tau_0 \) of \( \sigma \) and a subarc \( \delta_0 \) of \( \partial D_\beta \) such that \( \tau_0 \cup \delta_0 \) forms an innermost 2-gon in \( D \). It is then clear that \( \tau_0 \) neither can be contained in the boundary of a stripe of type \( \alpha^0 \) nor can be contained in the boundary of an elementary disk \( \beta^e \). Therefore, \( \tau_0 \) is properly embedded in \( C_{a,\beta} \) with \( \partial \tau_0 \in \partial D_\beta \). Then, each of \( \tau_{i-1}, \tau_{i+1} \) must be the boundary of a stripe \( \beta^0 \) or an elementary disk \( \beta^e \). Now the union

\[
\tau_{i-1} \cup \tau_i \cup \tau_{i+1}
\]

forms a subarc of type (1) of Definition 12 which is a contradiction, since \( D_a, D_\beta \) is a pair of geometrically minimal cutting disks for \( S \).

Hence we may assume that \( z_1, z_2 \in \partial D_a \) and \( \partial \tau_i \in \partial D_a \) for all \( i = 1, \ldots, m \). In the case \( m = 1 \) we have that \( \sigma_D \) is of the form

\[
\sigma_D = \sigma_1 \cup \sigma_{a\beta} \cup \tau_1 \cup \chi_{a\beta} \cup \sigma_2
\]

where \( \tau_1 \) is, necessarily, contained in the boundary of a stripe of type \( \alpha^0 \) or, is contained in the boundary of an elementary disk \( \alpha^e \). Hence, \( \sigma_D \) a subarc of type (3) of Definition 12 which is a contradiction, since \( D_a, D_\beta \) is a pair of geometrically minimal cutting disks for \( S \).

In the case \( m \geq 2 \), each of \( \tau_i, \tau_m \) is, necessarily, contained in the boundary of a stripe of type \( \alpha^0 \) or, is contained in the boundary of an elementary disk \( \alpha^e \). This follows from the fact that \( \sigma \cap \partial D_\beta = \emptyset \) and, hence, \( (\partial \sigma_{a\beta} \cup \partial \chi_{a\beta}) \cap \partial D_\beta = \emptyset \). Now the subarc \( \tau_2 \) is properly embedded in \( C_{a,\beta} \) and, as above, the union

\[
\tau_1 \cup \tau_2 \cup \tau_3
\]

forms a subarc of type (1) of Definition 12 which is a contradiction. This completes the proof of the proposition. \( \square \)

6. Strong isotopies

In this section we will further analyze an arbitrary isotopy between two incompressible surfaces. We will introduce the notion of a strong isotopy as well as the notion of elementary isotopies and we will show (see Theorem 23) that, up to finite applications of elementary isotopies, any isotopy between two incompressible surfaces is, in fact, a strong isotopy.

Assume \( S \) is a surface obtained by the gluing construction. Choose points \( x_1, x_2, x_3 \) and \( x_4 \) in \( S \cap \text{Int} D_a \) and arcs \( \overline{x_1x_2}, \overline{x_3x_2}, \overline{x_3x_4} \), so that the subarc \( \overline{x_1x_2} \) traces a stripe \( \alpha^1 \) (respectively \( \alpha^0 \)), the subarc \( \overline{x_3x_2} \) belongs to a component (i.e., an embedded disk) in \( C_{a,\beta} \) and the subarc \( \overline{x_3x_4} \) traces a stripe \( \alpha^1 \) (respectively \( \alpha^0 \)), which is not the same stripe traced by \( \overline{x_1x_2} \), in the opposite direction (cf. proof of Theorem 11). Assume further that there exists an arc \( \sigma_{\overline{x_1x_4}} \subset D_a \) joining \( x_1x_4 \) such that \( \sigma_{\overline{x_1x_4}} \cap S = \{ x_1, x_4 \} \). In this setup (which, in fact, means that the stripes containing \( \overline{x_1x_2} \) and \( \overline{x_3x_4} \) are parallel) we will describe an isotopy on \( S \) which will be called elementary isotopy of type \( \alpha^1 \) (respectively of type \( \alpha^0 \)) and will be used later on.

Choose points \( z \) and \( y \) in \( S \cap \text{Int} C_{a,\beta} \) close to \( x_1 \) and \( x_4 \) respectively, and form an arc \( \rho = \overline{zx_1x_2x_3x_4y} \) so that the arc \( \overline{x_1x_2x_3x_4} \) is a subarc of \( \rho \). Let \( U \) be a closed subset of \( \text{Int} S \) containing \( \rho \) and homeomorphic to the product \( \rho \times [-1, 1] \) (we agree that the arc \( \overline{x_1x_2x_3x_4} \) is mapped onto \( \rho \times \{ 0 \} \)). \( \partial U \) is a loop consisting of four arcs namely, \( z \times [-1, 1], \rho \times \{ 1 \}, y \times [-1, 1] \) and \( \rho \times \{-1 \} \).

Consider embedded arcs \( v_{-1}, v_1 \) with endpoints \( z \times \{-1 \}, y \times \{-1 \} \) and \( z \times \{ 1 \}, y \times \{ 1 \} \) respectively, such that

− The juxtaposition of the arcs \( v_{-1}, z \times \{-1 \}, v_1, y \times \{-1 \} \) is an embedded loop in \( C_{a,\beta} \).
− This loop bounds an embedded disk, say \( D_0 \), lying entirely in \( C_{a,\beta} \), with the property \( D_0 \cap S = \{ z, y \} \times [-1, 1] \).

To see that the arcs \( v_{-1}, v_1 \) can be chosen so that the latter equality holds it is sufficient to check that the arcs \( v_{-1}, \)
Let now \( f : U \times I \to M \) be a proper isotopy which

- is the identity on the arcs \( \nu_{-1}, \nu_1, y \times [-1, 1] \),
- transforms the disk \( U = \rho \times [-1, 1] \) onto the union \( D_{-1} \cup D_0 \cup D_1 \),
- for some \( s \in (0, 1) \), \( f \) has the property
  \[
  f\left((\rho \times [-s, +s]) \times \{1\}\right) \subset D_0.
  \]

The above situation is sketched in Fig. 6. Define

\[
F : S \times I \to M
\]

as follows: \( F \) is the identity on \((S \setminus U) \times I\) and \( F = f \) on \( U \). In particular, \( F \) is the identity on \( \partial U \times I \).

**Definition 18.** The isotopy \( F : S \times I \to M \) defined above will be called an elementary isotopy of type \( \alpha^1 \) (respectively of type \( \alpha^0 \)). In an identical way an elementary isotopy of type \( \beta^1 \) (respectively of type \( \beta^0 \)) is defined.

It is clear from the construction of the elementary isotopy that

1. (E1) If the disks \( D_\alpha, D_\beta \) are cutting disks for the surface \( S \) which means, in particular, that properties \((s_\alpha), (s_\beta)\) and \((s_{\alpha, \beta})\) hold for \( S \) then \( D_\alpha, D_\beta \) are cutting disks for the resulting surface \( F(S \times \{1\}) \) and properties \((s_\alpha), (s_\beta)\) and \((s_{\alpha, \beta})\) also hold for \( F(S \times \{1\}) \).

2. (E2) The boundary \( \partial S \) of \( S \) (in fact, a tubular neighborhood of \( \partial S \)) is point-wise fixed throughout the isotopy.

3. (E3) If the elementary isotopy is applied to arcs \( x_1x_2 \) and \( x_3x_4 \) contained in a pair of stripes (of type either \( \alpha^1 \) or \( \alpha^0 \)) which is not a reducible pair (see notation following Definition 12), then the number stripes of \( S \times \{0\} \) is equal to the number of stripes of \( F(S \times \{1\}) \). Moreover, the number of arcs in \((S \times \{0\}) \cap D_\alpha\) is equal to the number of arcs in \( F(S \times \{1\}) \cap D_\alpha \) and similarly for \( D_\beta \). The way these arcs are changed by the elementary isotopy is shown in Fig. 7. If, in addition, the surface \( S \) is incompressible then the number of components in \((S \times \{0\}) \cap C_{\alpha, \beta}\) is equal to the number of components in \( F(S \times \{1\}) \cap C_{\alpha, \beta}\).

