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IMBEDDINGS INTO TOPOLOGICAL GROUPS PRESERVING DIMENSIONS

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We give a negative answer to the following question of Bel'nov: Can every Tychonoff space X be imbedded as a subspace of a topological group G so that $\dim G \leqslant \dim X$? We show that if $n \neq 0, 1, 3, 7$, then the n-dimensional sphere S^n cannot be imbedded into an n-dimensional topological group G (no matter which dimension function, ind, Ind or dim, is considered). However, in case $\dim X = 0$ the answer to Bel'nov's question is "yes". We prove that, for every Tychonoff space X, $\dim X = 0$ implies (in fact, equivalent to) $\dim F^*(X) = 0$ and $\dim A^*(X) = 0$, where $F^*(X)$ ($A^*(X)$) is the free precompact (Abelian) group of X. As a corollary we obtain that every precompact group G is a quotient group of a precompact group H such that $\dim H = 0$ and $w(H) = w(G)$. A complete metric space X_1 and a pseudocompact Tychonoff space X_2 are constructed such that $\operatorname{ind} X_i = 0$, while $\operatorname{ind} F^*(X_i) \neq 0$ and $\operatorname{ind} A^*(X_i) \neq 0$ $(i = 1, 2)$. The equivalence of $\operatorname{ind} G = 0$ and $\dim G = 0$ for a precompact group G is established. We prove that $\dim H \leqslant \dim G$ whenever H is a precompact subgroup of a topological group G. We also show that for every Tychonoff topology $\mathcal{T}$ on a set X with $\operatorname{ind}(X, \mathcal{T}) = 0$ one can find a precompact Hausdorff group topology $\tilde{\mathcal{T}}$ on the free (Abelian) group $G(X)$ of X such that $w(G(X), \tilde{\mathcal{T}}) = w(X, \mathcal{T})$, $\tilde{\mathcal{T}}|_X = \mathcal{T}$ and $\dim(G(X), \tilde{\mathcal{T}}) = 0$.

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totally bounded group	precompact group	sphere
zero-dimensionality	free group	dim
covering dimension	imbedding	ind
topological group	dimension	

Introduction

In 1978 Bel'nov [7] showed that every Tychonoff space X can be imbedded as a closed subspace into a homogeneous Tychonoff space Y with $\dim Y \leqslant \dim X$. In connection with this result Bel'nov asked whether every Tychonoff space X can be imbedded into a Hausdorff topological group G with $\dim G \leqslant \dim X$ (see [52, Question III.23]). Answering this question in the negative we prove the following:

Theorem 0.1. *If $n \neq 0, 1, 3, 7$, then the n-dimensional sphere S^n cannot be imbedded into an n-dimensional Hausdorff group G (no matter which dimension function,* ind, Ind *or* dim, *is considered).*

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It turns out however that in case $\dim X = 0$ the answer to Bel'nov's question is "yes". This will be proved in Section 4. In particular, in this section we show that $\dim X = 0$ implies (in fact, equivalent to) $\dim F^*(X) = 0$ and $\dim A^*(X) = 0$, where $F^*(X)$ is the free precompact topological group of the space X, and $A^*(X)$ is the free precompact Abelian topological group of X. Examples are constructed serving to show that similar results for the dimension function ind do not hold. The main tool for proving these results is the technique of free precompact (Abelian) groups developed in Section 2. In Section 1 terminology, notations and preliminary facts are collected. In Section 3 some results concerning the dimension theory of compact-like topological groups are obtained. Finally, in Section 5 we give some applications of our results to closed imbeddings of Tychonoff spaces into Hausdorff topological groups which preserve zero-dimensionality.

Proof of Theorem 0.1. Suppose the contrary, and let G be a Hausdorff n-dimensional group containing S^n as a subspace. Let H be the smallest subgroup of G containing S^n. Since H is generated algebraically by the compact space S^n, H is σ-compact. So let $H = \bigcup \{K_i : i \in \mathbb{N}\}$, where each K_i is compact. Fix an $i \in \mathbb{N}$. If $\operatorname{ind} G \leqslant n$, then $\operatorname{ind} K_i \leqslant \operatorname{ind} G \leqslant n$, and since K_i is compact, $\dim K_i \leqslant \operatorname{ind} K_i \leqslant n$ [17, Theorem 7.2.7]. If $\operatorname{Ind} G \leqslant n$, then $\operatorname{ind} G \leqslant n$, and therefore $\dim K_i \leqslant n$, by the instance proved above. If $\dim G \leqslant n$, then $\dim \beta G \leqslant n$, where βG is the Čech-Stone compactification of G [17, Theorem 7.1.17]. Since K_i is closed in the compact (hence normal) space βG, $\dim K_i \leqslant \dim \beta G \leqslant n$ [17, Theorem 7.1.18]. Thus, in any case, $\dim K_i \leqslant n$ for every $i \in \mathbb{N}$. Being σ-compact, the space H is normal. Now the countable sum theorem (see Theorem 1.13 below) implies $\dim H \leqslant n$.

Consider the identity mapping $i : S^n \to S^n$. Since S^n is a closed subspace of the normal space H with $\dim H \leqslant n$, by Alexandroff's theorem [2, 23, 15] characterizing the dimension dim in terms of mappings into spheres, there exists a continuous mapping $r : H \to S^n$ extending i. Thus we conclude that S^n is a retract of H, which contradicts the following result of Uspenskii [54]: *If* $n \neq 0, 1, 3, 7$, *then* S^n *cannot be a retract of a topological group.* For completeness' sake let us present Uspenskii's argument here. Suppose that $r : G \to S^n$ is a retraction of a topological group G onto S^n. Without loss of generality we can assume that $e \in S^n$, where e is the neutral element of G. (Otherwise, fix a $g \in S^n$ and observe that $r_g : G \to g^{-1} \cdot S^n$, defined by $r_g(h) = g^{-1} \cdot r(g \cdot h)$ for every $h \in G$, would be a retraction of G onto $g^{-1} \cdot S^n$, which is homeomorphic to S^n, and that $e \in g^{-1} \cdot S^n$.) Now one can easily verify that the punctiform space (S^n, e), together with the continuous mapping $m : S^n \times S^n \to S^n$ defined by $m(x, y) = r(x \cdot y)$, is an H-space (see [11, 25]). So Adams' theorem [1] implies that $n = 0, 1, 3, 7$. □

Note that if $n = 0, 1, 3$, then S^n is a topological group itself, but the author was unable to decide whether S^7 can be imbedded into a topological group G with $\dim G = 7$.

1. Notations, terminology and preliminary results

Notations and terminology follow [17]. All topological spaces and topological groups considered are assumed to be Tychonoff. We fix βX for denoting the Čech–Stone compactification of X [17, Chapter 3.6], I is the unit interval $[0, 1]$ with the usual topology, $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$. We use S^1 for the quotient group $\mathbb{R}/\mathbb{Z}$ of the group $\mathbb{R}$ of real numbers by the subgroup $\mathbb{Z}$ of integers equipped with the quotient topology. Clearly, S^1 is homeomorphic to the unit circle. The bar denotes the closure of a set in a topological space. In what follows $|X|$ stands for the cardinality of a space X, $w(X)$ and $\mathrm{nw}(X)$ denote the weight and the net weight of a space X respectively [17]. We use $X \oplus Y$ for the disjoint sum of spaces X and Y [17, Chapter 2.2]. If Y is a subset of X, $\mathcal{T}$ is a topology on X and $\mathcal{U}$ is a uniformity on X, then $\mathcal{T}|_Y = \{U \cap Y: U \in \mathcal{T}\}$ is the *subspace topology* on Y induced by $\mathcal{T}$, and $\mathcal{U}|_Y = \{U \cap (Y \times Y): U \in \mathcal{U}\}$ is the *subspace uniformity* on Y induced by $\mathcal{U}$. If for each $\alpha \in A$ a continuous mapping $f_\alpha : X \to X_\alpha$ is fixed, then the mapping $f = \triangle\{f_\alpha: \alpha \in A\} : X \to \prod\{X_\alpha: \alpha \in A\}$ defined by $f(x) = \{f_\alpha(x)\}_{\alpha \in A}$ for $x \in X$, is continuous, and is said to be the *diagonal product* of the family $\{f_\alpha : \alpha \in A\}$ [17, Chapter 2.3]. The Katětov-Smirnov dimension of a space X defined by means of functionally open covers is denoted by $\dim X$; $\mathrm{ind}\, X$ and $\mathrm{Ind}\, X$ stand for the small and the large inductive dimensions of X respectively [17, Chapter 7.1]. A subspace Y of a space X is C^*-*embedded* in X provided that every continuous function $f: Y \to I$ has a continuous extension $\tilde{f}: X \to I$ over X.

We start with facts concerning topological groups.

Fact 1.1. *If H is a dense subgroup of a topological group G, then $w(G) = w(H)$.*

Proof. This follows from the equality $w(G) = \chi(G) \cdot d(G)$, which holds for every topological group G (see, for example, [13, Theorem 3.5(i)]), and from the equality $\chi(G) = \chi(H)$, which holds because H is a dense subspace of the (regular) space G. □

Fact 1.2 [38]. *Assume that τ is a cardinal, $(G, \mathcal{T})$ is a (Hausdorff) topological group, $\mathcal{E} \subset \mathcal{T}$, $\mathrm{nw}(G, \mathcal{T}) \leq \tau$ and $|\mathcal{E}| \leq \tau$. Then there exists a (Hausdorff) group topology $\tilde{\mathcal{T}}$ on G such that $\mathcal{E} \subset \tilde{\mathcal{T}} \subset \mathcal{T}$ and $w(G, \tilde{\mathcal{T}}) \leq \tau$.*

We use $F(X)$ for denoting the free group of X [22]. Recall that any $g \in F(X)$ except the neutral element e has the unique representation of the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $n \in \mathbb{N}^+$, $x_i \in X$, $\varepsilon_i = \pm 1$ $(i = 1, \ldots, n)$ and moreover, for every $i = 1, \ldots, n-1$ either $x_i \neq x_{i+1}$ or $\varepsilon_i = \varepsilon_{i+1}$. The n above is called the *length* of g and denoted by $l(g)$. The neutral element e of $F(X)$ is the unique element of $F(X)$ without any representation, and $l(e) = 0$ by the definition. For $n \in \mathbb{N}$ define $F_n(X) = \{g \in F(X): l(g) \leq n\}$ and $B_n(X) = \{g \in F(X): l(g) = n\}$. For a space X define $X^{-1} = \{x^{-1}: x \in X\}$ and $\hat{X} = X \oplus X^{-1} \oplus \{e\}$. Here we equip X^{-1} with the topology, which is copied from X by means of mapping $x \mapsto x^{-1}$ and $\{e\}$ with the discrete topology. For $n \in \mathbb{N}^+$ we define the mapping $\theta_n : \hat{X}^n \to F(X)$ by $\theta_n(y_1, \ldots, y_n) = y_1 \cdots y_n$ whenever $y_i \in \hat{X}$, $i = 1, \ldots, n$ (here the multiplication is taken in $F(X)$), and put $Z_n = \theta_n^{-1}(B_n(X)) \subset \hat{X}^n$.

