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Fraenkel's conjecture for six sequences

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Abstract

A striking conjecture of Fraenkel asserts that every decomposition of $\mathbb{Z}_{>0}$ into $m \geqslant 3$ sets $\{\lfloor \alpha_i n + \beta_i \rfloor\}_{n \in \mathbb{Z}_{>0}}$ with α_i and β_i real, $\alpha_i > 1$ and α_i's distinct for $i = 1, \ldots, m$ satisfies

$$\{\alpha_1, \ldots, \alpha_m\} = \left\{ \frac{2^m - 1}{2^k} \colon 0 \leqslant k < m \right\}.$$

Fraenkel's conjecture was proved by Morikawa if $m = 3$ and, under some condition, if $m = 4$. Proofs in terms of balanced sequences have been given for $m = 3$ by the author and for $m = 4$ by Altman, Gaujal and Hordijk. In the present paper we use the latter approach to establish Fraenkel's conjecture for $m = 5$ and for $m = 6$.

Keywords: Complementary sequences; Exact covers; Fraenkel's conjecture; Beatty sequences; Balanced words

1. Fraenkel's conjecture

For real numbers $\alpha \geqslant 1$ and β denote by $S(\alpha, \beta)$ the sequence $\{\lfloor n\alpha + \beta \rfloor \mid n = 1, 2, 3, \ldots\}$. A finite set $\{S(\alpha_i, \beta_i) \mid 1 \leqslant i \leqslant m\}$ is called an (eventual) exact cover if every (sufficiently large) positive integer occurs in exactly one $S(\alpha_i, \beta_i)$. If $\{S(\alpha_i, \beta_i)\}_{i=1}^m$ is an eventual exact cover, then $\sum_{i=1}^m \alpha_i^{-1} = 1$. Hence the only case with $m = 1$ is when $\alpha_1 = 1$. We call α the rate of $S(\alpha, \beta)$ and α^{-1} its density and β the rest.

Skolem [14] gave a criterion for exact covers in case $m = 2$. He proved that if α_1 is irrational, then $\{S(\alpha_i, \beta_i)\}_{i=1}^2$ is an eventual exact cover if and only if

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1 \quad \text{and} \quad \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \in \mathbb{Z}.$$

Fraenkel [6] proved the same result in case α_1 is rational. He further gave similar, but more complicated, necessary and sufficient conditions for $\{S(\alpha_i, \beta_i)\}_{i=1}^2$ to be an exact cover.

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For $m=3$ there exists an exact cover with distinct rates, namely

$$\{S(\tfrac{7}{4},0),S(\tfrac{7}{2},-1),S(7,-3)\}.$$

Fraenkel [7] noticed that there is an example for every m, namely

$$\left\{S\left(\frac{2^m-1}{2^{i-1}},1-2^{m-i}\right)\right\}_{i=1}^{m}. \tag{1}$$

He further made the following conjecture (cf. [5, p. 19]).

Fraenkel's Conjecture. If $\{S(\alpha_i,\beta_i)\}_{i=1}^m$ is an exact cover with distinct rates $\alpha_1,\ldots,\alpha_m$ and $m\geqslant 3$, then

$$\{\alpha_1,\ldots,\alpha_m\}=\left\{\frac{2^m-1}{2^k}\colon 0\leqslant k<m\right\}.$$

Graham [8] used Kronecker's approximation theorem and Skolem's work to show that if all the rates of an eventual exact cover of $m\geqslant 3$ sequences $S(\alpha_i,\beta_i)$ are distinct, then all the rates α_i are rational. It follows that each $S(\alpha_i,\beta_i)$ is periodic and that, when dealing with Fraenkel's conjecture, we need not distinguish between eventual exact covers and exact covers.

If the rate α of a sequence $S(\alpha,\beta)$ is rational, we may assume without loss of generality that β is a rational number with the same denominator as α. Suppose $\{S(a_i/c_i,b_i/c_i)\}_{i=1}^m$ is an exact cover with rates a_i/c_i where $\gcd(a_i,c_i)=1$ for $i=1,\ldots,m$ and $m\geqslant 3$. Berger et al. [4] proved that if $a_1\leqslant a_2\leqslant\cdots\leqslant a_m$, then $a_{m-1}=a_m$ and moreover if $c_{m-1}\neq c_m$, then $a_{m-2}=a_{m-1}=a_m$.