In particular, as reducible pairs of stripes do not exist when \( \alpha \) or, \( \alpha^0 \) which is not a reducible pair, then the number of stripes of \( S \), the number of components in \( S \cap (D_\alpha \cup D_\beta) \) and the number of components in \( S \cap C_{\alpha, \beta}\) is invariant under the application of an elementary isotopy.

4. (E4) If the elementary isotopy is applied to arcs \( x_1x_2 \) and \( x_3x_4 \) contained in a reducible pair (of type either \( \alpha^1 \) or \( \alpha^0 \)) then \( F(S \times \{1\}) \) can be isotoped in a trivial manner to a surface \( S' \) having one stripe less than \( S \). In particular, the number of arcs in \( S' \cap (D_\alpha \cup D_\beta) \) is two less than the number of arcs in \( S' \cap (D_\alpha \cup D_\beta) \).
Definition 19. A simplification of the above defined elementary isotopy can be applied to a surface $S$ in order to eliminate a stripe of type $\alpha^0$ (respectively $\beta^0$) and give rise to two elementary disks $\alpha^e$ (respectively $\beta^e$), see Remark 5, as follows: let $x_1, x_2, x_3$ and $x_4$ be points in $S \cap \text{Int} D_\alpha$ so that the arc $x_1 \overline{x_2 x_3 x_4}$ traces a single stripe $\alpha^0$ and, as above, form the arc $\rho = \overline{x_1 x_2 x_3 x_4 y}$ so that the arc $x_1 \overline{x_2 x_3 x_4}$ is a subarc of $\rho$. The definition of the elementary isotopy given above goes through in this new (and simpler) setup. It is clear that the resulting isotopy satisfies properties (E1), (E2) and (E3) above, hence, it will also be called an elementary isotopy of type $\alpha^e$ (respectively of type $\beta^e$).

We now define the notion of strong isotopy between two properly embedded surfaces in $M$. Start with a handlebody $M$ of genus 2 and fix two properly embedded disks $D_\alpha$ and $D_\beta$ satisfying property (c) of Definition 1. Recall that we denote by $S_{gm}$ the collection of connected surfaces obtained by the Main Construction with the additional requirements imposed by Definition 12. Let $S_0, S_1 \in S_{gm}$ be two surfaces. In particular, see Corollary 15, $D_\alpha$ and $D_\beta$ are geometrically minimal cutting disks for both $S_0, S_1$.

Definition 20. We will say that $S_0, S_1 \in S_{gm}$ are isotopic if and only if there exists a map $F : S_0 \times I \to M$ such that $F|_{S_0 \times \{0\}} = \text{Id}_{S_0}$, $F(S_0 \times \{1\}) = S_1$ and $F|_{S_0 \times \{t\}}$ is a proper embedding for all $t \in [0, 1]$.

We will say that $S_0, S_1 \in S_{gm}$ are strongly isotopic (briefly, s-isotopic) if and only if there exists a map $F$ as above which, in addition, satisfies

$$F(S_0 \times \{t\}) \in S_{gm}, \quad \forall t \in [0, 1].$$

In other words we require that for all times $t \in [0, 1]$, $D_\alpha$ and $D_\beta$ are cutting disks for the properly embedded surface $F(S_0 \times \{t\})$.

We will also need an other definition helpful in the study of strong isotopy classes of surfaces. Let $S \in S_{gm}$ and $D_\alpha, D_\beta$ geometrically minimal cutting disks.

Definition 21. The triplet of sets $(\partial S, S \cap D_\alpha, S \cap D_\beta)$ will be called the configuration of $S$ with respect to the cutting disks $D_\alpha$ and $D_\beta$. Notation, CONF($S, D_\alpha, D_\beta$).

Lemma 22. Let $S_0, S_1 \in S_{gm}$, $D_\alpha$ and $D_\beta$ be given as above. If $\partial S_0 = \partial S_1$ and the collection of arcs $S_0 \cap D_\alpha$ (respectively $S_0 \cap D_\beta$) is isotopic with endpoints fixed to the collection of arcs $S_1 \cap D_\alpha$ (respectively $S_1 \cap D_\beta$) then $S_0$ is s-isotopic to $S_1$.

In the case two surfaces $S_0, S_1$ satisfy the assumptions of the above lemma, we will be saying that the configuration of $S_0$ with respect to $D_\alpha, D_\beta$ is equivalent to the configuration of $S_1$ with respect to $D_\alpha, D_\beta$. Notation,

$$\text{CONF}(S_0, D_\alpha, D_\beta) \equiv \text{CONF}(S_1, D_\alpha, D_\beta).$$

Proof of Lemma 22. The isotopy given by the assumption may be extended to a strong isotopy

$$F_t : M \to M, \quad t \in [0, 1]$$

with the following properties
Proof. We now have that the boundary (which is a circle) of each component of $S_0 \cap C_{\alpha, \beta}$ is identical to the boundary of a component $S_1 \cap C_{\alpha, \beta}$. The components themselves intersect in circles forming 3-balls. Starting from the innermost such ball we may eliminate all such intersections so that an isotopy from $S_0 \cap C_{\alpha, \beta}$ onto $S_1 \cap C_{\alpha, \beta}$ will be straightforward. □

**Theorem 23.** Let $S_0, S_1$ be properly embedded incompressible surfaces in $M$ (hence elements of $S_{gm}$) obtained by the Main Construction. Then, $S_0$ is isotopic to $S_1$ if and only if there exist surfaces $S_0', S_1'$ such that

- $S_0', S_1'$ are s-isotopic.
- $S_0'$ (respectively $S_1'$) is obtained from $S_0$ (respectively $S_1$) by applying finitely many elementary isotopies.

**Proof.** Assume that $S_0, S_1$ are isotopic by an isotopy $F$. Apparently $F$ induces an isotopy

$$f : \partial S_0 \times [0, 1] \to \partial M$$

where $f(\partial S_0 \times \{0\}) = \partial S_0$ and $f(\partial S_0 \times \{1\}) = \partial S_1$.

Let $\partial S_0$ (respectively $\partial S_1$) be homeomorphic to the pairwise disjoint union of components $W_0^1, W_0^2, \ldots, W_0^k$ (respectively $W_1^1, W_1^2, \ldots, W_1^k$) where each component $W_0^i$ (respectively $W_1^i$) is homeomorphic to $S^1$ (where the index $k \geq 1$). After relabeling, if necessary, we may assume that $f(W_0^i \times \{1\}) = W_1^i$ for all $i = 1, 2, \ldots, k$.

**Claim 0.** We may alter $F$ and replace $S_0, S_1$ by surfaces $S_0', S_1'$ respectively so that $S_0$ is s-isotopic with $S_0'$, $S_1$ is s-isotopic with $S_1'$ and the boundary components of $S_0', S_1'$ (denoted again by $W_0^i, W_1^i$) satisfy $W_0^i \cap W_1^i = \emptyset$ for all $i, j = 1, 2, \ldots, k$. Moreover, if $A^i$ is the annulus bounded by the isotopic curves $W_0^i, W_1^i$, $i = 1, 2, \ldots, k$, then $A^i \cap A^j = \emptyset$ for $j \neq i$.

**Proof.** Assume that the components $W_0^i, W_1^j$ intersect for various values of $i, j$. We first claim that their intersection points come in pairs and form 2-gons. Recall that by saying a 2-gon we mean a disk $D \subset \partial M$ such that $\partial D = \partial_0 \cup \partial_1$ with $\partial_0 \subset W_0^i$ and $\partial_1 \subset W_1^j$. To see this, let $i_0, j_0$ be indices for which $W_0^{i_0} \cap W_1^{j_0}$ intersect. If the closed curves $W_0^{i_0}, W_1^{j_0}$ are isotopic then it follows from [3] that their intersection points come in pairs and form 2-gons. Suppose that $W_0^{i_0}, W_1^{j_0}$ are not isotopic and there exists an intersection point which is not the vertex of a 2-gon formed by $W_0^{i_0}, W_1^{j_0}$. Then every closed curve isotopic to $W_1^{j_0}$ must have such an intersection point with $W_0^{i_0}$. This must be true, in particular, for the component $W_0^{i_j}$ where $i_j$ is the unique index for which $f(W_0^{i_j} \times \{1\}) = W_1^{j_0}$. We have reached a contradiction since distinct boundary components of $S_0$ are disjoint.