Analogously, let $A(X)$ be the free Abelian group of X [22]. Recall that each $g \in A(X)$ except the zero element 0 has the unique (up to permutation) representation of the form $g = \varepsilon_1 \cdot x_1 + \cdots + \varepsilon_n \cdot x_n$ so that $n \in \mathbb{N}^+$, $x_i \in X$, $\varepsilon_i = \pm 1$ $(i = 1, \ldots, n)$, and if $x_i = x_j$ for some $i, j = 1, \ldots, n$, then $\varepsilon_i = \varepsilon_j$. The n above is the *length* of g, and it is denoted by $l(g)$. The zero element 0 has no representation, and $l(0) = 0$ by the definition. For $n \in \mathbb{N}^+$ we define $A_n(X) = \{g \in A(X) : l(g) \leq n\}$ and $B_n(X) = \{g \in A(X) : l(g) = n\}$. For $A(X)$ the space $\hat{X}^n$, its subspace Z_n and the mapping $\theta_n : \hat{X}^n \to A(X)$ can be defined similarly to these for $F(X)$.

Lemma 1.3. *Suppose that $\mathcal{T}$ is a group topology on $F(X)$ (on $A(X)$), which induces on X the original topology of the space X. Then for every $n \in \mathbb{N}^+$ mappings $\theta_n : \hat{X}^n \to (F(X), \mathcal{T})$ and $\theta_n|_{Z_n} : Z_n \to (B_n(X), \mathcal{T}|_{B_n(X)})$ are continuous (here Z_n is considered as a subspace of $\hat{X}^n$). Furthermore, $\theta_n(\hat{X}^n) = F_n(X)$, and $\theta_n|_{Z_n}$ is a bijection of Z_n onto $B_n(X)$.*

In particular, if X is compact, then $(F_n(X), \mathcal{T}|_{F_n(X)})$ is a compact subspace of $(F(X), \mathcal{T})$ $((A_n(X), \mathcal{T}|_{A_n(X)})$ is a compact subspace of $(A(X), \mathcal{T}))$.

A topological group G (and its topology) is said to be *precompact* iff G is isomorphic to a subgroup of some compact group or, equivalently, iff its two-sided uniformity completion is compact [57].

Fact 1.4 [57]. *If $\pi : G \to H$ is a continuous homomorphism, G is a precompact group and $H = \pi(G)$, then the group H is precompact too.*

Our next definition and proposition are folklore.

Definition 1.5. Let $(X, \mathcal{T})$ be a space and $\mathcal{T}_X^*$ a (Hausdorff) group topology on $F(X)$ (on $A(X)$). The topological group $(F(X), \mathcal{T}_X^*)$ (the topological group $(A(X), \mathcal{T}_X^*)$) is said to be the *free precompact group* of $(X, \mathcal{T})$ (the *free precompact Abelian group* of $(X, \mathcal{T})$) iff:

(i) $\mathcal{T}_X^*|_X = \mathcal{T}$,

(ii) $\mathcal{T}_X^*$ is precompact, and

(iii) whenever G is a compact group (a compact Abelian group) and $f : (X, \mathcal{T}) \to G$ is a continuous mapping, the natural homomorphic extension $\tilde{f} : (F(X), \mathcal{T}_X^*) \to G$ $(\tilde{f} : (A(X), \mathcal{T}_X^*) \to G)$ of the mapping f is continuous.

In this case $\mathcal{T}_X^*$ is called the *free precompact topology* (the *free precompact Abelian topology*) over the space $(X, \mathcal{T})$.

Proposition 1.6. *For every space $(X, \mathcal{T})$ there exists the free precompact group $(F(X), \mathcal{T}_X^*)$ of $(X, \mathcal{T})$ (the free precompact Abelian group $(A(X), \mathcal{T}_X^*)$ of $(X, \mathcal{T})$), and this group is unique.*

Denote by $\mathcal{H}_{(X,\mathcal{T})}$ *the set of all pairs* $\langle G_f, f\rangle$ *consisting of a compact (Abelian) group* G_f *with* $w(G_f) \leq 2^{\aleph_0\cdot|X|}$ *and of a continuous mapping* $f:(X, \mathcal{T}) \to G_f$. *For* $\langle G_f, f\rangle \in \mathcal{H}_{(X,\mathcal{T})}$ *let* $\tilde{f}: F(X) \to G_f$ ($\tilde{f}: A(X) \to G_f$) *be the homomorphism extending f. Then*

$$\mathcal{B}_{(X,\mathcal{T})} = \{\tilde{f}^{-1}(U): \langle G_f, f\rangle \in \mathcal{H}_{(X,\mathcal{T})} \text{ and } U \text{ is open in } G_f\}$$

is a base for $\mathcal{T}^*_X$.

In what follows the free precompact group and the free precompact Abelian group of a space X will be denoted by $F^*(X)$ and $A^*(X)$ respectively. All results will be formulated for $F^*(X)$ and the corresponding results for $A^*(X)$ will be formulated in round brackets. If proofs in both cases are similar to each other, we will give only a proof for $F^*(X)$ omitting that for $A^*(X)$.

Definition 1.7 [50]. A topological group G is said to be $\mathbb{R}$-*factorizable* iff one of the following equivalent conditions holds:

(i) for every continuous function $f: G \to \mathbb{R}$ there exist a topological group H with a countable base, a continuous homomorphism $\pi: G \to H$ and a continuous function $\varphi: H \to \mathbb{R}$ so that $f = \varphi \circ \pi$,

(ii) if Z is a space and $f: G \to Z$ is a continuous mapping, then there exist a topological group H, a continuous homomorphism $\pi: G \to H$ and a continuous mapping $\varphi: H \to Z$ such that $f = \varphi \circ \pi$ and $w(H) \leq w(Z)$.

Fact 1.8 [49, Theorem 3.8]. *Each precompact group is* $\mathbb{R}$-*factorizable.*

Fact 1.9 [5, Proposition 3.1]. *Suppose that G is a topological group, X is its subspace, which algebraically generates G. Then* $\mathrm{nw}(G) \leq \mathrm{nw}(X)$. *In particular, if* $\mathcal{T}$ *is a topology on X and* $\tilde{\mathcal{T}}$ *is a group topology on* $F(X)$ (*on* $A(X)$) *so that* $\tilde{\mathcal{T}}|_X = \mathcal{T}$ *and* $w(X, \mathcal{T}) \leq \tau$, *then* $\mathrm{nw}(F(X), \tilde{\mathcal{T}}) \leq \tau$ ($\mathrm{nw}(A(X), \tilde{\mathcal{T}}) \leq \tau$).

Fact 1.10 [53]. *Let G be a topological group the space of which is a Lindelöf Σ-space* (*see* [29]). *Then the closure* $\bar{B}$ *of each* G_δ-*subset B of G is a zero-set of G.*

Now we turn to facts from the dimension theory.

Fact 1.11 [17, Theorem 7.3.3]. *If Z is a space with a countable base, then* $\dim Z = \mathrm{ind}\, Z$. *In particular,* $\dim Y \leq \dim Z$ *for every subspace Y of the space Z.*

The following fact seems to be the part of the dimension theory folklore. Nevertheless, since it will be used very often in our further proofs, for the sake of completeness we give its proof here.

Fact 1.12. *For every space X the following conditions are equivalent*:

(i) $\dim X \leq n$,

(ii) *if Z is a space with a countable base and $f: X \to Z$ is a continuous mapping, then there exist a space Y with a countable base and continuous mappings $g: X \to Y$, $h: Y \to Z$ such that $f = h \circ g$ and* $\dim Y \leq n$.

Proof. (i)⇒(ii). Let $i: Z \to I^{\mathbb{N}}$ be a homeomorphic imbedding of Z into the Hilbert cube $I^{\mathbb{N}}$, and let $\tilde{f}: \beta X \to I^{\mathbb{N}}$ be the continuous extension of $i \circ f: X \to I^{\mathbb{N}}$ over βX. Since $\dim \beta X = \dim X \leq n$, using the Mardešić factorization theorem [26] we can find a compact space $\tilde{Y}$ having a countable base and continuous mappings $\tilde{g}: \beta X \to \tilde{Y}$, $\tilde{h}: \tilde{Y} \to I^{\mathbb{N}}$ such that $\tilde{f} = \tilde{h} \circ \tilde{g}$ and $\dim \tilde{Y} \leq n$. Define $Y = \tilde{g}(X)$, $g = \tilde{g}|_X$ and $h = i^{-1}|_{i(Z)} \circ \tilde{h}|_Y$. Then Y has a countable base, $f = h \circ g$ and $\dim Y \leq \dim \tilde{Y} \leq n$ (Fact 1.11).

(ii)⇒(i). Let $\gamma = \{U_1, \ldots, U_k\}$ be a covering of X consisting of functionally open sets. For $i = 1, \ldots, k$ fix an open subset V_i of $\mathbb{R}$ and a continuous function $f_i: X \to \mathbb{R}$ so that $U_i = f_i^{-1}(V_i)$, and define

$$V_i^* = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{i-1 \text{ times}} \times V_i \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k-i \text{ times}}.$$

Set $f = \triangle\{f_i : i = 1, \ldots, k\}: X \to \mathbb{R}^k$ and $Z = f(X)$. Choose Y, g and h in accordance with (ii). Since

$$\lambda = \{h^{-1}(V_1^*) \cap g(X), \ldots, h^{-1}(V_k^*) \cap g(X)\}$$

is a functionally open cover of $g(X)$ and $\dim g(X) \leq \dim Y \leq n$ (Fact 1.11), we can find a functionally open refinement of λ of order $\leq n+1$, say $\{W_1, \ldots, W_s\}$. Then $\{g^{-1}(W_1), \ldots, g^{-1}(W_s)\}$ is a functionally open refinement of γ of order $\leq n+1$. □

The countable sum theorem 1.13 [17, Theorem 7.2.1]. *Assume that $X = \bigcup\{F_j : j \in \mathbb{N}\}$, where each F_j is a closed subspace of X so that* $\dim F_j \leq n$. *If X is normal, then* $\dim X \leq n$.