A thorough study of exact covers $\{S(\alpha_i,\beta_i)\}_{i=1}^m$ was made by Morikawa in a series of five papers. In [9] he classified all exact covers with $m=3$. His results imply Fraenkel's conjecture for $m=3$. In [10] he classified all exact covers of the form $\{S(c/a_i,c/b_i)\}_{i=1}^4$ with $\gcd(a_i,c)=1$ for $i=1,2,3,4$ and verified Fraenkel's conjecture for this subclass.

In 1991 Simpson [13] showed that Fraenkel's conjecture is true if $\min_i\alpha_i\leqslant\frac{3}{2}$.

Quite recently, Altman et al. [1] established Fraenkel's conjecture in case $m=4$. Since they do so in the more general framework of balanced words, I shall deal with their paper in the next section.

2. Balanced words

Let Q be a finite set of 'letters', a so-called alphabet. A word on Q is defined as a surjective function $W:\mathbb{Z}_{>0}\to Q$. By a block we mean a finite set of consecutive positive integers. A block of N positive integers is called a block of length N or an N-block.

A sequence S of positive integers is called a Beatty sequence with density d if the number of terms of S in an arbitrary N-block equals $\lfloor Nd \rfloor$ or $\lceil Nd \rceil$.

A word $W = \{W(s)\}_{s=1}^{\infty}$ on Q is called a Beatty word if the inverse image of each letter of Q forms a Beatty sequence.

A set S of positive integers is called balanced if the numbers of elements of S in any two blocks of equal lengths differ by at most 1.

The word W is called balanced if the inverse image of each letter is a balanced set.

Usually, a Beatty sequence on the positive integers is defined as a sequence $\{\lfloor n\alpha \rfloor\}_{n=1}^{\infty}$ or as a sequence $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$ with $\beta < 1 \leqslant \alpha + \beta$ (cf. [2,3,15,13]). It follows from Lemma 1 that every such sequence $S(\alpha, \beta)$ is a Beatty sequence with $d = \alpha^{-1}$ as the density and that every Beatty sequence is of the form $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$ or of the form $\{\lceil n\alpha + \beta \rceil\}_{n=1}^{\infty}$. If α is rational, then every sequence of the latter type can be written in the form of the former type and vice versa. If α is irrational, a sequence of the latter type need not be of the former type; e.g. $\{\lceil n\pi + 2 - \pi \rceil\}_{n=1}^{\infty}$ cannot be represented as $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$.

Obviously, a Beatty word represents an exact cover of the positive integers by Beatty sequences, and similarly for balanced. By definition every Beatty sequence forms a balanced set. Hence every Beatty word is a balanced word. A result in the opposite direction is given in the Corollary of Lemma 2. Every balanced set has a density $d \in [0, 1]$ which is the limit of the number of elements $\leqslant x$ divided by x as $x \to \infty$ (see [11, p. 5; 17, Lemma 1]). In the case of a Beatty sequence the density is equal to the inverse of the rate. We define the rate of a balanced set as the inverse of its density.

Almost by definition the complement of a Beatty sequence is a Beatty sequence and the complement of a balanced set is a balanced set. So every balanced set defines a balanced word on two letters (defined by its characteristic function), and every balanced word on two letters is characterised by a balanced set (take the inverse image of one of both letters). Altman et al. [1, Lemma 2.26] used a generalisation of Graham's result by Hubert to show that every balanced word on $m \geqslant 3$ letters with distinct rates is periodic so that all the rates are rational.

The balanced words on three letters were characterised by the author [16]: a balanced word on the letters a, b, c originates from any balanced set $S = \{s_i\}_{i \in \mathbb{Z}_{>0}}$ with $s_i < s_{i+1}$ for all i by putting letters a on places s_{2i}, letters b on places s_{2i+1} and letters c on the remaining places, or is a periodic word with period cycle ($abacaba$). (Of course, the choice a, b, c is arbitrary.) Note that in the former case the letters a and b have the same rates and that the exceptional case is in agreement with Fraenkel's conjecture. Altman et al. [1, Theorem 2.19] gave a simplified proof of this result and generalised it to balanced words on four letters. To be precise, they proved that the only balanced words on four letters with distinct rates are the periodic words with cycle ($abacabadabacaba$) and rates $\frac{15}{8}, \frac{15}{4}, \frac{15}{2}, 15$. This established Fraenkel's conjecture in case $m = 4$, without the restriction imposed by Morikawa.