Let $i_s, j_s$ be indices so that $W_0^{i_s}, W_1^{j_s}$ form a 2-gon which is innermost among all 2-gons formed by $W_0^i, W_1^j$ for $i, j = 1, 2, \ldots, k$. Such an innermost 2-gon cannot be intersected by any component $W_0^i$ or $W_1^j$ for all $i \neq i_s$ and $j \neq j_s$. To see this assume that a component $W_0^{i_{s'}}$ intersects the 2-gon formed by $W_0^{i_s}$ and $W_1^{j_s}$. Since $W_0^{i_s} \cap W_0^{i_{s'}} = \emptyset$, $W_0^{i_{s'}}$ can only intersect $W_1^{j_s}$ hence, $W_0^{i_{s'}}$ and $W_1^{j_s}$ form a 2-gon which lies inside the 2-gon formed by $W_0^{i_s}$ and $W_1^{j_s}$ contradicting its innermost property.

Starting with the innermost 2-gon formed by $W_0^{i_s}, W_1^{j_s}$ we can alter $S_0, S_1$ and $F$ so that the 2-gon is eliminated. This elimination can be done so that the number of intersection points of each side of the innermost 2-gon with $\partial D_{\alpha}$ (respectively $\partial D_{\beta}$) is kept constant. It is then easy to see that the resulting, after elimination, surfaces are s-isotopic with the original ones. Continuing in this way we may eliminate all 2-gons. This completes the first part of the claim.

It is clear that for each $i = 1, \ldots, k$, $W_0^i, W_1^i$ are isotopic, hence by [2], bound an annulus $A^i$. Although $W_0^i \cap W_1^i = \emptyset$, for all $i, j$ it is not true that $A^i \cap A^j = \emptyset$ for $j \neq i$. However, we may alter $S_0, S_1$ and $F$ in order...
to make the intersection $A^i \cap A^j$ empty. Moreover, we can take care so that the number of components of $A^i \cap \partial D_{\alpha}$ and $A^j \cap \partial D_{\alpha}$ (respectively $A^i \cap \partial D_{\beta}$ and $A^j \cap \partial D_{\beta}$) is kept constant. The latter asserts, as before, that the resulting surfaces are 1-isotopic with the original ones. After a finite number of steps we obtain that all intersections $A^i \cap A^j$ are empty. This completes the proof of Claim 0. □

By standard isotopy extension properties, see for example [5, Theorem 1.3, Chapter 8], $f$ extends to an ambient isotopy on $\partial M$

$$G : \partial M \times [0, 1] \to \partial M$$

satisfying $G|_{\partial M \times \{0\}} = \text{Id}$ and $G(\partial S_0 \times \{1\}) = \partial S_1$.

Putting together the isotopies $F$ and $G$ we obtain an isotopy

$$H : (S_0 \cup \partial M) \times [0, 1] \to M$$

such that $H|_{S_0 \times [0, 1]} = F$ and $H|_{\partial M \times [0, 1]} = G$.

**Claim 1.** We may assume that $H$ is an isotopy keeping $\partial D_{\alpha}$ and $\partial D_{\beta}$ invariant.

**Proof.** Thicken the boundary $\partial M$ of $M$ by considering the product space $\partial M \times [0, \varepsilon]$ and identifying $\partial M$ with $\partial M \times \{0\}$. We will show the claim by extending $H$ to an isotopy defined on $(S_0 \cup (\partial M \times [0, \varepsilon])) \times [0, 1]$ which keeps $\partial D_{\alpha}$, $\partial D_{\beta} \subseteq \partial M \times [0, \varepsilon]$ invariant.

We have the isotopy $G_0 = G$ on $\partial M \times \{0\}$ and we proceed to construct an isotopy

$$G_\varepsilon : (\partial M \times \{\varepsilon\}) \times [0, 1] \to \partial M \times \{\varepsilon\}$$

satisfying

(S) $G_\varepsilon((\partial S_0 \times \{\varepsilon\}) \times \{0\}) = \partial S_0 \times \{\varepsilon\}$, $G_\varepsilon((\partial S_0 \times \{\varepsilon\}) \times \{1\}) = \partial S_1 \times \{\varepsilon\}$

and

(D) $\partial D_{\alpha} \times \{\varepsilon\}$ and $\partial D_{\beta} \times \{\varepsilon\}$ are invariant under $G_\varepsilon$.

**Subclaim.** For each $i = 1, 2, \ldots, k$ the cardinality of the set $W_0^i \cap \partial D_{\alpha}$ (respectively $W_0^i \cap \partial D_{\beta}$) is equal to the cardinality of the set $W_1^i \cap \partial D_{\alpha}$ (respectively $W_1^i \cap \partial D_{\beta}$).

Moreover, there exist natural 1–1 correspondences

$$f_\alpha : W_0^i \cap \partial D_{\alpha} \to W_1^i \cap \partial D_{\alpha} \quad \text{and} \quad f_\beta : W_0^i \cap \partial D_{\beta} \to W_1^i \cap \partial D_{\beta}$$

so that $f_\alpha(x) = F(x, 1)$ and similarly for $f_\beta$.

**Proof.** We will prove the subclaim for the components $W_0^1, W_1^1$. Recall that the components $W_0^1, W_1^1$ have the property $G(W_0^1 \times \{1\}) = W_1^1$. The proof is identical for all pairs $W_0^i, W_1^i, i = 2, \ldots, k$. Since $D_{\alpha}, D_{\beta}$ are cutting disks for both $S_0, S_1$ we may decompose $W_0^1$ (respectively $W_1^1$) into subarcs $w_{01}, w_{02}, \ldots, w_{0m_0}$ (respectively $w_{11}, w_{12}, \ldots, w_{1m_1}$) such that

$$\bigcup_{i=1}^{m_0} w_{0i} = W_0^1, \quad \partial w_{0i} \in \partial D_{\alpha} \cup \partial D_{\beta} \quad \text{and} \quad \text{Int } w_{0i} \cap (\partial D_{\alpha} \cup \partial D_{\beta}) = \emptyset, \quad \forall i = 1, \ldots, m_0,$$

where each $w_{0i}$ belongs to one of the following types:

(a) $w_{0i}$ is contained in the boundary of a stripe of type $\alpha^1$ or, a stripe of type $\alpha^0$ or, an elementary disk $\alpha^e$.

(b) $w_{0i}$ is contained in the boundary of a stripe of type $\beta^1$ or, a stripe of type $\beta^0$ or, an elementary disk $\beta^e$.

(C$\alpha$) $w_{0i} \subseteq C_{\alpha}$ with $w_{0i} \cap \partial D_{\beta} = \emptyset$.

(C$\beta$) $w_{0i} \subseteq C_{\beta}$ with $w_{0i} \cap \partial D_{\beta} = \emptyset$.