Definition 1.14 (Filippov). Let X be a normal space. By induction we will define $\operatorname{Ind}_0 X$ as follows:

(i) $\operatorname{Ind}_0 X = -1$ iff $X = \emptyset$,

(ii) $\operatorname{Ind}_0 X \leq n$ iff for every closed set $F \subset X$ and any open set $V \subset X$ that contains F there exists an open set $U \subset X$ such that $F \subset U \subset V$, $\operatorname{Ind}_0 \operatorname{Fr} U \leq n-1$ and $\operatorname{Fr} U$ is a G_δ-subset of X (here $\operatorname{Fr} U = \bar{U} \cap \overline{X \setminus \bar{U}}$),

(iii) $\operatorname{Ind}_0 X = n$ iff $\operatorname{Ind}_0 X \leq n$ and the inequality $\operatorname{Ind}_0 X \leq n-1$ does not hold, and

(iv) $\operatorname{Ind}_0 X = \infty$ iff the inequality $\operatorname{Ind}_0 X \leq n$ does not hold for any n.

For basic properties of the dimension function Ind_0 see [24].

Recall that a space Z is said to be *perfectly $\varkappa$-normal* iff the closure of every open subset of Z is a zero-set of Z [9, 37].

Fact 1.15 [18, Proposition 1]. *Suppose that X is normal and each closed G_δ-subset of X is perfectly $\varkappa$-normal in the subspace topology. Then* $\mathrm{Ind}_0 X = \mathrm{Ind}\, X$.

A subset Y of a topological space X is *z-embedded* in X [9, 8, 10] iff each zero-set of Y is the restriction to Y of a zero-set of X.

Fact 1.16 [9, Theorem 5.1]. *For any space X the following conditions are equivalent*:

(i) *X is perfectly $\varkappa$-normal,*

(ii) *every open subset of X is z-embedded in X,*

(iii) *each dense subset of X is z-embedded in X.*

Fact 1.17. *If Y is z-embedded in X, then*:

(i) $\dim Y \leqslant \dim X$ [30, Theorem 1.3], *and*

(ii) $\mathrm{Ind}_0 Y \leqslant \mathrm{Ind}_0 X$ [12, Corollary 3.10].

Lemma 1.18. (i) *If for every continuous mapping $f: X \to Z$ from X to a space Z with a countable base one can find a space Y with a countable base and continuous mappings $g: X \to Y$, $h: Y \to Z$ such that $f = h \circ g$ and* $\mathrm{ind}\, Y \leqslant \mathrm{ind}\, X$, *then* $\dim X \leqslant \mathrm{ind}\, X$.

(ii) *Suppose G is an $\mathbb{R}$-factorizable group. If for every group H with a countable base and for every continuous homomorphism $\pi: G \to H$ there exist a group G^* with a countable base and continuous homomorphisms $g: G \to G^*$, $h: G^* \to H$ such that $f = h \circ g$ and* $\mathrm{ind}\, G^* \leqslant \mathrm{ind}\, G$, *then* $\dim G \leqslant \mathrm{ind}\, G$.

Proof. (i) Assume that X, Y, Z, f, g and h satisfy (i). Since Y has a countable base, $\dim Y = \mathrm{ind}\, Y \leqslant \mathrm{ind}\, X$(Fact 1.11), and now it suffices to apply (ii)$\Rightarrow$(i) of Fact 1.12.

(ii) Suppose that $f: G \to Z$ is a continuous mapping of G to a space Z with a countable base. Choose H, φ and π in accordance with (ii) of Definition 1.7. Now, if G^*, g and h satisfy the assumption of item (ii) of our lemma, then item (i) of it implies that $\dim G \leqslant \mathrm{ind}\, G$. □

2. Some properties of free precompact groups

The aim of this section is to investigate a topological structure and basic properties of free precompact (Abelian) groups.

Let $\mathcal{B}_e(X) = \{U\colon U \text{ is open in } F^*(X) \text{ and } e \in U\}$. For $U \in \mathcal{B}_e(X)$ define

$$\Phi_U = \{(g, h) \in F(X) \times F(X)\colon gh^{-1} \in U \text{ and } g^{-1}h \in U\}.$$

The family $\{\Phi_U : U \in \mathcal{B}_e(X)\}$ constitutes a base of the two-sided uniformity $\mathcal{F}_X$ on $F(X)$. It is clear that $\mathcal{F}_X$ generates the original topology of $F^*(X)$. Similarly, let $\mathcal{B}_0(X) = \{U\colon U \text{ is open in } A^*(X) \text{ and } 0 \in U\}$. For $U \in \mathcal{B}_0(X)$ define

$$\Phi_U = \{(g, h) \in A(X) \times A(X)\colon g - h \in U\}.$$

The family $\{\Phi_U : U \in \mathscr{B}_0(X)\}$ forms a base of the uniformity $\mathscr{A}_X$ on $A^*(X)$, which generates the original topology of $A^*(X)$.

Let $C^*(X)$ be the set of all continuous functions $f: X \to I$. For $f \in C^*(X)$ and $\varepsilon > 0$ define

$$C^X_{f,\varepsilon} = \{(x, y) \in X \times X : |f(x) - f(y)| < \varepsilon\} \subset X \times X.$$

The family $\{C^X_{f,\varepsilon} : f \in C^*(X), \varepsilon > 0\}$ forms a subbase of the uniformity $\mathscr{C}^*(X)$ on X (see [17, Example 8.1.19]).

Lemma 2.1. *Suppose $f \in C^*(X)$, $\varepsilon \in \mathbb{R}$ and $0 < \varepsilon < 1$. Then there exists a $\Phi_U \in \mathscr{F}_X$ ($\Phi_U \in \mathscr{A}_X$) so that $\Phi_U \cap (X \times X) = C^X_{f,\varepsilon}$.*

Proof. Let $H = \{2n : n \in \mathbb{Z}\} \subset \mathbb{R}$ and let $\pi : \mathbb{R} \to \mathbb{R}/H$ be the natural quotient mapping. Since π is open, the set $W = \pi((-\varepsilon, \varepsilon))$ is an open neighbourhood of zero of $\mathbb{R}/H$. Let $\varphi : F^*(X) \to \mathbb{R}/H$ be the continuous homomorphic extension of $\pi \circ f : X \to \mathbb{R}/H$ over $F^*(X)$. One can easily check that for $U = \varphi^{-1}(W)$ we have $\Phi_U \cap (X \times X) = C^X_{f,\varepsilon}$. □

Proposition 2.2. *For every space X, $\mathscr{F}_X|_X = \mathscr{C}^*(X)$ and $\mathscr{A}_X|_X = \mathscr{C}^*(X)$.*

Proof. Lemma 2.1 implies that $\mathscr{C}^*(X) \subset \mathscr{F}_X|_X$. Let us verify the reverse inclusion. Since $F^*(X)$ is a precompact group, the uniformity $\mathscr{F}_X$ is totally bounded. The uniform space $(X, \mathscr{F}_X|_X)$ is totally bounded as a subspace of the totally bounded uniform space $(F(X), \mathscr{F}_X)$. Furthermore, $\mathscr{F}_X$ induces the original topology of $F^*(X)$, and X is a subspace of $F^*(X)$, so the uniformity $\mathscr{F}_X|_X$ induces the original topology of X. Now inclusion $\mathscr{F}_X|_X \subset \mathscr{C}^*(X)$ follows from the fact that $\mathscr{C}^*(X)$ is the finest uniformity among all totally bounded uniformities on X generating the original topology of X [36]. □

Theorem 2.3. *For every space X, the group $F^*(X)$ is a subgroup of $F^*(\beta X)$, and $A^*(X)$ is a subgroup of $A^*(\beta X)$.*

Proof. In view of Proposition 1.6, it suffices to show that $\mathscr{B}_X = \{F(X) \cap U : U \in \mathscr{B}_{\beta X}\}$. First of all note that $\langle G_f, f|_X\rangle \in \mathscr{H}_X$ provided that $\langle G_f, f\rangle \in \mathscr{H}_{\beta X}$, so $\{F(X) \cap U : U \in \mathscr{B}_{\beta X}\} \subset \mathscr{B}_X$. On the other hand, assume that $\langle G_f, f\rangle \in \mathscr{H}_X$. Since G_f is a compact space and $f: X \to G_f$ is continuous, there exists the continuous mapping $g : \beta X \to G_f$ extending f. If $\tilde{f} : F^*(X) \to G_f$ and $\tilde{g} : F^*(\beta X) \to G_f$ are continuous homomorphisms extending f and g respectively, then $\tilde{g}|_{F^*(X)} = \tilde{f}$. Since $\langle G_f, g\rangle \in \mathscr{H}_{\beta X}$, this yields $\mathscr{B}_X \subset \{F(X) \cap U : U \in \mathscr{B}_{\beta X}\}$. □

It is worth comparing Theorem 2.3 with the following result of Nummela [32] and Pestov [33]: The free (Abelian) topological group $G(X)$ of a space X is a subgroup of the free (Abelian) topological group $G(\beta X)$ of its Čech–Stone compactification βX iff X is pseudocompact (for the definition of $G(X)$ see [27, 19]).

Lemma 2.4. *For every space X and for every $n \in \mathbb{N}^+$, the set $F_n(X)$ is closed in $F^*(X)$, and the set $A_n(X)$ is closed in $A^*(X)$.*

Proof. Fix an $n \in \mathbb{N}^+$. By the above theorem, $F^*(X)$ is a subgroup of $F^*(\beta X)$. Since $F_n(X) = F_n(\beta X) \cap F(X)$, to prove our lemma it suffices to check that $F_n(\beta X)$ is closed in $F^*(\beta X)$. Since βX is compact, from Lemma 1.3 it follows that $F_n(\beta X)$ is a compact subspace of $F^*(\beta X)$. Since $F^*(\beta X)$ is Hausdorff, $F_n(\beta X)$ is closed in $F^*(\beta X)$. □

The following proposition is the precompact analogue of the well-known result of Arhangel'skiĭ obtained by him for free topological groups ([3, 4]; see also [5, 5.1]).