In the present paper we extend the method to establish Fraenkel's conjecture in the cases $m = 5$ and $m = 6$. By Lemma 1 it suffices to prove the following assertion.

Theorem. *The only Beatty words W on m letters $a_1,\dots,a_m$ with distinct rates $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ and $3 \leqslant m \leqslant 6$ are the periodic words with cycle (F_m) where $F_1 = a_1$ and $F_m = F_{m-1}a_mF_{m-1}$ for $m = 2,3,\dots$. It follows that $\alpha_j = (2^m-1)/(2^{m-j})$ for $j = 1,2,\dots,m$.*

Section 3 contains some fundamental properties of Beatty sequences, balanced sequences and balanced words. Lemma 3 states that if a word is balanced on letters with distinct rates and the letter a has rate $\leqslant 3$, then the word remains balanced if all the a's are omitted. This is an extension of a result of Altman et al. [1, Lemma 2.27], who proved the result when a has rate $\leqslant 2$. Lemma 4, already contained in [17], states that if the rates of the letters of a balanced word are distinct, then the integer parts of the rates are distinct. The Theorem is proved in Section 4 in case all rates are at least 3 and in Section 5 in the remaining case.

For more information on covers and Fraenkel's conjecture I refer to [12] which contains a survey of the theory of covers up to 1981. Paper [19] consists of a less complete, but more up-to-date survey of the literature on exact covers, a direct proof for Fraenkel's conjecture in case $m = 5$ and proofs of some related results.

3. Some basic principles

In this section we derive some fundamental properties which are used in the proofs of the theorem. We denote by $|V|$ and $c_x(V)$ the number of letters of the finite subword V of W and the number of times that the letter x occurs in V, respectively. We further denote by r_a the rate of the letter a (in a given word W). Observe that r_a is the average distance between consecutive a's, where the distance between consecutive letters is 1.

Lemma 1. *Every sequence of the form $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$ with $\alpha \geqslant 1$ and $\beta < 1 \leqslant \alpha + \beta$ or of the form $\{\lceil n\alpha + \beta \rceil\}_{n=1}^{\infty}$ with $\alpha \geqslant 1$ and $\beta \leqslant 0 < \alpha + \beta$ is a Beatty sequence. Conversely, every Beatty sequence is of one of these forms.*

Proof. The number of terms of the sequence $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$ in an N-block of positive integers $(x, x+N]$ equals $\lfloor (x+N-\beta)/\alpha \rfloor - \lfloor (x-\beta)/\alpha \rfloor$ and this difference equals $N/\alpha + \delta$ with $|\delta| < 1$ and therefore $\lfloor N/\alpha \rfloor$ or $\lceil N/\alpha \rceil$. The proof that the sequence $\{\lceil n\alpha + \beta \rceil\}_{n=1}^{\infty}$ in a Beatty sequence is similar.

Conversely, every Beatty sequence is a balanced sequence. The balanced sequences have been classified by Morse and Hedlund [11, pp. 8–19] and consist of three classes: periodic, rational and skew. The skew sequences are not Beatty. The periodic and irrational sequences are of the form $\{\lfloor n\alpha + \beta \rfloor\}_{n=1}^{\infty}$ or $\{\lceil n\alpha + \beta \rceil\}_{n=1}^{\infty}$ with α rational in the periodic case and α irrational in the irrational case. (cf. [18, Theorem 2]). $\square$

Lemma 2. *Let $S = \{s_i\}_{i \in \mathbb{Z}_{>0}}$ be an infinite balanced set of rate r. Let k be a positive integer. If $kr \notin \mathbb{Z}$, then $s_{i+k} - s_i$ equals $\lfloor kr \rfloor$ or $\lceil kr \rceil$ for all i. If $kr \in \mathbb{Z}$, then either*

$s_{i+k} - s_i \in \{kr, kr+1\}$ for all i or $s_{i+k} - s_i \in \{kr-1, kr\}$ for all i and it can occur for only finitely many i that $s_{i+k} - s_i - kr$ is non-zero.