(d) $w_{0i}$ is neither of type (C$\alpha$), nor of type (C$\beta$) but it is contained in the 3-ball $C_{\alpha, \beta}$. 
Identical properties are required to hold for the subarcs \( w_{11}, w_{12}, \ldots, w_{1m_1} \) which decompose \( W_1' \). Note that each arc of type \((C_\alpha) \) and \((C_\beta) \) forms a back and forth with \( \partial D_\alpha \) and \( \partial D_\beta \) respectively. Moreover, all such arcs are contained in \( C_{\alpha, \beta} \). Pick a subarc of type \((C_\alpha) \), say \( w_{0i0} \), which forms a back and forth with \( \partial D_\alpha \) which is innermost among all such. By eliminating the back and forth of \( w_{0i0} \) with \( \partial D_\alpha \) we may isotope the union \( w_{0(i_0-1)} \cup w_{0i0} \cup w_{0(i_0+1)} \) into an arc \( w_{0i0} \) with

\[
 w_{0i0} \subset T_\alpha \quad \text{and} \quad \partial w_{0i0} \in \partial D_\alpha.
\]

By the properties of the decomposition above, the arcs \( w_{0(i_0-1)}, w_{0(i_0+1)} \) are of type \((\alpha) \). Since \( D_\alpha \) is geometrically minimal, \( w_{0i0} \) and \( \partial D_\alpha \) do not form a back and forth. We proceed and eliminate all back and forths formed by arcs of type \((C_\alpha) \) with \( \partial D_\alpha \) and \((C_\beta) \) with \( \partial D_\beta \). We then obtain a simple closed curve \( W_0' \) (respectively \( W_1' \)) which is freely isotopic to \( W_0' \) (respectively \( W_1' \)) and a decomposition of \( W_0' \) (respectively \( W_1' \)) into subarcs \( w_{0i1}', w_{0i2}', \ldots, w_{0im_0}' \) (respectively \( w_{1i1}', w_{1i2}', \ldots, w_{1im_1}' \)) such that

\[
 \bigcup_{i=1}^{m_0'} w_{0i}' = W_0', \quad \partial w_{0i}' \in \partial D_\alpha \cup \partial D_\beta \quad \text{and} \quad \text{Int} w_{0i}' \cap (\partial D_\alpha \cup \partial D_\beta) = \emptyset, \quad \forall i = 1, \ldots, m_0',
\]

and each \( w_{0i}' \) belongs to one of the following types:

1. \((\alpha')\) \( w_{0i}' \subset T_\alpha \).
2. \((\beta')\) \( w_{0i}' \subset T_\beta \).
3. \((d)\) \( w_{0i}' \) is contained in the 3-ball \( C_{\alpha, \beta} \) and \( \partial w_{0i}' \) intersects both \( \partial D_\alpha \), \( \partial D_\beta \).

Identical properties hold for the subarcs \( w_{1i1}', w_{1i2}', \ldots, w_{1im_1}' \) which decompose \( W_1' \). Since the arcs of type \((\alpha')\), \((\beta')\) do not form a back and forth with \( \partial D_\alpha \), \( \partial D_\beta \) it is clear that each of the curves \( W_0', W_1' \) has the minimal (in their isotopy class) number of intersection points with \( \partial D_\alpha \cup \partial D_\beta \). In other words, \( m_0' = m_1' \) or, equivalently, the first part of the subclaim holds for the curves \( W_0', W_1' \). Moreover, \( W_0', W_1' \) bound an annulus which is cut into stripes by \( \partial D_\alpha \), \( \partial D_\beta \) with each such stripe being contained in \( T_\alpha \), \( T_\beta \) or \( C_{\alpha, \beta} \). This is also a consequence of the fact that \( W_0', W_1' \) do not form a back and forth with \( \partial D_\alpha \), \( \partial D_\beta \). Therefore, for each subarc \( w_{0i}' \) of \( W_0' \) there corresponds a subarc, \( w_{1i}' \), of \( W_1' \) so that \( w_{0i}' \), \( w_{1i}' \) are properly (with respect to their boundary points) isotopic. In particular, the second part of the subclaim holds for the curves \( W_0', W_1' \), i.e., there exist 1–1 correspondences

\[
 f'_\alpha: W_0' \cap \partial D_\alpha \rightarrow W_1' \cap \partial D_\alpha \quad \text{and} \quad f'_\beta: W_0' \cap \partial D_\beta \rightarrow W_1' \cap \partial D_\beta.
\]

We may view \( D_\alpha \) as a subset of \( \partial T_\alpha \) and collapse it to a point, say \( *_\alpha \). We then get a torus, again denoted by \( \partial T_\alpha \), with base point \( *_\alpha \) and all arcs of type \((\alpha')\) are now loops based at \( *_\alpha \). An arc of type \((\alpha)\) which belongs to the boundary of a stripe of type \( \alpha^1 \) is now a loop based at \( *_\alpha \) and we denote by \((\alpha^1, 0)\) its homotopy class in \( \pi_1(\partial T_\alpha, *_\alpha) \). An arc of type \((\alpha)\) which belongs to the boundary of a stripe of type \( \alpha^0 \) or, an elementary disk \( \alpha^e \), is now a loop based at \( *_\alpha \) and denote by \((0, \alpha^e)\) its homotopy class in \( \pi_1(\partial T_\alpha, *_\alpha) \). Apparently, \( \pi_1(\partial T_\alpha, *_\alpha) \) is the free Abelian group on \((\alpha^1, 0)\) and \((0, \alpha^e)\).

Consider now the three subarcs \( w_{0(i_0-1)} \), \( w_{0i0} \) and \( w_{0(i_0+1)} \) of \( W_1' \), mentioned above, with \( w_{0i0} \) forming an innermost back and forth with \( \partial D_\alpha \) (similarly for \( \partial D_\beta \)). By eliminating this back and forth we obtain an arc \( w_{0i0}' \) which, in the torus \( (\partial T_\alpha, *_\alpha) \), is a loop based at \( *_\alpha \). Since \( w_{0i0}' \) and \( \partial D_\alpha \) do not form a back and forth we have that the homotopy class \([w_{0i0}']\) of \( w_{0i0}' \) in \( \pi_1(\partial T_\alpha, *_\alpha) \) is given by

\[
 [w_{0i0}'] = \kappa(\alpha^1, 0) + \lambda(0, \alpha^e),
\]

where \(|\kappa| \leq 1, |\lambda| \leq 1\) and \(|\kappa| + |\lambda| = 2\).

We may show inductively that for each arc \( w_{0i}' \) of type \((\alpha')\) (respectively of type \((\beta')\)) in the decomposition of \( W_0' \), its homotopy class \([w_{0i}']\) in \( \pi_1(\partial T_\alpha, *_\alpha) \) (respectively in \( \pi_1(\partial T_\beta, *_\beta) \)) is given by

\[
 [w_{0i}'] = \kappa_0(\alpha^1, 0) + \lambda_0(0, \alpha^e) \quad \text{respectively} \quad [w_{0i}'] = \kappa_0(\beta^1, 0) + \lambda_0(0, \beta^e), \quad (H)
\]
where $|\kappa_0| + |\lambda_0|$ is the number of arcs of type $(\alpha)$ (respectively of type $(\beta)$) used to produce $w^0_{\lambda_0}$ (after eliminating back and forths).

For, if $w^i_{\lambda_0}$ and $w^0_{\lambda_0}$ are three consecutive subarcs with $w^i_{\lambda_0}$ forming an innermost back and forth with $\partial D_\alpha$ whose elimination gives rise to a subarc $w^i_{\lambda_0}$ of type $(\alpha')$ and

$$[w^i_{\lambda_0}] = \kappa_{i0}(\alpha^1, 0) + \lambda_{i0}(0, \alpha^\epsilon)$$

and

$$[w^0_{\lambda_0}] = \kappa_{0}(\alpha^1, 0) + \lambda_{0}(0, \alpha^\epsilon)$$

we must show that $\kappa_{i0} + \kappa_{0} = \kappa_{i0}$ and $\lambda_{i0} + \lambda_{0} = \lambda_{i0}$. If $\kappa_{i0}$, $\kappa_{0}$ are both nonnegative or both nonpositive then the equation $\kappa_{i0} + \kappa_{0} = \kappa_{i0}$ holds trivially in the Abelian group $\pi_1(\partial T_\alpha, \ast_\alpha)$. If $\kappa_{i0} \cdot \kappa_{0} < 0$ then there must exist a subarc of the original component $W^0_1$ forming a back and forth with respect to $\partial D_\alpha$. As $D_\alpha, D_\beta$ are geometrically minimal this is impossible by Proposition 17. To see that the equation $\lambda_{i0} + \lambda_{0} = \lambda_{i0}$ holds, in a similar manner, that if $\lambda_{i0} \cdot \lambda_{0} < 0$ then a subarc of the component $W^0_1$ must be of type (2) of Definition 12 which is impossible since $D_\alpha, D_\beta$ are geometrically minimal.