Proposition 2.5. *For every $n \in \mathbb{N}^+$ consider the set $Z_n \subset \hat{X}^n$ with the subspace topology inherited from $\hat{X}^n$ and the set $B_n(X) \subset F^*(X)$ with the subspace topology inherited from $F^*(X)$ (see Section 1). Then $\theta_n|_{Z_n} : Z_n \to B_n(X)$ is a homeomorphism. The similar statement holds also for $A^*(X)$.*

Proof. Taking into account Theorem 2.3, we conclude that it suffices to prove our proposition only for a compact space X. In this case let $\mathcal{T}$ be the topology of the free topological group of X [27, 19]. Since $\mathcal{T}$ induces the original topology on X, $(F_n(X), \mathcal{T}|_{F_n(X)})$ is a compact space (Lemma 1.3). The topology $\mathcal{T}_X^*$ of the topological group $F^*(X)$ is Hausdorff and $\mathcal{T}_X^* \subset \mathcal{T}$, so $\mathcal{T}|_{F_n(X)} = \mathcal{T}_X^*|_{F_n(X)}$, and therefore $\mathcal{T}|_{B_n(X)} = \mathcal{T}_X^*|_{B_n(X)}$. Now it remains to apply Arhangel'skiĭ's result cited above according to which $\theta_n|_{Z_n} : Z_n \to (B_n(X), \mathcal{T}|_{B_n(X)})$ is a homeomorphism. □

Lemma 2.6. *For every space X and for any $i \in \mathbb{N}^+$ there exists a decomposition $Z_i = \bigcup \{L_{i,\varepsilon} : \varepsilon \in \{-1, 1\}^i\}$ of the subspace Z_i of the space $\hat{X}^i$ such that each $L_{i,\varepsilon}$ is closed in Z_i and homeomorphic to a subspace of X^i.*

Proof. Without loss of generality we can assume that $X \neq \emptyset$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_i) \in \{-1, 1\}^i$ define

$$L_{i,\varepsilon} = (X^{\varepsilon_1} \times \cdots \times X^{\varepsilon_i}) \cap Z_i$$

and note that $L_{i,\varepsilon}$ do the job. □

Proposition 2.7. *Suppose that X is a space with a countable base. Then $F^*(X) = \bigcup \{K_n : n \in \mathbb{N}\}$, where each K_n is closed in $F^*(X)$ and homeomorphic to a subspace of X^{k_n} for some $k_n \in \mathbb{N}^+$. The analogous result holds also for $A^*(X)$.*

Proof. Fix an $i \in \mathbb{N}^+$. Since $F^*(X)$ has a countable network (Fact 1.9), the set F_{i-1}, being closed in $F^*(X)$ (Lemma 2.4), is a G_δ-subset of $F^*(X)$. Let $F(X) \setminus F_{i-1} = \bigcup \{\Phi_{i,j} : j \in \mathbb{N}^+\}$, where each $\Phi_{i,j}$ is closed in $F^*(X)$. Since $F_i(X)$ is also closed in

$F^*(X)$ (Lemma 2.4), $P_{i,j} = \Phi_{i,j} \cap F_i(X)$ is closed in $F^*(X)$ for any $j \in \mathbb{N}^+$. Observe that $B_i(X) = \bigcup \{P_{i,j} : j \in \mathbb{N}^+\}$. Let $Z_i = \bigcup \{L_{i,\varepsilon} : \varepsilon \in \{-1, 1\}^i\}$ be the decomposition constructed in Lemma 2.6. Since $\theta_i|_{Z_i} : Z_i \to B_i(X)$ is a homeomorphism (Proposition 2.5), for each $\varepsilon \in \{-1, 1\}^i$ the set $\theta_i(L_{i,\varepsilon})$ is closed in $B_i(X)$ and homeomorphic to a subspace of X^i. Therefore, for $j \in \mathbb{N}^+$ and $\varepsilon \in \{-1, 1\}^i$ the set $Q_{i,j,\varepsilon} = \theta_i(L_{i,\varepsilon}) \cap P_{i,j}$ is closed in $F^*(X)$ and homeomorphic to a subspace of X^i. Thus $F^*(X) = \bigcup \{Q_{i,j,\varepsilon} : i, j \in \mathbb{N}^+, \varepsilon \in \{-1, 1\}^i\} \cup \{e\}$ is the desired decomposition. □

Proposition 2.8. *Assume that X is a space, G is a precompact (Abelian) group, $f: X \to G$ is a quotient mapping and $f(X) = G$. Then the homomorphism $\tilde{f}: F^*(X) \to G$ ($\tilde{f}: A^*(X) \to G$) extending f is open.*

Proof. Define $\mathcal{T}$ to be the quotient topology on G with respect to f. Then $\mathcal{T}$ is finer than the original topology of G. Since X is a subspace of $F^*(X)$, the mapping $f: X \to G$ remains continuous if one replaces the original topology of G by $\mathcal{T}$. But since f is quotient and $f(X) = G$, we conclude that $\mathcal{T}$ coincides with the original topology of G. Therefore, f is quotient and hence open, being a homomorphism of topological groups. □

Proposition 2.9. *Every space X is C^*-embedded in $F^*(X)$ (in $A^*(X)$). In particular,* $\dim X \leq \dim F^*(X)$ ($\dim X \leq \dim A^*(X)$).

Proof. Let $f: X \to I$ be a continuous function. Fix a homeomorphic imbedding $j: I \to S^1$ and a retraction $r: S^1 \to j(I)$ of S^1 onto $j(I)$. If $\varphi: F^*(X) \to S^1$ is the continuous homomorphism extending $j \circ f: X \to S^1$, then $\tilde{f} = j^{-1}|_{j(I)} \circ r \circ \varphi : F^*(X) \to I$ is the continuous extension of f over $F^*(X)$. Thus, X is C^*-embedded in $F^*(X)$. The last conclusion of our proposition follows from [17, Exercise 7.2.A].

3. Some dimension theory results for topological groups close to being compact

We start with the factorization theorem.

Theorem 3.1. *Suppose that G is an $\mathbb{R}$-factorizable group (see Definition 1.7), H is a topological group, $\pi: G \to H$ is a continuous homomorphism and* $\operatorname{ind} G = 0$. *Then there exist a topological group G^* and continuous homomorphisms $g: G \to G^*$, $h: G^* \to H$ so that $\pi = h \circ g$,* $\operatorname{ind} G^* = 0$ *and $w(G^*) \leq w(H)$.*

Proof. Let $w(H) = \tau$. By induction we will define, for every $n \in \mathbb{N}$, a group G_n and continuous homomorphisms $g_n: G \to G_n$, $h_n: G_n \to G_{n-1}$ as follows:

(i) $G_0 = H$ and $g_0 = \pi$,

(ii) $w(G_n) \leqslant \tau$ for every $n \in \mathbb{N}$,

(iii) $g_n = h_{n+1} \circ g_{n+1}$ for every $n \in \mathbb{N}$,

(iv) for any $n \in \mathbb{N}$ and for every U, an open neighbourhood of the neutral element of G_n, there exists a W, a clopen neighbourhood of the neutral element of G_{n+1}, so that $W \subset h_{n+1}^{-1}(U)$.

Define G_0 and g_0 in accordance with (i), and suppose that G_i, g_i and h_i have already been defined for every $i \leqslant n$ so that (i)-(iv) hold. Now we will define G_{n+1}, g_{n+1} and h_{n+1}.

By the inductive hypothesis, $w(G_n) \leqslant \tau$, so let $\{U_n^\alpha : \alpha < \tau\}$ be a base of open neighbourhoods of the neutral element of G_n. Fix an $\alpha < \tau$. Since $\operatorname{ind} G = 0$, there exists a V_n^α, a clopen neighbourhood of the neutral element of G, so that $V_n^\alpha \subset g_n^{-1}(U_n^\alpha)$. Define $f_{n+1}^\alpha : G \to \mathbb{R}$ by $f_{n+1}^\alpha(x) = 0$ if $x \in V_n^\alpha$ and $f_{n+1}^\alpha(x) = 1$ otherwise. Since V_n^α is clopen, f_{n+1}^α is continuous. Since G is $\mathbb{R}$-factorizable, we can find a group G_{n+1}^α with a countable base, a continuous homomorphism $g_{n+1}^\alpha : G \to G_{n+1}^\alpha$ and a continuous mapping $\varphi_{n+1}^\alpha : G_{n+1}^\alpha \to \mathbb{R}$ such that $f_{n+1}^\alpha = \varphi_{n+1}^\alpha \circ g_{n+1}^\alpha$. Define $g_{n+1} = \triangle\{g_{n+1}^\alpha : \alpha < \tau\} \triangle g_n$, $G_{n+1} = g_{n+1}(G)$, and let $h_{n+1} : G_{n+1} \to G_n$, $\rho_{n+1}^\alpha : G_{n+1} \to G_{n+1}^\alpha$ $(\alpha < \tau)$ be the natural projections. Then (ii) and (iii) hold trivially. Furthermore, for every $\alpha < \tau$ the set

$$W_n^\alpha = (\varphi_{n+1}^\alpha \circ \rho_{n+1}^\alpha)^{-1}(0)$$

is a clopen neighbourhood of the neutral element of G_{n+1}, and

$$g_{n+1}^{-1}(W_n^\alpha) = g_{n+1}^{-1}((\varphi_{n+1}^\alpha \circ \rho_{n+1}^\alpha)^{-1}(0)) = (\varphi_{n+1}^\alpha \circ \rho_{n+1}^\alpha \circ g_{n+1})^{-1}(0)$$

$$= (\varphi_{n+1}^\alpha \circ g_{n+1}^\alpha)^{-1}(0) = (f_{n+1}^\alpha)^{-1}(0) = V_n^\alpha \subset g_n^{-1}(U_n^\alpha),$$

so $W_n^\alpha \subset g_{n+1}(g_n^{-1}(U_n^\alpha)) = h_{n+1}^{-1}(U_n^\alpha)$, which yields (iv).