Proof. See [11, p. 17] or [19, Lemma 2]. □

We call a word W an eventually Beatty (balanced) word if $W(s)$ for s greater than some s_0 coincides with a Beatty (balanced) word. It is obvious that every eventually Beatty word is an eventually balanced word. As an immediate consequence of Lemma 2 we have the converse result.

Corollary. *Every eventually balanced word is an eventually Beatty word.*

It is not true that every balanced word is a Beatty word. For example, $a(ab)^\infty$ is balanced, but not Beatty. Note that the rate of a sequence is determined by its asymptotic behaviour. In the proof of the Theorem we shall use that if $\{s_i\}_{i=1}^\infty$ is an eventually balanced set of rate r, then $\lfloor r \rfloor \leqslant s_{i+1} - s_i \leqslant \lceil r \rceil$ for i greater than some i_0.

Lemma 3. *Let W be a Beatty word on a finite alphabet $\{a, b, \ldots\}$. Suppose the rate of a in W is at most 3. Then the word W' obtained from W by omitting all a's is balanced.*

A similar result if a has rate at most 2 is due to Altman et al. [1, Lemma 2.27].

Proof. Suppose W is a Beatty word. Choose two subwords V_1', V_2' of length n in W'. Let V_1 and V_2 be subwords of W such that the first and last letters are not a's and that the words reduce to V_1', V_2', respectively, if the a's are omitted. Let $k := |V_1| - n$ and $l := |V_2| - n$ denote the number of appearances of the letter a in V_1 and V_2, respectively. We may assume $l \geqslant k$.

Case 1: $l = k$. Since the number of occurrences of any letter in V_1 and V_2 differs by at most 1, the same is true for V_1' and V_2'. Hence W' is balanced with respect to V_1' and V_2'.

Case 2: $l > k+1$. Note that V_1 has $n+k$ letters among which exactly k letters a. Let $\hat{V}_2$ be the word obtained from V_2 by truncating the first letter and the last $l-k-1$ letters. Then $\hat{V}_2$ has also $n+k$ letters, but the number of a's in $\hat{V}_2$ is at least $l-(l-k-2) = k+2$. This contradicts that W is balanced.

Case 3: $l = k+1$. Suppose there is a $b \in Q$ such that $|c_b(V_1') - c_b(V_2')| > 1$. Then $b \neq a$ and $|c_b(V_1) - c_b(V_2)| > 1$. Since V_1 and V_2 are subwords of the balanced word W and $|V_2| = |V_1| + 1$, we obtain that $c_b(V_2) = c_b(V_1) + 2$ and that $V_2 = bVb$ for some subword V of W. By a similar reasoning we further deduce that W contains bV_1b as a subword. Hence between the last a preceding bV_1b and the first a succeeding it there are at least $|V_1| + 2 = n+k+2$ letters. On the other hand, between the first and the last a in V_2 there are at most $|V| - 2 = n+k-3$ letters. Since W is a Beatty word, V_1 contains k letters a and V_2 contains $k+1$ letters a, we find that on the one hand $(k+1)r_a > n+k+2$ and on the other hand $kr_a < n+k-1$. On combining both inequalities, we get $r_a > 3$. This contradiction shows that $|c_b(V_1') - c_b(V_2')| \leqslant 1$ for all $b \in Q$. Hence W' is balanced. □

Remark. If in Lemma 3 the rates of the letters in W are distinct, then the rates of the letters in W' are distinct and we have in fact for the rate r_b' of $b \neq a$ in W' that $r_b' = r_b(1 - r_a^{-1})$. This follows from the observations that the ratios of rates of letters $\neq a$ are the same in W and W' and that the sum over all letters of the inverses of the rates equals 1.