A statement identical to (H) holds for each arc $w^i_{\lambda_0}$ of type $(\alpha')$ (respectively of type $(\beta')$) in the decomposition of $W^i_1$. Recall that $m^i_0 = m^i_1$ and, since $w^i_{\lambda_0}, w^i_{\lambda_1}$ are properly isotopic, it follows that

$$|\kappa_0| = |\kappa_1|$$

and

$$|\lambda_0| = |\lambda_1|, \quad \forall i = 1, \ldots, m^i_0.$$

The latter implies that the number of arcs of type $(\alpha)$ (respectively of type $(\beta)$) used to produce $w^i_{\lambda_0}$ is equal to the number of arcs of type $(\alpha)$ (respectively of type $(\beta)$) used to produce $w^i_{\lambda_1}$. Moreover, the number of arcs of type $(C_\alpha)$ (respectively of type $(C_\beta)$) are also equal. Since these statements hold for all $i = 1, \ldots, m^i_0$ and $m^i_0 = m^i_1$ it follows that $m_0 = m_1$ as required.

To obtain the maps $f_\alpha$ and $f_\beta$ all we have to do is to use the maps $f'_\alpha$, $f'_\beta$ and trace back the isotopies performed on $w^0_0, w^0_{\lambda_0}, w^1_1, w^1_{\lambda_1}$ in order to obtain the curves $W^0_0$ and $W^1_1$. This completes the proof of the subclaim for the boundary components $W^0_0, W^1_1$ and we work similarly for the pairs $W^i_0, W^i_1, i = 2, \ldots, k$. \Box

We continue to complete the proof of Claim 1. Using the maps $f_\alpha$ and $f_\beta$ as a guide we may construct an ambient isotopy

$$G_\varepsilon : (\partial M \times \{\varepsilon\}) \times [0, 1] \to \partial M \times \{\varepsilon\}$$

satisfying properties (S) and (D) above. Denote by $g_0$ (respectively $g_\varepsilon$) the restriction of $G_\varepsilon$ (respectively $G_\varepsilon$) on $\partial S_0 \times [0]$ (respectively $\partial S_0 \times \{\varepsilon\}$). Hence we have an isotopy

$$g_0 \cup g_\varepsilon : \left(\left(\partial S_0 \times \{0\}\right) \cup \partial S_0 \times \{\varepsilon\}\right) \times [0, 1] \to \partial M \times [0, \varepsilon].$$

The isotopy $g_0 \cup g_\varepsilon$ is the restriction of the isotopy $G_0 \cup G_\varepsilon$ which is ambient (on $\partial M \times \{0, \varepsilon\}$). Hence, the isotopy $g_0 \cup g_\varepsilon$ can be extended on a set of the form $\partial M \times (\{0, \lambda\} \cup \{\lambda', \varepsilon\})$ for some $\lambda, \lambda'$ with $0 < \lambda < \lambda' < \varepsilon$. In other words, $g_0 \cup g_\varepsilon$ is locally ambient (in $\partial M \times [0, \varepsilon]$). By the isotopy extension theorem (see [8, Theorem 4.24]) $g_0 \cup g_\varepsilon$ is ambient in $\partial M \times [0, \varepsilon]$. In particular, for each pair of boundary components $W^i_0, W^i_1, i = 1, 2, \ldots, k$, there exists a family of annuli $A^i_t, t \in [0, \varepsilon]$ such that $g_0(W^i_0 \times \{t\})$ and $g_\varepsilon(W^i_0 \times \{t\})$ are the two boundary components of $A^i_t$. This family of annuli permits us to extend the isotopy $H$ to an isotopy defined on $(S_0 \cup (\partial M \times [0, \varepsilon])) \times [0, 1]$ satisfying the desired property namely, keeps $\partial D_\alpha$ and $\partial D_\beta$ invariant. This completes the proof of Claim 1. \Box

**Claim 2.** There is a surface $S'_0$ which is $s$-isotopic to $S_0$ such that $\partial S'_0 = \partial S_1$. Hence, we may assume that $\partial S_0 = \partial S_1$. Moreover, we may assume that the common boundary of $S_0$ and $S_1$ is point-wise fixed by $H$.

**Proof.** Start with the set $\partial S_1$ and form a surface $S'_0$ by specifying that its boundary $\partial S'_0$ is equal to $\partial S_1$ and by declaring which points in $\partial S'_0 \cap (D_\alpha \cup D_\beta)$ belong to the same embedded disk in $C_{\alpha, \beta}$. To do the latter, we use the 1–1 correspondences

$$f_\alpha : \partial S_0 \cap \partial D_\alpha \to \partial S_1 \cap \partial D_\alpha \quad \text{and} \quad f_\beta : \partial S_0 \cap \partial D_\beta \to \partial S_1 \cap \partial D_\beta$$

explained in the proof of Claim 1 above. For short notation, we write $c$ for the map

$$f_\alpha \cup f_\beta : \partial S_0 \cap (D_\alpha \cup D_\beta) \to \partial S_1 \cap (D_\alpha \cup D_\beta)$$
induced by \(c_\alpha\) and \(c_\beta\). Then, two points \(x, y \in \partial S_0' \cap (D_\alpha \cup D_\beta) = \partial S_1 \cap (D_\alpha \cup D_\beta)\) belong to the same embedded disk in \(C_{\alpha,\beta}\) if and only if \(c^{-1}(x)\), \(c^{-1}(y)\) belong to the same embedded disk in \(C_{\alpha,\beta}\) for the surface \(S_0, S_0'\) is by construction the desired surface. To complete the proof of the claim we need to alter \(H\) so that the common boundary \(\partial S_0 = \partial S_1\) is invariant. This can be done by a process identical to the one performed in Claim 1 except that \(G_\varepsilon\) (and, hence, \(g_\varepsilon\)) is the constant isotopy. This completes the proof of Claim 2.

**Claim 3.** We may further assume that for each \(t \in [0, 1]\), the surface \(S_t = H(S_0 \times \{t\})\) has the property that \(S_t \cap D_\alpha\) (respectively \(S_t \cap D_\beta\)) is a union of arcs with at most two arcs having exactly one point in common.

**Proof.** For each \(t \in [0, 1]\) we can find a tubular neighborhood \(U_t \subset M\) of \(S_t\) diffeomorphic to \(S_t \times (-1, 1)\) and an open interval \(J_t \subset [0, 1]\) such that: \(\forall t' \in J_t\) the surface \(S_{t'}\) lies in \(U_t\) and is parallel to \(S_t\), i.e., each fiber \(\{x\} \times (-1, 1), x \in S_t\) intersects \(S_{t'}\) at a single point. By compactness of \([0, 1]\), we may find \(t_0 = 0 < t_1 < \cdots < t_n = 1\) so that \(\bigcup_{i=1}^n J_{t_i} \supset [0, 1]\). Furthermore, we may assume that \(D_\alpha\) and \(D_\beta\) are transverse to \(S_{t_i}\) for \(i = 0, 1, \ldots, n\). This can be done by slightly perturbing the disks \(D_\alpha\) and \(D_\beta\) or, equivalently, by slightly altering the isotopy.

By moving along the fibers \(\{x\} \times (-1, 1)\) we can construct an isotopy \(K_0 : S_0 \times [0, 1] \to U_0\) such that

- each fiber \(\{x\} \times (-1, 1), x \in S_0\) of \(U_0\) intersects the surface \(K_0(S_0 \times \{t\})\) transversely at a single point, for every \(t \in [0, 1]\).
- \(K_0|_{S_0 \times [0, 1]}\) and \(K_0|_{S_0 \times [\frac{1}{2}, 1]}\) are embeddings.
- \(K_0|_{S_0 \times \{0\}} = \text{Id}, K_0(S_0 \times \{\frac{1}{2}\}) \cap S_1 = \emptyset\) and \(K_0(S_0 \times \{1\}) = S_1\).
- \(\partial D_\alpha\) and \(\partial D_\beta\) are transverse to \(K_0(\partial S_0 \times \{t\})\) for all \(t \in [0, 1]\).