Now define $g = \triangle\{g_n : n \in \mathbb{N}\}$, $G^* = g(G)$, and let $h : G^* \to H$ be the natural projection (here we use (i)). Then $\pi = h \circ g$, and $w(G^*) \leqslant \tau$ by (ii). On the other hand, (iii) implies that G^* is a subgroup of the group $\tilde{G}$, the limit of the inverse sequence $\{G_n, \sigma_m^n : n, m \in \mathbb{N}, m \leqslant n\}$, where $\sigma_m^n = h_{m+1} \circ h_{m+2} \circ \cdots \circ h_{n-1} \circ h_n$ if $m < n$, and σ_n^n is the identity mapping of G_n for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $\sigma_n : \tilde{G} \to G_n$ be the limit projection. Then $\sigma_n = h_{n+1} \circ \sigma_{n+1}$ for every $n \in \mathbb{N}$. Now choose a V, an open neighbourhood of the neutral element of G. Then there exist an $n \in \mathbb{N}$ and a U, an open neighbourhood of the neutral element of G_n, such that $\sigma_n^{-1}(U) \subset V$ [17, Proposition 2.5.5]. Use (iv) and choose a W, a clopen neighbourhood of the neutral element of G_{n+1}, so that $W \subset h_{n+1}^{-1}(U)$. Hence $\sigma_{n+1}^{-1}(W)$ is the clopen neighbourhood of the neutral element of G, and

$$\sigma_{n+1}^{-1}(W) \subset \sigma_{n+1}^{-1}(h_{n+1}^{-1}(U)) = (h_{n+1} \circ \sigma_{n+1})^{-1}(U)$$

$$= \sigma_n^{-1}(U) \subset V.$$

Therefore, $\operatorname{ind} G^* \leqslant \operatorname{ind} \tilde{G} = 0$. □

Corollary 3.2. *The conclusion of Theorem* 3.1 *remains valid if one replaces "G is* $\mathbb{R}$*-factorizable" by "G is precompact" in its assumption.*

Proof. Use Fact 1.8 and Theorem 3.1. □

Theorem 3.3. ind $G=0$ *is equivalent to* dim $G=0$ *for each* $\mathbb{R}$*-factorizable topological group G.*

Proof. For "ind $G=0$ implies dim $G=0$" use Theorem 3.1 and Lemma 1.18(ii). On the other hand, the reverse implication holds for any Tychonoff space. □

Corollary 3.4. *If G is a precompact topological group, then* ind $G=0$ *is equivalent to* dim $G=0$.

Proof. Combine Fact 1.8 and Theorem 3.3. □

Now we turn to subspace theorems.

Theorem 3.5. *Suppose that H is a subgroup of a topological group G. If H is precompact, then* dim $H \leq$ dim G.

Proof. We advise the reader to consult Fig. 1 while reading the proof.

Let Z be a space with a countable base and $f: H \to Z$ a continuous mapping. Use Fact 1.8 and fix a group H^* with a countable base, a continuous homomorphism $\pi: H \to H^*$ and a continuous mapping $\varphi: H^* \to Z$ such that $f = \varphi \circ \pi$. Let $\tilde{G}$, $\tilde{H}$ and $\tilde{H}^*$ be two-sided uniformity completions of groups G, H and H^* respectively. Clearly, $\tilde{H}$ is a subgroup of $\tilde{G}$. Let $\tilde{\pi}: \tilde{H} \to \tilde{H}^*$ be the continuous homomorphism

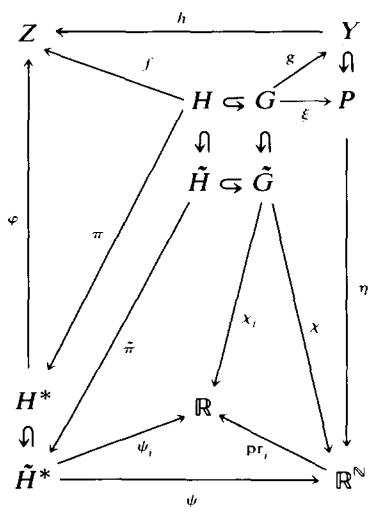

Fig. 1.

extending π [20]. Since H^* has a countable base, $\tilde{H}^*$ has a countable base too (Fact 1.1). So let $\psi_i : \tilde{H}^* \to \mathbb{R}$, for every $i \in \mathbb{N}$, be a continuous function such that

$$\psi = \triangle\{\psi_i : i \in \mathbb{N}\} : \tilde{H}^* \to \mathbb{R}^{\mathbb{N}}$$

is a homeomorphic imbedding. Since H is precompact, $\tilde{H}$ is compact, and so for every $i \in \mathbb{N}$ there exists a continuous function $\chi_i : \tilde{G} \to \mathbb{R}$ such that $\chi_i|_{\tilde{H}} = \psi_i \circ \tilde{\pi}$ [46]. From (i)⇒(ii) of Fact 1.12 it follows that for

$$\chi = \triangle\{\chi_i : i \in \mathbb{N}\} : \tilde{G} \to \mathbb{R}^{\mathbb{N}}$$

there exist a space P with a countable base and continuous mappings $\xi : G \to P$, $\eta : P \to \mathbb{R}^{\mathbb{N}}$ such that $\chi|_G = \eta \circ \xi$ and $\dim P \leqslant \dim G$. One can easily see that $\chi|_{\tilde{H}} = \psi \circ \tilde{\pi}$, so

$$\begin{aligned}\pi = \tilde{\pi}|_H &= \psi^{-1}|_{\psi(H^*)} \circ \psi|_{H^*} \circ \tilde{\pi}|_H = \psi^{-1}|_{\psi(H^*)} \circ \chi|_H \\ &= \psi^{-1}|_{\psi(H^*)} \circ (\eta \circ \xi)|_H.\end{aligned}$$

(We used here the fact that ψ is a homeomorphic imbedding.) Hence, if we set

$$Y = \xi(H), \qquad g = \xi|_H,$$

$$h = \varphi \circ \psi^{-1}|_{\psi(H^*)} \circ \eta|_Y,$$

then h would be defined correctly. Now observe that $g : H \to Y$ and $h : Y \to Z$ are continuous mappings, $f = h \circ g$ and $\dim Y \leqslant \dim P \leqslant \dim G$, because $Y \subset P$ and P has a countable base (Fact 1.11). Now (ii)⇒(i) of Fact 1.12 yields $\dim H \leqslant \dim G$. □

Corollary 3.6. *If H is a subgroup of a precompact group G, then* $\dim H \leqslant \dim G$.

For definition of Lindelöf Σ-spaces see [29]. Recall that each σ-compact space is a Lindelöf Σ-space. For definition of the dimension function Ind_0 see Definition 1.14.

Theorem 3.7. *Suppose that G is a topological group the space of which is a Lindelöf Σ-space (in particular, a σ-compact space), and X is its dense subspace.*

(i) *If U is an open subset of X, then* $\dim U \leqslant \dim X$ *and* $\mathrm{Ind}_0\, U \leqslant \mathrm{Ind}_0\, X$.

(ii) *If D is a dense subspace of X, then* $\dim D \leqslant \dim X$ *and* $\mathrm{Ind}_0\, D \leqslant \mathrm{Ind}_0\, X$.

Proof. From Fact 1.10 it follows that the space G is perfectly $\varkappa$-normal, and so is X as a dense subspace of a perfectly $\varkappa$-normal space. Now the result follows from Facts 1.16 and 1.17. □

Corollary 3.8. *If B is either an open or dense subset of a precompact group G, then* $\dim B \leqslant \dim G$ *and* $\operatorname{Ind}_0 B \leqslant \operatorname{Ind}_0 G$.

Corollary 3.9. *If D is a dense subspace of a Lindelöf Σ-group G (in particular, of a σ-compact group G), then* $\dim D \leqslant \dim G$ *and* $\operatorname{Ind}_0 D \leqslant \operatorname{Ind}_0 G = \operatorname{Ind} G$.

Proof. From Fact 1.10 one can easily deduce that each closed G_δ-subset of G is perfectly $\varkappa$-normal in the subspace topology, so $\operatorname{Ind}_0 G = \operatorname{Ind} G$ by Fact 1.15. The rest follows from Theorem 3.7. □

Corollary 3.10. *If H is a subgroup of a Lindelöf Σ-group G, then* $\dim H \leqslant \dim G$ *and* $\operatorname{Ind}_0 H \leqslant \operatorname{Ind}_0 G = \operatorname{Ind} G$.

Proof. Since $\bar{H}$, the closure of H in G, is a closed subspace of the normal space G, $\dim \bar{H} \leqslant \dim G$ and $\operatorname{Ind}_0 \bar{H} \leqslant \operatorname{Ind}_0 G$. Being a closed subgroup of the Lindelöf Σ-group G, $\bar{H}$ is a Lindelöf Σ-group. Now the result follows from Corollary 3.9. □

Corollary 3.11. *If H is a subgroup of a σ-compact group G, then* $\dim H \leqslant \dim G$ *and* $\operatorname{Ind}_0 H \leqslant \operatorname{Ind}_0 G = \operatorname{Ind} G$.

Theorem 3.12. *If H is a subgroup of a locally compact group G, then* $\dim H \leqslant \dim G$ *and* $\operatorname{Ind}_0 H \leqslant \operatorname{Ind}_0 G = \operatorname{Ind} G$.

Proof. Let U be an open neighbourhood of the neutral element of G having compact closure $\bar{U}$. Then G^*, the smallest subgroup of G containing $\bar{U}$, is a clopen σ-compact subgroup of G, and so G can be covered by disjoint clopen copies of G^*. Hence $\dim G^* = \dim G$ and $\operatorname{Ind}_0 G^* = \operatorname{Ind}_0 G$. Since $\operatorname{Ind}_0 G^* = \operatorname{Ind} G^*$ (see Corollary 3.11), we conclude that $\operatorname{Ind} G = \operatorname{Ind} G^* = \operatorname{Ind}_0 G^* = \operatorname{Ind}_0 G$. Now from Corollary 3.11 it follows that $\dim H^* \leqslant \dim G^*$ and $\operatorname{Ind}_0 H^* \leqslant \operatorname{Ind}_0 G^*$, where $H^* = H \cap G^*$. Since H^* is a clopen subgroup of H, H can be covered by disjoint clopen copies of H^*, and therefore $\dim H = \dim H^*$ and $\operatorname{Ind}_0 H = \operatorname{Ind}_0 H^*$. □

Corollary 3.13. *If H is a subgroup of a locally pseudocompact group G, then* $\dim H \leqslant \dim G$ *and* $\operatorname{Ind}_0 H \leqslant \operatorname{Ind}_0 G$.