Lemma 4. *Let W be an eventually balanced word on a finite alphabet $\{a,b,\ldots\}$. Let k and l be integers such that $\lfloor kr_a \rfloor = \lfloor lr_b \rfloor$. Then $kr_a = lr_b$.*

Proof. Suppose $kr_a < lr_b$. Then the situation occurs infinitely often that there are two a's with exactly $k-1$ letters a and fewer than l letters b in between. By Lemma 2 there are infinitely often at least $\lfloor kr_a \rfloor - 1$ letters between the two a's. Then there are at least $\lfloor kr_a \rfloor + 1$ letters between the last b preceding the first a and the first b succeeding the last a. Thus $\lceil lr_b \rceil \geqslant \lfloor kr_a \rfloor + 2 = \lfloor lr_b \rfloor + 2$ which is a contradiction. □

Corollary. *If $\lfloor r_a \rfloor = \lfloor r_b \rfloor$, then $r_a = r_b$. If the rates of an eventually balanced word are distinct, then each letter is characterised by the integer part of its rate.*

Remark. In the proof of the Theorem we shall often denote a letter by the integer part of its rate. By the corollary this characterises the letter uniquely.

4. Fraenkel's conjecture if all rates exceed 3

In this section we assume that W is a Beatty word on a finite alphabet Q such that the rates of the letters are distinct and at least 3. We shall show that Q contains at least 7 different letters. By the Corollary of Lemma 4 the letters are characterised by the integer parts of their rates. We shall denote a letter by the integer part of its rate if we know the value and it is less than 10, and otherwise by some other symbol. An $*$ will denote some unknown letter which may be equal to one of the already used letters. By r_x we denote the rate of x. Recall that the distance between two consecutive x's in W is $\lfloor r_x \rfloor$ or $\lceil r_x \rceil$. Thus if a letter is denoted by 3, then the distance between consecutive 3's is 3 or 4. Sometimes we make an assumption by using symmetry. Then it may occur that the letters in the argument are written in the inverse order. Since the arguments concern finite blocks, it does not effect their validity.

Lemma 5. *If each of the letters* 3, 4 *and* 5 *occur in W, then* $r_3 > \frac{15}{4}$, $r_4 > \frac{9}{2}$, $r_5 > \frac{17}{3}$ *and all other letters have rates at least* 10.

Proof. Suppose the letters 3, 4 and 5 occur in W. Since $r_5 > r_4 > r_3$, the patterns 4 3 $*$ $*$ 3 4 and 5 4 $*$ $*$ $*$ 4 5 occur. In the latter pattern the middle $*$ has to be 3, since 3 can have only jumps 3 and 4. Thus we have the pattern 3 5 4 $\circ$ 3 $\circ$ 4 5 3 where $\circ$ denotes letters different from 3, 4 and 5. The two consecutive jumps of length 4 of the letter 3 imply $r_3 > \frac{7}{2}$. It follows that the former pattern has to be extended as follows: 3 $*$ $*$ 4 3 $*$ $*$ 3 4 $*$ $*$ 3. One of the middle $*$'s has to be 5. Because of symmetry we may assume that we have 5 3 $\circ$ $\circ$ 4 3 5 $\circ$ 3 4 $\circ$ 5 3. The letter before the first 5 has to be 4 and we obtain $r_4 > \frac{9}{2}$. Thus we have the patterns

4 5 3 $\circ$ $\circ$ 4 3 5 $\circ$ 3 4 $\circ$ 5 3 and 4 $\circ$ $\circ$ 3 5 4 $\circ$ 3 $\circ$ 4 5 3 $\circ$ $\circ$ 4.

The latter pattern can only be extended as

5 3 4 $\circ$ $\circ$ 3 5 4 $\circ$ 3 $\circ$ 4 5 3 $\circ$ $\circ$ 4 3 5

and we obtain $r_3 > \frac{15}{4}$ and $r_5 > \frac{17}{3}$. Applying this information to the other pattern, we find that the patterns have a common extension

5 3 4 $\circ$ $\circ$ 3 5 4 a 3 b 4 5 3 c $\circ$ 4 3 5 $\circ$ 3 4 $\circ$ 5 3

where a, b and c are letters with rates at least 6. If some letter 6, 7, 8 or 9 occurs, it has to be a, b or c, It is easy to check that $a \neq 6, 7, 8, 9$, that $b \neq 6, 7, 8, 9$ and that $c \neq 7, 8, 9$. The only remaining possibility is $c = 6$, but then $a = 6$, which has been excluded. □

Remark. Since three consecutive places are blocked for 1's and 2's, it follows from the proof that the letters 1 and 2 do not occur. The same remark applies to Lemmas 6, 7 and 9.