The surfaces \(K_0(S_0 \times \{t\}), t \in [0, 1]\) induce a partial foliation on a compact subset of \(M\). By a standard procedure (see for example [1, Chapter 7]) we may slightly perturb the disks \(D_\alpha\) and \(D_\beta\) or, equivalently, perturb the isotopy, in order to make the disks \(D_\alpha\) and \(D_\beta\) in general position with respect to the foliation. In other words, these surfaces foliate \(D_\alpha\) and \(D_\beta\) and the singularities of this foliation are centers or saddle points. A further perturbation we may assume that each pair of the induced leaves on \(D_\alpha\) (respectively \(D_\beta\)) have at most one point in common. By the process described in the proof of property (d) in Section 2 we may eliminate all centers. Therefore, \(K_0\) provides an isotopy from \(S_0\) to \(S_1\) which satisfies the conclusion of Claim 3. We work similarly for all times \(t_1, \ldots, t_n = 1\) and put together the corresponding isotopies to complete the proof of the claim. □

In the sequel, we may assume given surfaces \(S_0, S_1\), with respect to the cutting disks \(D_\alpha, D_\beta\), have no stripes of type \(\alpha^0\) neither of type \(\beta^0\). We can do this because we can apply elementary isotopies of type \(\alpha^e, \beta^e\) with respect to the cutting disks \(D_\alpha, D_\beta\) (see Definition 19) in order to obtain surfaces \(S'_0, S'_1\), which, with respect to the cutting disks \(D_\alpha, D_\beta\), have no stripes of type \(\alpha^0\) nor of type \(\beta^0\). Moreover, under these elementary isotopies, the boundaries of \(S_0, S_1\) remain fixed. Hence, the cutting disks \(D_\alpha, D_\beta\) are geometrically minimal for both \(S'_0, S'_1\) so the assumption of the theorem is satisfied.

Choose properly embedded disks \(\Delta_\alpha, \Delta_\beta\) satisfying properties (\(\Delta 1\))--(\(\Delta 4\)) (see Section 5). Following the same procedure as in Claim 1 we may alter \(H\) so that

(\(\Delta 5\)) \(\partial \Delta_\alpha\) and \(\partial \Delta_\beta\) are invariant under \(H\).

This latter property combined with the invariance of \(\partial S_0 = \partial S_1\) under \(H\) implies that

(\(\Delta 6\)) the (finite, discrete) set of points \(\partial \Delta_\alpha \cap \partial S_0\) is fixed by \(H\) for all \(t \in [0, 1]\) and similarly for the set \(\partial \Delta_\beta \cap \partial S_0\).

Claim 3 applies verbatim to the disks \(\Delta_\alpha\) and \(\Delta_\beta\). In other words, following the same procedure as above we may assume that

(\(\Delta 7\)) for all \(t \in [0, 1]\), \(H(S_0 \times \{t\}) \cap (\Delta_\alpha \cup \Delta_\beta)\) is a union of arcs with at most two arcs having one point in common.
Let \( t_1, t_2, \ldots, t_m \) be the distinct times in \((0, 1)\) such that \( H(S_0 \times \{t_i\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) is a union of arcs with exactly two arcs having one point in common. Denote by \( \mu_i, \nu_i \) the arcs in \( H(S_0 \times \{t_i\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) having one point in common. Choose small positive numbers \( \varepsilon_i, i = 1, 2, \ldots, m \) so that
\[
\varepsilon_1 < t_1, \quad \varepsilon_m < 1 - t_m \quad \text{and} \quad t_i + \varepsilon_i < t_{i+1} - \varepsilon_{i+1} \quad \text{for all} \ i = 1, 2, \ldots, m - 1,
\]
and set \( t_i^- = t_i + \varepsilon_i, t_i^+ = t_i - \varepsilon_i \).

We start with \( t_1 \), i.e., the first time an intersection point between the arcs in \( H(S_0 \times \{t\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) occurs.

Without loss of generality, we may assume that \( \mu_1, \nu_1 \) belong to \( H(S_0 \times \{t_1\}) \cap \Delta_\alpha \) and let \((\alpha^1)_\mu_1, (\alpha^1)_\nu_1 \) be the stripes of type \( \alpha^1 \) uniquely determined by the arcs \( \mu_1, \nu_1 \), respectively.

**Notation 24.** The disks \( D_\alpha, D_\beta \) decompose \( M \) into a 3-ball \( C_{\alpha, \beta} \), a torus \( T_\alpha \) containing \( \Delta_\alpha \), and a torus \( T_\beta \) containing \( \Delta_\beta \). Set \( C_\alpha = T_\alpha \cup C_{\alpha, \beta} \) and \( C_\beta = T_\beta \cup C_{\alpha, \beta} \).

There exists a unique component of \( H(S_0 \times \{t_1^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \), say \( K_{\mu_1, \nu_1} \), such that \( K_{\mu_1, \nu_1} \cap C_\alpha \) contains an embedded arc in \( C_{\alpha, \beta} \) joining \((\alpha^1)_\mu_1\) and \((\alpha^1)_\nu_1\). We want to use this arc to perform an elementary isotopy on the stripes \((\alpha^1)_\mu_1\) and \((\alpha^1)_\nu_1\).

To do this we also need an embedded arc \( \sigma_1 \subset D_\alpha \) joining the arcs \((\alpha^1)_\mu_1 \cap D_\alpha\) and \((\alpha^1)_\nu_1 \cap D_\alpha\) with \( \delta_1 \cap S_0 = \emptyset \). To check the existence of such an arc observe that there exists an embedded arc \( \sigma \subset \Delta_\alpha \) which does not intersect any of the arcs in \( S_0 \cap \Delta_\alpha \) (otherwise \( \mu_1, \nu_1 \) would not realize the first, with respect to time, intersection point in \( H(S_0 \times [0, 1]) \cap (\Delta_\alpha \cup \Delta_\beta) \)). Since the collection of arcs \( S_0 \cap \Delta_\alpha \) has remained unchanged, with respect to isotopy with endpoints fixed, during the time interval \([0, t_1)\) it follows that the required embedded arc \( \sigma_1 \subset D_\alpha \) with \( \delta_1 \cap S_0 = \emptyset \) exists. We now perform an elementary isotopy on the stripes \((\alpha^1)_\mu_1\) and \((\alpha^1)_\nu_1\) and, thus, we obtain an isotopy
\[
\Phi_{t_1} : S_0 \times [0, t_1^+] \to M
\]
such that

\[
A(t_1) : D_\alpha, D_\beta \text{ are cutting disks for the surface } \Phi_{t_1} : S_0 \times \{t_1^+\},
\]
\[
B(t_1) : \text{the collection of (disjoint) arcs } \Phi_{t_1} : (S_0 \times \{t_1^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \text{ is isotopic to the collection } H(S_0 \times \{t_1^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \text{ with end points fixed},
\]
\[
C(t_1) : \text{the isotopy in property } B(t_1) \text{ extends to a component-wise homeomorphism between}
\]
\[
\Phi_{t_1} : (S_0 \times \{t_1^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \quad \text{and} \quad H(S_0 \times \{t_1^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta).
\]

Note that for the time interval \([t_1^+, t_2^-]\) the components of \( H(S_0 \times \{t\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \) remain, up to homeomorphism, invariant under \( H \), hence, property \( C(t_1) \) above provides a component-wise homeomorphism
\[
D(t_1) : \Phi_{t_1} : (S_0 \times \{t_1^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \to H(S_0 \times \{t_2^-\}) \setminus (\Delta_\alpha \cup \Delta_\beta).
\]

We now look at the arcs \( \mu_2, \nu_2 \in H(S_0 \times \{t_2\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) having one point in common. If \( \mu_2, \nu_2 \in H(S_0 \times \{t_2\}) \cap \Delta_\alpha \) (respectively \( H(S_0 \times \{t_2\}) \cap \Delta_\beta \) there exists a unique component of \( H(S_0 \times \{t_2^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \), say \( K_{\mu_2, \nu_2} \), such that \( K_{\mu_2, \nu_2} \cap C_\alpha \) (respectively \( K_{\mu_2, \nu_2} \cap C_\beta \)) contains a path in \( C_{\alpha, \beta} \) joining \((\alpha^1)_\mu_2\) and \((\alpha^1)_\nu_2\) (respectively joining \((\beta^1)_\mu_2\) and \((\beta^1)_\nu_2\)).