Proof. The two-sided uniformity completion $\tilde{G}$ of G is a locally compact group. Moreover, $\dim \tilde{G} = \dim G$ and $\operatorname{Ind} \tilde{G} = \operatorname{Ind}_0 G$ [51]. Now apply Theorem 3.12. □

Theorem 3.14. *Assume that G is a Lindelöf Σ-group (in particular, a σ-compact group) and B is a G_δ-subset of G. Then* $\dim B \leqslant \dim G$ *and* $\operatorname{Ind}_0 B \leqslant \operatorname{Ind}_0 G = \operatorname{Ind} G$. *If we additionally suppose that B is normal in the subspace topology, then* $\operatorname{Ind} B \leqslant \operatorname{Ind} G$.

Proof. We need here the following result of Chigogidze [12, Theorem 6.11]: *If the closure of each G_δ-subset of X is a zero-set of X, then* $\dim B \leqslant \dim X$ *and* $\operatorname{Ind}_0 B \leqslant \operatorname{Ind}_0 X$ *provided that B is a G_δ-subset of X.* For the reader's convenience we give its proof here. Really, let B be a G_δ-subset of X and F a zero-set of B. Then F is a G_δ-subset of X and therefore, $\bar{F}$ is a zero-set of X. Since $\bar{F} \cap B = F$, we conclude that B is z-embedded in X. Now the reference to Fact 1.17 finishes the proof.

The first part of our theorem follows from Fact 1.10 and Chigogidze's result cited above (the inequalities), and from Corollary 3.10 (the equality $\operatorname{Ind}_0 G = \operatorname{Ind} G$). The second part follows from Facts 1.10 and 1.15. □

Theorem 3.15. *Suppose that $\{G_\alpha : \alpha \in A\}$ is a family consisting of precompact groups and* $\dim G_\alpha = 0$ *for every $\alpha \in A$. Then* $\dim \prod \{G_\alpha : \alpha \in A\} = 0$.

Proof. Since $\operatorname{ind} G_\alpha = 0$ for every $\alpha \in A$, $\operatorname{ind} \prod \{G_\alpha : \alpha \in A\} = 0$. The product $\prod \{G_\alpha : \alpha \in A\}$ is a precompact group, and now the result follows from Corollary 3.4. □

Remark 3.16. Theorem 3.15 is the particular case of the following results recently obtained by the author [42, 43]:

(i) *if $G_1, \ldots, G_k$ are precompact groups, then* $\dim G_1 \times \cdots \times G_k \leqslant \dim G_1 + \cdots + \dim G_k$,

(ii) *if $\{G_\alpha : \alpha \in A\}$ is a family consisting of precompact groups, and* $\dim \prod \{G_\alpha : \alpha \in B\} \leqslant n$ *for every finite $B \subset A$, then* $\dim \prod \{G_\alpha : \alpha \in A\} \leqslant n$.

4. Zero-dimensionality of free precompact groups

The main result of this section is the following:

Theorem 4.1. *For every space X the following conditions are equivalent:*

(i) $\dim X = 0$,

(ii) $\dim F^*(X) = 0$,

(iii) $\dim A^*(X) = 0$,

(iv) $\operatorname{ind} F^*(X) = 0$, *and*

(v) $\operatorname{ind} A^*(X) = 0$.

Before turning to the proof of this theorem let us deduce some corollaries from it.

Corollary 4.2. *Let X be the complete metric space with* $\operatorname{ind} X = 0$ *and* $\dim X \neq 0$ *constructed by Roy* [35]. *Then* $\operatorname{ind} F^*(X) \neq 0$ *and* $\operatorname{ind} A^*(X) \neq 0$. *Therefore,* $\operatorname{ind} X = 0$ *does not imply neither* $\operatorname{ind} F^*(X) = 0$ *nor* $\operatorname{ind} A^*(X) = 0$ *even for the metric space X.*

In connection with this corollary see also Theorem 4.5 and Corollaries 4.6 and 4.7 below.

In somewhat the same time and independently from each other Tkačenko [51] and the author made the following easy observation: *If* $\pi: G \to H$ *is a continuous homomorphism of a pseudocompact group* G *onto a (pseudocompact) group* H, *then* $\dim H \leqslant \dim G$. Indeed, let $\tilde{\pi}: \tilde{G} \to \tilde{H}$ be the continuous homomorphism extending π, where $\tilde{G}$ and $\tilde{H}$ are the two-sided uniformity completions of G and H respectively [20]. Since G and H are pseudocompact, $\tilde{G} = \beta G$ and $\tilde{H} = \beta H$ ([14]; see also [13, Theorem 6.5]). So $\dim \tilde{G} = \dim G$ and $\dim \tilde{H} = \dim H$. Being a continuous mapping defined on a compact space, $\tilde{\pi}$ is quotient. Now observe that $\dim \tilde{H} \leqslant \dim \tilde{G}$ because the dimension of a quotient group of a compact group does not exceed the dimension of the initial group [58].

It turns out however that if we pass from pseudocompact groups to precompact ones, the situation changes entirely (recall that pseudocompact groups are precompact, see [14], or [13, Lemma 6.3]).

Corollary 4.3. *Every precompact (Abelian) group* G *is a quotient group of a precompact (Abelian) group* H *so that* $\dim H = 0$ *and* $w(H) = w(G)$.

Proof. Let $\xi: D \to G$ be a mapping of the discrete space D onto G and $\tilde{\xi}: \beta D \to \beta G$ be its continuous extension over βD. Define $X = \tilde{\xi}^{-1}(G)$ and $f = \tilde{\xi}|_X : X \to G$. Then f is perfect, and hence quotient. Now Proposition 2.8 implies that the continuous homomorphic extension $\tilde{f}: F^*(X) \to G$ of f over $F^*(X)$ is open. Since $D \subset X \subset \beta D$, $\dim X = \dim \beta D = 0$, and so $\operatorname{ind} F^*(X) = 0$ by Theorem 4.1. Using Corollary 3.2 we can find a group H and continuous homomorphisms $\omega: F^*(X) \to H$, $\pi: H \to G$ such that $\tilde{f} = \pi \circ \omega$, $H = \omega(F^*(X))$, $w(H) \leqslant w(G)$ and $\operatorname{ind} H = 0$. The group H is precompact (Fact 1.4), so $\dim H = 0$ by Corollary 3.4. Now it remains to note that π is open, since $\tilde{f}$ is open [17, Proposition 2.1.3], and hence $w(G) \leqslant w(H)$. □

Recall that $\mathcal{T}^*_X$ denotes the topology of the free precompact (Abelian) group $F^*(X)$ ($A^*(X)$) of a topological space X (see Definition 1.5).

Proof of Theorem 4.1. (i)⇒(iv). Denote by $\mathcal{T}$ the topology of the free topological group of βX [27, 19]. Obviously, $\mathcal{T}^*_{\beta X} \subset \mathcal{T}$. Since $\dim \beta X = \dim X = 0$, from Graev's theorem[1] [19] it follows that $\operatorname{ind}(F(\beta X), \mathcal{T}) = 0$ (see also [48] for another proof of Graev's theorem). Fix an $n \in \mathbb{N}^+$. Since βX is a compact subspace of $(F(\beta X), \mathcal{T})$, from Lemma 1.3 it follows that $(F_n(\beta X), \mathcal{T}|_{F_n(\beta X)})$ is compact, and hence

$$\begin{aligned}\dim(F_n(\beta X), \mathcal{T}|_{F_n(\beta X)}) &\leqslant \operatorname{ind}(F_n(\beta X), \mathcal{T}|_{F_n(\beta X)}) \\ &\leqslant \operatorname{ind}(F(\beta X), \mathcal{T}) = 0 \end{aligned} \tag{1}$$

(in the first inequality of (1) we used [17, Theorem 7.2.7]). Furthermore, since $\mathcal{T}^*_{\beta X} \subset \mathcal{T}$ and $\mathcal{T}^*_{\beta X}$ is Hausdorff, $\mathcal{T}|_{F_n(\beta X)} = \mathcal{T}^*_{\beta X}|_{F_n(\beta X)}$. Therefore, (1) yields

[1] Graev's theorem states that the free (Abelian) topological group of a zero-dimensional compact space is ind-zero-dimensional.

$\dim(F_n(\beta X), \mathcal{T}^*_{\beta X}|_{F_n(\beta X)}) = 0$. Since $(F_n(\beta X), \mathcal{T}^*_{\beta X}|_{F_n(\beta X)})$ is compact for every $n \in \mathbb{N}$, $(F(\beta X), \mathcal{T}^*_{\beta X})$ is σ-compact (and hence normal). Now Theorem 1.13 implies that $\dim(F(\beta X), \mathcal{T}^*_{\beta X}) = 0$, i.e., $\dim F^*(\beta X) = 0$. In particular, $\operatorname{ind} F^*(\beta X) = 0$. Therefore, $\operatorname{ind} F^*(X) \leq \operatorname{ind} F^*(\beta X) = 0$ by Theorem 2.3.

(i)$\Rightarrow$(v) is analogous to (i)$\Rightarrow$(iv).

(ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) follow from Proposition 2.9.

(ii)$\Leftrightarrow$(iv) and (iii)$\Leftrightarrow$(v) follow from Corollary 3.4. □

The rest of this section is devoted to strengthening the author's results from [40] concerning the "$\operatorname{ind} X = 0$ implies $\operatorname{ind} F(X) = 0$" problem.

If X is a topological space and $\mathcal{U}$ is a uniformity on the set X generating the original topology of X, then we will say that $\mathcal{U}$ is *zero-dimensional* provided that for every $U \in \mathcal{U}$ there exists a Q, a clopen subset of $X \times X$, such that $\Delta_X = \{(x, x): x \in X\} \subset Q \subset U$. For definitions of $C^*(X)$, $\mathcal{C}^*(X)$, Φ_U and $C^X_{f,\varepsilon}$ used below see Section 2.