Corollary. *If* 3, 4 *and* 5 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Suppose $|Q| \leqslant 6$. Then according to Lemma 5 the sum of the densities of the letters is less than

$$\frac{4}{15} + \frac{2}{9} + \frac{3}{17} + \frac{3}{10} < 1$$

which is a contradiction. □

Lemma 6. *If* 3, 4, 6 *and* 7 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Suppose 3, 4, 6 and 7 occur. Then 5 does not occur in W by the Corollary of Lemma 5. Since $r_7 > r_6$, the following pattern is in W: 7 6 $*$ $*$ $*$ $*$ $*$ 6 7. If there are two 4's between the two 6's, then we have 7 6 4 $*$ $*$ $*$ 4 6 7 and the 3's cannot be placed. Thus, there is only one 4 in between and we have 4 7 6 $\circ$ $\circ$ 4 $\circ$ $\circ$ 6 7 4 where $\circ$ denotes a letter different from 4, 6, 7. The only possible extension is

a 3 4 7 6 3 b 4 c 3 6 7 4 3 d,

where a, b, c, d are letters with rates at least 8. If $|Q| \leqslant 6$, then $a = c \neq b = d$ and a and b are at most 8. This is impossible by Lemma 4. □

Lemma 7. *If* 3, 4 *and* 6 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Suppose 3, 4 and 6 occur and $|Q| \leqslant 6$. Then 5 and 7 do not occur in W by Lemmas 5 and 6. Because of symmetry we may assume that at least one of the following patterns is in W:

case (a) $a\ 4\ 3\ 6\ b\ 3\ 4\ c$

case (b) $6\ 4\ 3\ a\ b\ 3\ 4\ 6$,

where a, b, c are distinct and at least 8.

Case (a). We have $Q = \{3, 4, 6, a, b, c\}$. Denoting by $(x|y)$ that only xy and yx are possible, the pattern (a) can only be extended as

$$(4|6)(3|c)\ a\ 4\ 3\ 6\ b\ 3\ 4\ c\ *. \tag{2}$$

Hence $8 \leqslant c \leqslant 9$ and $*$ is 3 or a. If $* = a$, then $a = 8$, $c = 9$ and we obtain

$$(4|6)(3|9)\ 8\ 4\ 3\ 6\ b\ 3\ 4\ 9\ 8\ 3\ 6\ 4\ (3|b)\ *\ *\ *\ *.$$

Here the four $*$'s have to comprise 3, 4, 6, 8 and 9, which is impossible. If $* = 3$ in (2), then $r_3 < \frac{7}{2}$ and we have

$$*\ b\ 3\ (4|6)(3|c)\ a\ 4\ 3\ 6\ b\ 3\ 4\ c\ 3\ *.$$

Hence $9 < r_b < 11$. Since the leftmost $*$ can only be 4, we get

$$4\ b\ 3\ 6\ 4\ (3|c)\ a\ 4\ 3\ 6\ b\ 3\ 4\ c\ 3\ *$$

and $r_4 < \frac{9}{2}$. This yields

$$4\ b\ 3\ 6\ 4\ (3|c)\ a\ 4\ 3\ 6\ b\ 3\ 4\ c\ 3\ 6\ 4\ (3|a)\ b$$

whence $10 < r_a < 13$ and $b = 9$. We obtain

$$*\ 3\ 4\ 9\ 3\ 6\ 4\ (3|c)\ a\ 4\ 3\ 6\ 9\ 3\ 4\ c\ 3\ 6\ 4\ (3|a)\ 9$$

and $*$ can only be $c = 8$. Note that the three $*$'s in

$$8\ 3\ 4\ 9\ 3\ 6\ 4\ 3\ 8\ a\ 4\ 3\ 6\ 9\ 3\ 4\ 8\ 3\ 6\ 4\ (3|a)\ 9\ *\ *\ *$$

have to comprise a 3, 4, 6 and 8. This is impossible.