By the homeomorphism in property \( D(t_1) \), \( K_{\mu_2, \nu_2} \) determines a unique component in \( \Phi_{t_1} : (S_0 \times \{t_1^+\}) \setminus (\Delta_\alpha \cup \Delta_\beta) \) along with a path joining \((\alpha^1)_\mu_2\) and \((\alpha^1)_\nu_2\) (respectively joining \((\beta^1)_\mu_2\) and \((\beta^1)_\nu_2\)). In order to perform an elementary isotopy on the stripes, say, \((\alpha^1)_\mu_2\) and \((\alpha^1)_\nu_2\) (we work similarly with \((\beta^1)_\mu_2\) and \((\beta^1)_\nu_2\)) we need, as before, an embedded arc \( \sigma_2 \subset D_\alpha \) joining the arcs \((\alpha^1)_\mu_2 \cap D_\alpha\) and \((\alpha^1)_\nu_2 \cap D_\alpha\) with \( \delta_2 \cap S_0 = \emptyset \). Equivalently, we need to show the existence of an arc embedded in \( \Delta_\alpha \) joining \( \mu_2 \) with \( \nu_2 \) whose interior does not intersect any of the arcs in the collection \( \Phi_{t_1} : (S_0 \times \{t_1^+\}) \cap \Delta_\alpha \). If the latter were not true, then there must exist a stripe \((\alpha^1)_\nu_2\) so that the arc \( t_2 = (\alpha^1)_\nu_2 \cap \Delta_\alpha \) decomposes \( \Delta_\alpha \) into two half-disks, one containing \( \mu_2 \) and the other containing \( \nu_2 \). Using property \( B(t_1) \) we can see that this is impossible as \( \mu_2 \) and \( \nu_2 \) will intersect at time \( t^1 = t_2 \) and no other arcs in
which satisfies the following properties

\[ H_t(S_0 \times \{t\}) \cap (\Delta_\alpha \cup \Delta_\beta) \] will intersect for \( t < t_2 \). We may now perform an elementary isotopy on the stripes \((\alpha^1)_{\mu_2}\) and \((\alpha^1)_{\nu_2}\) (respectively \((\beta^1)_{\mu_2}\) and \((\beta^1)_{\nu_2}\)) in order to extend \( \Phi_t \) to an isotopy

\[ \Phi_{t_2} : S_0 \times [0, t_2^+] \to M \]

which satisfies the following properties

\[ A(t_2) : D_\alpha, D_\beta \text{ are cutting disks for the surface } \Phi_{t_2}(S_0 \times \{t_2^+\}), \]

\[ B(t_2) : \text{the collection of (disjoint) arcs } \Phi_t(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \text{ is isotopic to the collection } H(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \text{ with end points fixed,} \]

\[ C(t_2) : \text{the isotopy in property } B(t_2) \text{ extends to a component-wise homeomorphism between} \]

\[ \Phi_{t_2}(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \] and \( H(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta), \]

\[ D(t_2) : \Phi_{t_2}(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) = H(S_0 \times \{t_2^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \]

is component-wise homeomorphic induced by the homomorphism in \( C(t_2) \).

Proceeding in the same way we obtain, after \( m \) steps, an isotopy

\[ \Phi = \Phi_{t_m} : S_0 \times [0, t_m^+] \to M. \]

Since no pair of arcs from the collection \( H(S_0 \times \{t\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) intersect for \( t > t_m^+ \) we see that \( H(S_0 \times \{t_m^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) is component-wise homeomorphic with \( H(S_0 \times \{1\}) \cap (\Delta_\alpha \cup \Delta_\beta) = S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \). Thus, by property \( C(t_m) \), there exists a component-wise homeomorphism between \( \Phi(S_0 \times \{t_m^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) and \( S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \). Moreover, the collection of (disjoint) arcs \( \Phi(S_0 \times \{t_m^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) is isotopic to the collection \( S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \) with end points fixed. The collection of embedded arcs \( S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \) determines uniquely, up to isotopy with endpoints fixed, the collection \( S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \) simply because \( D_\alpha, D_\beta \) are cutting disks for \( S_1 \). A similar statement holds for \( \Phi(S_0 \times \{t_m^+\}) \cap (\Delta_\alpha \cup \Delta_\beta) \) and \( S_1 \cap (\Delta_\alpha \cup \Delta_\beta) \) for the same reason. It now follows that the surfaces \( \Phi(S_0 \times \{t_m^+\}) \) and \( S_1 \) have equivalent configurations with respect to \( D_\alpha, D_\beta \) and, hence, by Lemma 22 are \( s \)-isotopic. This completes the proof of the theorem. \( \square \)

The proof of the following corollary is straightforward as its assumptions guarantee that no elementary isotopy can be performed on neither \( S_0 \) nor \( S_1 \).

**Corollary 25.** Let \( S_0, S_1 \) be elements of \( S_{gm} \). We may assume that with respect to \( D_\alpha, D_\beta \) both \( S_0, S_1 \) have no stripes of type \( \alpha^0, \beta^0 \) (see Remark 5 and Definition 19). If \( S_0 \) does not contain any pair of parallel stripes of type \( \alpha^1 \) or \( \beta^1 \) and similarly for \( S_1 \), then \( S_1 \) are isotopic if and only if they are \( s \)-isotopic.

**Remark 26.** As a corollary to the above theorem we will complete the proof of Proposition 16, i.e., will show that given an incompressible surface \( S_0 \) and cutting disks \( D_\alpha, D_\beta \) then, if \( D_\alpha, D_\beta \) are geometrically minimal for \( S_0 \) it follows that \( D_\alpha, D_\beta \) satisfy the minimality condition posited in property (3) of Lemma 3.

**Proof.** Let \( S_0 \) be an incompressible surface with geometrically minimal cutting disks \( D_\alpha, D_\beta \). Let \( D_\alpha', D_\beta' \) be cutting disks satisfying properties (1), (2) and (3) of Lemma 3. We may find an isotopy \( G : M \times I \to M \) such that \( G \mid_{M \times \{0\}} = Id_M \) and \( G(D_\alpha' \times \{1\}) = D_\alpha, G(D_\beta' \times \{1\}) = D_\beta \) (we first find an isotopy \( \partial M \times I \to \partial M \) taking \( \partial D_\alpha' \) onto \( \partial D_\alpha \) and \( \partial D_\beta' \) onto \( \partial D_\beta \) and then extend it to \( M \) by [8, Theorem 4.24]). Let \( S_1 \) be the surface \( G(S_0 \times \{1\}) \). Since \( D_\alpha', D_\beta' \), viewed as cutting disks for \( S_0 \), satisfy property (3) of Lemma 3, it follows that \( D_\alpha, D_\beta \), viewed as cutting disks for \( S_1 \), also satisfy property (3) of Lemma 3. By the direction of Proposition 16 already proved in Section 5, \( D_\alpha, D_\beta \), are geometrically minimal cutting disks for \( S_1 \). Hence, \( D_\alpha, D_\beta \) are geometrically minimal cutting disks for both \( S_0, S_1 \) and we may apply Theorem 23 to \( S_0 \) and \( S_1 \) to obtain that, up to applications of elementary isotopies, \( S_0, S_1 \) are \( s \)-isotopic with respect to the cutting disks \( D_\alpha, D_\beta \). Since the application of an elementary isotopy on an incompressible surface \( S \) does not alter the cardinality of \( S \cap (D_\alpha \cup D_\beta) \) provided that \( D_\alpha, D_\beta \) are geometrically minimal, it follows that the number of components of \( S_0 \cap (D_\alpha \cup D_\beta) \) is equal to the number of components of \( S_1 \cap (D_\alpha \cup D_\beta) \). This shows that the cutting disks \( D_\alpha, D_\beta \) for \( S_0 \) satisfy property (3) of Lemma 3. \( \square \)
7. Examples

In this section we use the Main Construction described above in order to produce several examples of surfaces properly embedded in the genus 2 handlebody $H_2$. Our first example is intended to show that the assumption of Theorem 11 is not always necessary.

Example 27. In Fig. 8 an incompressible surface is drawn which does not satisfy the assumption of Theorem 11, in other words it does contain a pair of parallel stripes. To see this observe that in order to construct this surface by the gluing process presented in the Main Construction, one needs to use one stripe of type $\alpha^1$, two stripes of type $\beta^1$, one elementary disk $\alpha^e$, one elementary disk $\beta^e$ and one stripe of type $\beta^0$. The components in the 3-ball are indicated by $K_1, K_2$ and $K_2'$. By moving the cutting disk $D_{\beta}$ the stripe $\beta^0$ is eliminated and it gives rise to two elementary disks $\beta^e$ and the components $K_2$ and $K_2'$ are unified into one component. It is now clear that two parallel stripes of type $\beta^1$ are attached to the (unified) component $K_2$. Thus, this is an incompressible surface which does not satisfy the assumption of Theorem 11.