Lemma 4.4. *Assume that $(X, \mathcal{T})$ is a topological space, $\tilde{\mathcal{T}}$ is a group topology on $F(X)$ (on $A(X)$) satisfying $\mathcal{T}^*_X \subset \tilde{\mathcal{T}}$, $\tilde{\mathcal{T}}|_X = \mathcal{T}$ and* $\operatorname{ind}(F(X), \tilde{\mathcal{T}}) = 0$ ($\operatorname{ind}(A(X), \tilde{\mathcal{T}}) = 0$). *Then the uniformity $\mathcal{C}^*(X)$ is zero-dimensional.*

Proof. Fix $f_1, \ldots, f_n \in C^*(X)$ and an $\varepsilon > 0$. Without loss of generality we can assume that $\varepsilon < 1$. By Lemma 2.1, one can choose $U_i \in \mathcal{T}^*_X$ so that $e \in U_i$ and $C^X_{f_i,\varepsilon} = \Phi_{U_i} \cap (X \times X)$, $i = 1, \ldots, n$. Set $U = \bigcap \{U_i: i = 1, \ldots, n\}$. Since $e \in U \in \mathcal{T}^*_X \subset \tilde{\mathcal{T}}$ and $\operatorname{ind}(F(X), \tilde{\mathcal{T}}) = 0$, there is a $\tilde{\mathcal{T}}$-clopen set V with $e \in V \subset U$. Mappings $(g, h) \mapsto g^{-1}h$ and $(g, h) \mapsto gh^{-1}$ being $\tilde{\mathcal{T}}$-continuous, Φ_V is a clopen subset of $(F(X), \tilde{\mathcal{T}}) \times (F(X), \tilde{\mathcal{T}})$ satisfying $\Delta_X \subset \Phi_V$. Now $\Delta_X \subset Q = \Phi_V \cap (X \times X) \subset C_{f_1,\varepsilon} \cap \cdots \cap C_{f_n,\varepsilon}$, and Q is clopen in $X \times X$. □

In [40] the following spaces were constructed:

(i) *a normal space X_1 such that* $\operatorname{ind} X_1 = 0$ *but the uniformity $\mathcal{C}^*(X_1)$ is not zero-dimensional* (in fact, X_1 is the famous Dowker space [16]; see also [17, Example 6.2.20]), and

(ii) *a pseudocompact space X_2 so that* $\operatorname{ind} X_2 = 0$ *but $\mathcal{C}^*(X_2)$ is not zero-dimensional.*

Now Lemma 4.4 yields:

Theorem 4.5. *Let $i = 1, 2$ and X_i be as above. Suppose that $\mathcal{T}_{X_i}$ is a group topology on $F(X_i)$ (on $A(X_i)$) inducing the original topology of X_i and satisfying $\mathcal{T}^*_{X_i} \subset \mathcal{T}_{X_i}$. Then* $\operatorname{ind}(F(X_i), \mathcal{T}_{X_i}) \neq 0$ ($\operatorname{ind}(A(X_i), \mathcal{T}_{X_i}) \neq 0$).

Corollary 4.6. *There exists a pseudocompact space X so that* $\operatorname{ind} X = 0$ *but* $\operatorname{ind} F^*(X) \neq 0$ *and* $\operatorname{ind} A^*(X) \neq 0$.

Note that if X is compact (even Lindelöf) and $\operatorname{ind} X = 0$, then $\dim X = 0$ [17, Theorem 7.2.7], and thus $\operatorname{ind} F^*(X) = \operatorname{ind} A^*(X) = 0$ by Theorem 4.1.

Corollary 4.7 [40]. *There exist a normal space X and a pseudocompact space Y such that* $\operatorname{ind} X = \operatorname{ind} Y = 0$, *while* $\operatorname{ind} G(X) \neq 0$ *and* $\operatorname{ind} G(Y) \neq 0$, *where $G(Z)$ denotes either the free topological group or the free Abelian topological group of a space Z* [27, 19].

5. Closed imbeddings into precompact groups preserving zero-dimensionality

Theorem 5.1. *Assume that $(X, \mathcal{T})$ is a space with* $\operatorname{ind}(X, \mathcal{T}) = 0$. *Then there exists a (Hausdorff) group topology $\tilde{\mathcal{T}}$ on the free group $F(X)$ of X satisfying the following properties*:

(i) $\tilde{\mathcal{T}}|_X = \mathcal{T}$,
(ii) $(F(X), \tilde{\mathcal{T}})$ *is a precompact group*,
(iii) $w(F(X), \tilde{\mathcal{T}}) = w(X, \mathcal{T})$,
(iv) $\dim(F(X), \tilde{\mathcal{T}}) = 0$,
(v) *the set X is $\tilde{\mathcal{T}}$-closed*,
(vi) *for every $n \in \mathbb{N}$ the set $F_n(X)$ is $\tilde{\mathcal{T}}$-closed*,
(vii) *for every $n \in \mathbb{N}$ the mapping $\theta_n|_{Z_n} : (Z_n, \mathcal{T}_n) \to (B_n(X), \tilde{\mathcal{T}}|_{B_n(X)})$ is a homeomorphism (here $\mathcal{T}_n$ is the subspace topology on Z_n induced by the topology of the Cartesian product $((X, \mathcal{T}) \oplus (X, \mathcal{T})^{-1} \oplus \{e\})^n$).*

An analogous result also holds for the free Abelian group $A(X)$ of X.

Proof. Let $w(X, \mathcal{T}) = \tau$. First of all let us consider the special case $\dim(X, \mathcal{T}) = 0$.

Fix an $n \in \mathbb{N}^+$. By Proposition 2.5, the mapping

$$\theta_n|_{Z_n} : (Z_n, \mathcal{T}_n) \to (B_n(X), \mathcal{T}^*_X|_{B_n(X)})$$

is a homeomorphism, so, since $w(Z_n, \mathcal{T}_n) = w(X, \mathcal{T}) = \tau$, we can choose a $\gamma_n \subset \mathcal{T}^*_X$ such that $|\gamma_n| = \tau$, and $\{U \cap B_n(X) : U \in \gamma_n\}$ forms a base of the topology $\mathcal{T}^*_X|_{B_n(X)}$. Set

$$\mathcal{E} = \bigcup \{\gamma_n : n \in \mathbb{N}^+\} \cup \{F(X) \setminus F_n(X) : n \in \mathbb{N}^+\} \cup \{F(X) \setminus X\}. \tag{2}$$

Then $|\mathcal{E}| \leq \tau$ and $\mathcal{E} \subset \mathcal{T}^*_X$. Fact 1.9 yields $\operatorname{nw}(F(X), \mathcal{T}^*_X) \leq w(X, \mathcal{T}) = \tau$. Hence we can apply Fact 1.2 to find a (Hausdorff) group topology $\mathcal{T}'$ on $F(X)$ so that

$$\mathcal{E} \subset \mathcal{T}' \subset \mathcal{T}^*_X \tag{3}$$

and $w(F(X), \mathcal{T}') \leq \tau$. Since $\dim(X, \mathcal{T}) = 0$, $\operatorname{ind}(F(X), \mathcal{T}^*_X) = 0$ (Theorem 4.1). From (3) it follows that the identity homomorphism $i : (F(X), \mathcal{T}^*_X) \to (F(X), \mathcal{T}')$ is continuous. Now we can use Corollary 3.2 to choose a (Hausdorff) group topology $\tilde{\mathcal{T}}$ on $F(X)$ such that

$$\mathcal{T}' \subset \tilde{\mathcal{T}} \subset \mathcal{T}^*_X, \tag{4}$$

$w(X, \tilde{\mathcal{T}}) \leq w(X, \mathcal{T}') \leq \tau$ and $\operatorname{ind}(F(X), \tilde{\mathcal{T}}) = 0$. We claim that $\tilde{\mathcal{T}}$ is as required.

Let us verify (i)–(vii) for $\tilde{\mathcal{T}}$. From our choice of γ_1 and (2)–(4) it follows that $\mathcal{T} \subset \tilde{\mathcal{T}}|_X \subset \mathcal{T}^*_X|_X = \mathcal{T}$, so (i) holds. Since $\tilde{\mathcal{T}} \subset \mathcal{T}^*_X$ by (4), precompactness of $\mathcal{T}^*_X$

implies precompactness of $\tilde{\mathscr{T}}$ (Fact 1.4), and hence (ii) holds. Item (iii) follows from our construction. As for (iv), note that $\operatorname{ind}(F(X), \tilde{\mathscr{T}}) = 0$, and $\tilde{\mathscr{T}}$ is precompact, hence $\dim(F(X), \tilde{\mathscr{T}}) = 0$ by Corollary 3.4. Items (v), (vi) follow from (2)-(4). To verify (vii) observe that (2)-(4) and our choice of γ_n imply that $\tilde{\mathscr{T}}|_{B_n(X)} = \mathscr{T}_X^*|_{B_n(X)}$, and now the result follows from Proposition 2.5.

Now let us consider the general case, i.e., $\operatorname{ind}(X, \mathscr{T}) = 0$. In this case there exists a homeomorphic imbedding $j : (X, \mathscr{T}) \to D^\tau$ of $(X, \mathscr{T})$ into the Cantor cube D^τ. Let $\mathscr{T}_0$ be the topology of D^τ. Then $\dim(D^\tau, \mathscr{T}_0) = 0$, and so our special case considered above guarantees the existence of the group topology $\tilde{\mathscr{T}}_0$ on $F(D^\tau)$ satisfying the properties (i)-(vii), in which the subscript 0 should be added to $\mathscr{T}$ and $\tilde{\mathscr{T}}$. Observe that the homomorphic extension $\tilde{j} : F(X) \to F(D^\tau)$ of j is the isomorphism between groups $F(X)$ and $\tilde{j}(F(X))$. Define $\tilde{\mathscr{T}} = \{\tilde{j}^{-1}(U \cap \tilde{j}(F(X))) : U \in \tilde{\mathscr{T}}_0\}$. Then the properties (i)-(iii), (v)-(vii) for $\tilde{\mathscr{T}}$ follow from the corresponding properties for $\tilde{\mathscr{T}}_0$ and formulas $X = \tilde{j}^{-1}(D^\tau \cap \tilde{j}(F(X)))$ and $F_n(X) = \tilde{j}^{-1}(F_n(D^\tau) \cap \tilde{j}(F(X)))$, $n \in \mathbb{N}^+$. Moreover,

$$\operatorname{ind}(F(X), \tilde{\mathscr{T}}) = \operatorname{ind}(\tilde{j}(F(X)), \tilde{\mathscr{T}}_0|_{\tilde{j}(F(X))}) \leq \operatorname{ind}(F(D^\tau), \tilde{\mathscr{T}}_0) = 0.$$

Now $\dim(F(X), \tilde{\mathscr{T}}) = 0$ follows from precompactness of $\tilde{\mathscr{T}}$ and Corollary 3.4. (Another way is to use Corollary 3.6, which yields

$$\dim(F(X), \tilde{\mathscr{T}}) = \dim(\tilde{j}(F(X)), \tilde{\mathscr{T}}_0|_{\tilde{j}(F(X))}) \leq \dim(F(D^\tau), \tilde{\mathscr{T}}_0) = 0.) \qquad \square$$

Corollary 5.2. *Every space X with* $\operatorname{ind} X = 0$ *can be imbedded as a closed subspace into a precompact Abelian group G so that* $\dim G = 0$ *and* $w(G) = w(X)$.