Case (b). We assume $a < b$. The pattern has to be extended as

$$3\ 4\ (3|c)\ 6\ 4\ 3\ a\ b\ 3\ 4\ 6\ (3|c)\ 4$$

where $c \neq 3, 4, 6, a, b$. Hence $Q = \{3, 4, 6, a, b, c\}$ and $r_4 < \frac{14}{3}$. The left $(3|c)$ should be $c\ 3$ and we obtain $r_3 < \frac{10}{3}$ and $9 \leqslant r_c < 12$. The only possible extension is

$$(3|4)\ 6\ *\ 3\ 4\ c\ 3\ 6\ 4\ 3\ a\ b\ 3\ 4\ 6\ (3|c)\ 4\ (3|*),$$

where $*$ is a or b. Since $b > a \geqslant 8$, the right $*$ should be a. Then the left $*$ has to be a too and we find $a = 8$. Now the only possible further extension is

$$*\ *\ *\ *\ *\ b\ (3|4)\ 6\ 8\ 3\ 4\ c\ 3\ 6\ 4\ 3\ 8\ b\ 3\ 4\ 6\ (3|c)\ 4\ (3|8).$$

The five $*$'s have to contain two 3's and one 4, one 6, one 8 and one c. This is a contradiction. $\square$

Lemma 8. *If* 3 *and* 4 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Assume 3 and 4 occur and $|Q| \leqslant 6$. In view of Lemmas 5 and 7 we may assume that 5 and 6 are not in Q. Then the pattern

$$a\ 4\ 3\ b\ c\ 3\ 4\ d$$

occurs in W, where a, b, c, d are distinct unless $a = d = 7$. We may assume $b < c$.

Case (a). $a = d = 7$. Then $c > b \geqslant 8$. Hence we have

$$(3|e)\ 7\ 4\ 3\ b\ c\ 3\ 4\ 7\ (3|e),$$

where $e \neq 3, 4, 7, b, c$ and $Q = \{3, 4, 7, b, c, e\}$. The only possible extension is

$$*\ *\ (b|c)\ 3\ 4\ (3|e)\ 7\ 4\ 3\ b\ c\ 3\ 4\ 7\ (3|e).$$

The two $*$'s should contain both 3 and 4 and 7 which is impossible.

Case (b). $Q = \{3, 4, a, b, c, d\}$ and the only possible extension is

$$(3|d)\ a\ 4\ 3\ b\ c\ 3\ 4\ d\ (3|a)\ *\ *\ *\ *.$$

Hence $7 \leqslant a \leqslant 9$ and $7 \leqslant d \leqslant 9$. Since $7 \leqslant b < c$, we have $r_c \geqslant 10$. Note that the four $*$'s cannot be a, c or d and contain at most one 3, one 4 and one b. This does not suffice. □

Lemma 9. *If* 3, 5 *and* 6 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Suppose 3, 5 and 6 occur and $|Q| \leqslant 6$. By Lemma 7 we have $4 \notin Q$. Since $r_5 < r_6$, the pattern

$$6\ 5\ 3\ a\ b\ 3\ 5\ 6$$

occurs in W. The only possible extension is

$$*\ (3|c)\ 6\ 5\ 3\ a\ b\ 3\ 5\ 6\ (3|c),$$

where $c \neq 3, 5, 6, a, b$. We may assume $a < b$, whence $a \geqslant 7$, $b \geqslant 8$, but now there is no letter available for $*$. □

Lemma 10. *If* 3 *and* 5 *occur in* W, *then* $|Q| \geqslant 7$.

Proof. Suppose 3 and 5 occur and $|Q| \leqslant 6$. Then we may assume $4, 6 \notin Q$ in view of Lemmas 8 and 9, so that the other letters are at least 7. Since letter 5 has sometimes jumps 5, we may assume that in W one of the following patterns occurs:

(a) $a\ 5\ 3\ b\ c\ 3\ 5\ d$,
(b) $5\ a\ 3\ b\ c\ 5\ 3\ d$.

If $a = d$ in case (a), then we have $a = 7$, $b \geqslant 8$ and as only possible extension

$$7\ 5\ 3\ b\ c\ 3\ 5\ 7\ (3|e)\ *$$

with $e \neq 3,5,7,b,c$. Hence no letter is available for $(*)$. We may therefore assume that $3,5,a,b,c,d$ are distinct and $Q=\{3,5,a,b,c,d\}$.

In case (a) we may assume $7 \leqslant b < c$ and the only possible extension is

$$* \ (3|d) \ a \ 5 \ 3 \ b \ c \ 3 \ 5 \ d \ (3|a)$$

and no letter is available for $*$ which is a contradiction.