Example 28. The surfaces $S_0, S_1$ drawn in Fig. 9 are homeomorphic with isotopic boundaries (each of $\partial S_0, \partial S_1$ has 3 components pair wise isotopic to the curves $C_1, C_2, C_3$ shown in Fig. 9(c)) but they are not isotopic. To see this, first observe that there are no stripes of type $\alpha^0$ or $\beta^0$. Moreover, there is no pair of parallel stripes. By Corollary 25, $S_0, S_1$ are isotopic if and only if they are $s$-isotopic. Assume that $S_0, S_1$ are $s$-isotopic and let $D_{\alpha}, D_{\beta}$ be a pair of cutting disks for both $S_0, S_1$ satisfying properties (1), (2) and (3) of Lemma 3. Then $D_{\alpha} \cap S_0$ must be properly isotopic to $D_{\alpha} \cap S_1$. It is easy to see that both $D_{\alpha} \cap S_0$ and $D_{\beta} \cap S_0$ are, up to proper isotopy, as shown in Fig. 4(a). On the contrary, $D_{\alpha} \cap S_1$ and $D_{\beta} \cap S_1$ are, up to proper isotopy, as shown in Fig. 4(a'). Since the configurations shown in Figs. 4(a) and 4(a') are not properly isotopic we have a contradiction.

Example 29. W. Jaco in [6] provided a class of non-separating incompressible surfaces $J_{n,n} \geq 1$ of arbitrarily high genus, 1 or 2 boundary components and properly embedded in $H_2$. We demonstrate here how these examples can be obtained by the main construction and how Theorem 11 can be used in an elementary way to show that these surfaces are incompressible. Start with the surface $J_0$ (this is just an annulus) as shown in Fig. 10(a). To construct $J_0$ via the Main Construction one needs two disks $K_1, K_2$ embedded in the 3-ball $C_{\alpha,\beta}$, two stripes (one of type $\alpha^1$ and one of type $\beta^1$) and two elementary disks (one $\alpha^e$ and one $\beta^e$). We may obtain a surface $J_1$ out of $J_0$ as follows: let $K_2$ be the component (disk) whose intersection with $D_{\alpha}$ has two components ($K_1 \cap D_{\alpha}$ has four components). Attach a stripe of type $\alpha^1$ to $K_2$ using one boundary arc of $K_2$ joining $D_{\alpha}$ with $D_{\beta}$ and a stripe of type $\beta^1$ using the second boundary arc of $K_2$ joining $D_{\alpha}$ with $D_{\beta}$. Then create a third component $K_3$ in order to glue together these stripes. This is shown in Fig. 10(b) where the new stripes are indicated by arrows. The resulting surface $J_1$ has three disks $K_1, K_2, K_3$ embedded in the 3-ball $C_{\alpha,\beta}$ and two stripes of type $\alpha^1$ with one being inside the other, i.e., the new stripe is contained in the region bounded by the original stripe and the $\partial M$. Similarly for the stripes of type $\beta^1$.

Performing the same procedure on the component $K_3$ we obtain a surface $J_2$ and continuing in the same way (Fig. 10(d) shows the crucial part of the surface which reproduces itself so the next step is performed in an identical way) we obtain surfaces $J_n$, for all $n$. For each $n$, the surface $J_n$ satisfies the assumptions of Theorem 11 because
Fig. 9. $S_0, S_1$ are not $s$-isotopic.

exactly one stripe of type $\alpha^1$ (respectively $\beta^1$) is attached to each component $K_i$, $i = 2, \ldots, n + 2$ of $J_n$ and, hence, $J_n$ does not contain any pair of parallel stripes. Thus $J_n$ is incompressible for all $n$.

**Example 30.** Our next example concerns the surfaces constructed by R. Qiu in [7]. These are separating incompressible surfaces $Q_n$, $n \geq 1$ of arbitrarily high genus, one boundary component and properly embedded in $H_2$. We demonstrate how these examples can be obtained by the Main Construction and how Theorem 11 can be used to show that these surfaces are incompressible. Start with the surface $Q$ as shown in Fig. 11(a). This surface can be obtained by the main construction by using two stripes of type $\alpha^1$, four stripes of type $\beta^1$, two stripes of type $\beta^0$ and seven components (disks) $K_i$, $i = 1, \ldots, 7$ inside the 3-ball $C_{\alpha,\beta}$. We may construct a surface $Q_1$ out of $Q$ as shown in Fig. 11(b). In fact, this can be done by using an (not properly) embedded disk, denoted by $D_{Q_1}$ in order to connect two boundary components of $K_5$. The surface $Q_1$ has 4 stripes of type $\alpha^1$, 8 stripes of type $\beta^1$ and 4 stripes of type $\beta^0$. Observe that the number of stripes of type $\alpha^1$ is multiplied by 2 (and similarly for $\beta^1$). This is because the embedded disk $D_{Q_1}$ used to construct $Q_1$ out of $Q$ “traces” all stripes of type $\alpha^1$ and $\beta^1$ used to construct $Q$. Moreover, for the same reason, the number of components (disks) inside the 3-ball $C_{\alpha,\beta}$ is increased by 5. $Q$ does not contain any pair of parallel stripes and, by construction, the same is true for $Q_1$. As in the previous example, we may perform the same procedure with a disk $D_{Q_2}$ (see Fig. 11(d)) to obtain a surface $Q_2$. With a repeated application of this attaching procedure we obtain the family $Q_n$, $n \geq 1$. As above, Theorem 11 guarantees that $Q_n$ is incompressible for all $n$. 
All surfaces $Q_n$, $n \geq 1$ above have one boundary component. The construction in the above example can be modified in order to produce separating incompressible surfaces $S_n$, $n \geq 1$ of arbitrarily high genus properly embedded in $H_2$ each of which has two boundary components. This can be done by starting with the surface $Q_0$ and adding one stripe of type $\alpha^1$ (this stripe is indicated by an arrow in Fig. 12) to get a surface $S_0$ with two boundary components. Then proceed in a way identical to Example 30 to construct the surfaces $S_n$, $n \geq 1$. It is clear that the modified surface $S_0$ has two boundary components and these two boundary components are preserved throughout the whole construction, i.e., $S_n$ has two boundary components for all $n$. Moreover, the lack of parallel stripes is also preserved hence, as in the previous example, Theorem 11 guarantees that $S_n$, $n \geq 1$ is an incompressible surface in $H_2$.

The technique used to adopt the examples of W. Jaco (Example 29) and R. Qiu (Example 30) in our Main Construction can be used into various setting to produce new examples of incompressible surfaces in $H_2$. We demonstrate one such setting: consider an annulus $R_0$ properly embedded in $H_2$ as shown in Fig. 13. To construct $R_0$ via the Main
Construction one needs one disk $K_1$ embedded in the 3-ball $C_{\alpha,\beta}$, one stripe of type $\alpha^1$ and one elementary disk $\beta^e$. The cardinality of $R_0 \cap \partial D_\alpha$ is 1 and the cardinality of $R_0 \cap \partial D_\beta$ is 2. Since the cardinality of $J_0 \cap \partial D_\alpha$ is 3 and the cardinality of $J_0 \cap \partial D_\beta$ is 3 (see Example 29), it is immediate that $R_0$ is not s-isotopic to the annulus $J_0$. Hence, by Theorem 23, $R_0$ and $J_0$ are not properly isotopic in $H_2$. Then proceed as in Examples 29 and 30 to obtain surfaces $R_n$, $n \geq 1$ properly embedded in $H_2$, with 1 or 2 boundary components, according to whether $n$ is odd or even. It is clear
that, for each $n$, $R_n$ has the same topological type with $J_n$. However, as shown in the case $n = 0$, $R_n$ is not $s$-isotopic to $J_n$ and, hence, by Theorem 23, $R_n$ is not properly isotopic to $J_n$ in $H_2$.

References