Our next result was obtained by the author in [39], but its proof in [39] contained some misleading typographical errors, so we give the proof here.

Theorem 5.3. *Each space X with* $\operatorname{ind} X = 0$ *can be imbedded as a closed subspace into a pseudocompact Abelian group G such that* $\dim G = 0$ *and* $w(G) = \aleph_1 \cdot w(X)$.

Proof. Define $\tau = w(X)$. Let H be the Cantor cube D^τ regarded as the topological group. Define

$$Z = \{z = \{z_\alpha : \alpha \in \omega_1\} \in H^{\omega_1} : |\{\alpha \in \omega_1 : z_\alpha \neq 0\}| \leq \aleph_0\}.$$

Being the Σ-product of compact spaces, Z is countably compact. Let $j : H \to H^{\omega_1}$ be the diagonal mapping sending each $h \in H$ to the element $j(h) \in H^{\omega_1}$ all coordinates of which coincide with h. Then j is a homeomorphic imbedding. Since $\operatorname{ind} X = 0$, we can fix a homeomorphic imbedding $i : X \to H$. The subspace $Y = j(i(X))$ of H^{ω_1} is homeomorphic to X. Let G be the smallest subgroup of H^{ω_1} generated by $Y \cup Z$. Since G contains dense countably compact subspace Z, G is pseudocompact. Note that $\bar{Y} \subset \overline{j(H)} = j(H)$, where the bar denotes the closure in H^{ω_1}. Moreover, if $g \in G$, then $g = j(i(x)) + z$ for some $x \in X$ and $z \in Z$. This yields that some coordinate of g coincides with $i(x)$, because $z \in Z$. Thus, we conclude that $\bar{Y} \cap G = Y$, i.e., Y is closed in G. $\square$

Remark 5.4. Since pseudocompact groups are precompact, Theorem 5.3 is a strengthening of Corollary 5.2 in case $w(X) > \aleph_0$. The last restriction is essential. In fact, in the equality $w(G) = \aleph_1 \cdot w(X)$ of Theorem 5.3 the cardinal $\aleph_1$ *cannot* be dropped, because a pseudocompact space with a countable base is compact, and hence a noncompact space X with a countable base cannot be imbedded as a closed subspace into a pseudocompact group G with a countable base.

It is easy to see that every precompact group is a closed subgroup of a pseudocompact group[2]. Comparing this fact with Theorem 5.3 leads to the following question: Is every zero-dimensional precompact group a (closed) subgroup of a zero-dimensional pseudocompact group?

Example 5.5. *Suppose that G is a countable dense subgroup of* S^1. *Then G is precompact,* $\dim G = 0$, *but* $\dim H > 0$ *whenever H is a (locally) pseudocompact group containing G as a subgroup.* Indeed, let $\tilde{G}$ and $\tilde{H}$ be the two-sided uniformity completions of G and H respectively. Note that $\tilde{G}$ is a subgroup of $\tilde{H}$ and $\tilde{G} = S^1$, so $\dim \tilde{H} \geqslant \dim \tilde{G} = 1$ by Corollary 3.13. Since H is locally pseudocompact, $\dim H = \dim \tilde{H}$ [51].

Remark 5.6. Theorem 5.1 improves Sipachöva's result from [44], which states that for every space $(X, \mathscr{T})$ with $\mathrm{ind}(X, \mathscr{T}) = 0$ there exists a group topology $\tilde{\mathscr{T}}$ on $F(X)$ satisfying $\mathrm{ind}(F(X), \tilde{\mathscr{T}}) = 0$ and the properties (i), (iii), (v), (vi) of Theorem 5.1.

6. Open problems and concluding remarks

We start with the following

Question 6.1 (Arhangel'skiĭ [6]). Assume that $\dim X = 0$. Is then $\dim G(X) = 0$? (Here $G(X)$ denotes either the free topological group or the free Abelian topological group of X.)

Tkačenko [47] showed that if $\dim X = 0$, then the free Abelian topological group of X is ind-zero-dimensional, and Sipachöva [45] obtained the analogous result for the free topological group of X.

One can easily see that a topological group G is precompact iff for every U, an open neighbourhood of the neutral element of G, there exists a *finite* set $F_U \subset G$ with $G = F_U \cdot U = \{xu : x \in F_U, u \in U\}$. Recall that a topological group G is said to be $\aleph_0$-*bounded* [21] iff the set F_U as above can be chosen to be *at most countable.* Therefore, precompact groups are $\aleph_0$-bounded. Guran [21] showed that G is an $\aleph_0$-bounded group if and only if there exists a family $\{G_\alpha : \alpha \in A\}$ consisting of topological groups with a countable base so that G is isomorphic (algebraically and topologically) to a subgroup of $\prod \{G_\alpha : \alpha \in A\}$ (see [55] for a nice proof of this

[2] This observation seems to be due to M.I. Ursul and W.W. Comfort.

result), and that the class of $\aleph_0$-bounded groups forms a variety (see [28] for the basic theory of varieties of topological groups). Let $F_{\aleph_0}(X)$ ($A_{\aleph_0}(X)$) be the free $\aleph_0$-bounded (Abelian) topological group of a space X, i.e., the free object in the variety of $\aleph_0$-bounded (Abelian) groups. Now Theorem 4.1 leads to the following:

Question 6.2. Does $\dim X = 0$ imply $\dim F_{\aleph_0}(X) = 0$ and $\dim A_{\aleph_0}(X) = 0$?

The proof of the main result of [48] can be easily modified to obtain:

Theorem 6.3. *If* $\dim X = 0$, *then* $\operatorname{ind} F_{\aleph_0}(X) = 0$ *and* $\operatorname{ind} A_{\aleph_0}(X) = 0$.

Therefore, the answer to Question 6.2 will be "yes" if our next question should be settled positively.

Question 6.4. Is $\operatorname{ind} G = 0$ equivalent to $\dim G = 0$ for every $\aleph_0$-bounded group G?

Note that Theorem 3.3 cannot be applied to settle this question, because Tkačenko constructed an $\aleph_0$-bounded topological group which is not $\mathbb{R}$-factorizable.

In [41] the author found an example of a precompact Abelian group G with $\dim G = 1$ and $\operatorname{ind} G = \infty$, so Corollary 3.4 cannot be improved to "$\dim G = \operatorname{ind} G$ for every precompact group G". Nevertheless, the following question is open.

Question 6.5. Does $\dim G \leq \operatorname{ind} G$ for every precompact (and more generally, $\mathbb{R}$-factorizable or $\aleph_0$-bounded) group G?

Theorem 3.1 and Corollary 3.2 lead to the following:

Question 6.6. Assume that G is a precompact ($\mathbb{R}$-factorizable, $\aleph_0$-bounded) topological group, $\pi : G \to H$ is a continuous homomorphism from G to a topological group H. Are there a topological group G^* and continuous homomorphisms $g : G \to G^*$, $h : G^* \to H$ such that $\pi = h \circ g$, $w(G^*) \leq w(G)$ and $\operatorname{ind} G^* \leq \operatorname{ind} G$?

From Fact 1.8 and Lemma 1.18(ii) it follows that "yes" to Question 6.6 for precompact ($\mathbb{R}$-factorizable) groups implies "yes" to Question 6.5 for precompact ($\mathbb{R}$-factorizable) groups.

Our Theorems 3.5, 3.12, Corollaries 3.6, 3.10, 3.11 and 3.13 are related to the following question, which was claimed to be an old one by Zambakhidze.

Question 6.7. Is $\dim H \leq \dim G$ whenever H is a subgroup of a topological group G?

The next question is motivated by Theorem 0.1.

Question 6.8. Can every space X with $\dim X < \infty$ be imbedded as a (closed) subspace into a topological group G with $\dim G < \infty$? Is it true that every space X with $\dim X \leq n$ can be imbedded as a (closed) subspace into a topological group

G so that $\dim G \leqslant 2n+1$? What would be the answer if one additionally assumes that X is compact?

By Nöbeling–Pontrjagin's theorem [31, 34], if X has a countable base and $\dim X \leqslant n$, then X can be imbedded into the group $\mathbb{R}^{2n+1}$, so the half of Question 6.8 is settled positively for spaces having a countable base. Nevertheless, the other half of it, i.e., whether every space X with a countable base so that $\dim X \leqslant n$ can be imbedded as a *closed* subspace into a topological group G with $\dim G \leqslant 2n+1$, seems to be unsolved.

Now assume that the answer to Question 6.8 is "yes". Vopenka [56] constructed, for every integer $n \geqslant 1$, a compact space Y_n so that $\dim Y_n = 1$ and $\operatorname{ind} Y_n \geqslant n$. Let X be the one-point compactification of $\oplus\{Y_n : n \in \mathbb{N}^+\}$. Then $\dim X = 1$ by Theorem 1.13, and $\operatorname{ind} X = \infty$. For this X use the positive answer to Question 6.8, and let G be a topological group containing X as a subspace so that $\dim G < \infty$. Let H be the smallest subgroup of G that contains X. Then H is σ-compact, and the same argument as in the proof of Theorem 0.1 shows that $\dim H \leqslant \dim G < \infty$. On the other hand, $\operatorname{ind} H \geqslant \operatorname{ind} X = \infty$. Therefore, H is a σ-compact group satisfying $\dim H \neq \operatorname{ind} H$, and so the positive answer to Question 6.8 yields the negative answer to Question 3.6 of [41].

Note added in proof

(i) For any Abelian group G let $G^{\#}$ denote the group G endowed with the smallest topology which makes every homomorphism $\pi : G \to S^1$ continuous. E.K. van Douwen in his paper entitled "*The maximal totally bounded group topology on G and the biggest minimal G-space, for Abelian group G*" (Topology Appl. 34 (1990) 69–91) showed that $\operatorname{ind} G^{\#} = 0$ for every Abelian group G (the same result was also announced in: W.W. Comfort and F.J. Trigos, *The maximal totally bounded group topology*, Abstracts Amer. Math. Soc. 9 (1988) 420–421 (Abstract #88T-22-195)) and asked whether always $\dim G^{\#} = 0$ (Question 4.10). Since $G^{\#}$ is precompact, the positive answer to this question follows from Corollary 3.4 of our paper.

(ii) Some additional open questions related to this paper can be found in: D.B. Shakhmatov, *A survey of current researches and open problems in the dimension theory of topological groups*, Questions Answers in Gen. Topology 8 (1990) 101–128.

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