In case (b) a,b,c,d are distinct. The only possible extension is

$$c \ (3|d) \ 5 \ a \ 3 \ b \ c \ 5 \ 3 \ d \ a.$$

Hence $a=c=7$. This contradicts the corollary of Lemma 4. □

Proposition. *If W is a Beatty word on an alphabet Q so that the rates of the letters are distinct and all at least* 3, *then* $|Q| \geqslant 7$.

Proof. Suppose $|Q| \leqslant 6$. If $3 \notin Q$, then, by Lemma 4, Corollary,

$$1=\sum_{q\in Q} r_q^{-1} \leqslant \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9} < 1.$$

For an exact cover the sum of the densities of the sequences has to be equal to 1. Thus $3 \in Q$. By Lemmas 8 and 10 we may assume that 4 and 5 are not in Q. Hence, by Lemma 4, Corollary,

$$1=\sum_{q\in Q} r_q^{-1} \leqslant \frac{1}{3}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10} < 1.$$

5. Fraenkel's conjecture with at most six letters

In this section we prove the Theorem by induction on m.

Proof of the Theorem. For the case $m=3$ see [16], [1, Theorem 2.19] or [19]. Use that every Beatty word is balanced.

Induction step. Suppose $3 < m \leqslant 6$, the statement of the Theorem is true for $m-1$ and we are given a Beatty word W on m letters $a_1,\dots,a_m$ with distinct rates $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, respectively. Then, by the Proposition, we have $\alpha_1 < 3$. By Lemma 3 the word W', obtained from W by omitting all a_1's, is balanced. By Lemma 2, Corollary, W' is an eventually Beatty sequence. Denoting the rate of a_j in W' by α'_j, we have $\alpha_i/\alpha_j = \alpha'_i/\alpha'_j$ for $1 < i < j \leqslant m$ so that $\alpha'_2 < \alpha'_3 < \cdots < \alpha'_m$. By the induction hypothesis applied to some suitable tail of W', we obtain that W' is eventually a periodic word with cycle (F'_{m-1}) where $F'_1 = a_2$ and $F'_j = F'_{j-1} a_{j+1} F'_{j-1}$ for $2 \leqslant j < m$. Moreover $\alpha'_j = (2^{m-1}-1)/2^{m-j}$ for $j=2,\dots,m$. Since $\alpha'_2 < 2$, the pattern a_2a_2 occurs in W'. Then the pattern $a_2a_1^ka_2$ appears in W for some k.

If $k=0$, then a_2a_2 appears in W, whence $\alpha_2 < 2$ and by Lemma 4, Corollary, $\alpha_1 < 1$. This is impossible.

If $k=1$, then $a_2a_1a_2$ appears in W. Hence $\alpha_2<3$ and $\alpha_1<2$. So every pair of consecutive letters contains an a_1. Then a_3 is surrounded by a_1's, in contradiction with $\alpha_2<3$.

We conclude that $k\geqslant 2$ and $\alpha_2\geqslant 3$. Since a_1 is balanced in W, every subword of k letters contains at least $k-1$ letters a_1 and therefore a_3 is surrounded by a_1^{k-1}'s. Since a_2 is balanced in W and $a_2a_1^ka_2$ and $a_1^{k-1}a_3a_1^{k-1}$ occur, we obtain $k+2>\alpha_2\geqslant 2k-1$. Thus $k=2$ and $a_2a_1a_1a_2$ appears in W. It follows that $\alpha_1<2$ and $3\leqslant\alpha_2<4$. Then in between two consecutive a_2's in W there are two or three letters, the first and last of which are equal to a_1. We conclude that in a block of two a_2's in W' exactly two a_1's are inserted in W and between any other pair of consecutive letters in W' exactly one a_1. This means that W is eventually equal to the mth Fraenkel word (F_m). Hence $\alpha_j=(2^m-1)/2^{m-j}$ for $j=1,2,\ldots,m$. Hence, by Lemma 1 the inverse image of each letter is of the form $\{\lfloor n\alpha+\beta\rfloor\}_{n=1}^{\infty}$ or $\{\lceil n\alpha+\beta\rceil\}_{n=1}^{\infty}$ with α rational. Since these sequences are periodic, W itself is a periodic word with period cycle (F_m).

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