On the chromatic equivalence class of graphs

G.L. Chia*

Department of Mathematics, University of Malaya, Kuala Lumpur 50603, Malaysia

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Abstract

In this paper, the set of all graphs having the same chromatic polynomial as that of the graph consisting of the join of \( m \) copies of the paths each of length 3 is determined. Also, it is shown that, for \( i = 0, 1 \), the set of all graphs having the same chromatic polynomial \((\lambda - i)P(G; \lambda)\) is obtained by applying the two operations, described in the paper, to every graph which has the same chromatic polynomial as that of \( G \). This result is used in the final section to prove a characterization for \( T \)-chromatically unique graphs.

I. Introduction

Let \( G \) be a finite undirected graph having no multiple edges or loops, and let \( P(G; \lambda) \) denote its chromatic polynomial. The set of all graphs having the same chromatic polynomial \( P(G; \lambda) \) is denoted by \( \mathcal{C}(G) \) and is called the chromatic equivalence class of \( G \). The graph \( G \) is said to be chromatically unique if \( \mathcal{C}(G) = \{ G \} \). Although chromatically unique graphs have been the subject of many recent papers, relatively fewer results concerning the chromatic equivalence class of graphs are known. If \( T \) is a tree on \( n \) vertices, then \( \mathcal{C}(T) \) is the set of all trees on \( n \) vertices. This result has been generalized to 2-trees in [18] and then to \( q \)-trees in [2]. Other results on the chromatic equivalence class of graphs can be found in [1, 5, 7, 17].

In Section 3, two operations which will generate the chromatic equivalence class of a graph whose chromatic polynomial is \((\lambda - i)P(G; \lambda)\), where \( G \) is some given graph and \( i = 0, 1 \), will be presented. This result which generalizes the main result in [3, 14] is used in Section 4 to prove a characterization for \( T \)-chromatically unique graphs.

In what follows, we let \( K_n \) denote the complete graph on \( n \) vertices, \( K_{n_1, n_2, \ldots, n_t} \) the complete \( t \)-partite graph having \( n_i \) vertices in the \( i \)th partite set, \( P_n \) the path on \( n \) vertices and \( C_n \) the cycle on \( n \) vertices.

* E-mail: j2chia@cc.um.edu.my.

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The join of two graphs \( G \) and \( H \), denoted by \( G + H \), is the graph obtained from the union of \( G \) and \( H \) by joining every vertex of \( G \) to every vertex of \( H \).

Let \( J \) be a graph and let \( G = J + J + \cdots + J \) denote the join of \( m \) (\( \geq 2 \)) copies of \( J \). We wish to determine \( \mathcal{C}(G) \). Let \( \mathcal{F}_m(J) \) denote the set of all graphs \( H \) which are of the form

\[
H = H_1 + H_2 + \cdots + H_m,
\]

where \( H_i \in \mathcal{C}(J), \ i = 1, 2, \ldots, m \). Then evidently, \( \mathcal{F}_m(J) \subseteq \mathcal{C}(G) \) because \( H \in \mathcal{C}(G) \).

**Problem.** What are those graphs \( J \) for which \( \mathcal{F}_m(J) = \mathcal{C}(G) \)?

Note that in the event that \( J \) is chromatically unique, \( \mathcal{F}_m(J) = \mathcal{C}(G) \) is equivalent to the fact that \( G = J + J + \cdots + J \) is chromatically unique.

In Section 2, we solve the above problem for the case when \( J \) is the path \( P_4 \) (see Theorem 1). Note that \( \mathcal{C}(P_4) = \{ P_4, K_{1,3} \} \) and in particular, \( \mathcal{F}_2(P_4) \) is the set of three graphs \( P_4 + P_4, P_4 + K_{1,3} \) and \( K_{1,3} + K_{1,3} \).

### 2. A chromatic equivalence class

We shall need the following method of computing the chromatic polynomial of a graph introduced by Frucht [8].

A spanning subgraph is called *special* if its connected components are complete graphs. Let \( G \) be a graph on \( p \) vertices. Let \( s_i(G) \) denote the number of special spanning subgraphs of \( G \) with \( i \) connected components, \( i = 1, 2, \ldots, p \). Then

\[
P(G; \lambda) = \sum_{i=1}^{p} s_i(G)(\lambda)_i,
\]

where \( (\lambda)_i = \lambda(\lambda - 1)\cdots(\lambda - i + 1) \) is a falling factorial and \( \overline{G} \) is the complement of \( G \). In this case, we say that \( P(G; \lambda) \) is expressed in *factorial basis*.

It is easy to see that \( s_p(\overline{G}) = 1 \) and that \( s_{p-1}(\overline{G}) = q \) if \( G \) has \( q \) edges. Note that if \( G \) has chromatic number \( \chi \), then \( s_i(\overline{G}) = 0 \) for all \( i < \chi \).

A graph \( G \) is *uniquely \( \chi \)-colorable* if it has chromatic number \( \chi \) and there is precisely one way of partitioning its vertex set into \( \chi \) independent subsets.

The following lemma (due to Liu [12]) is needed.

**Lemma 1** (Liu [12]). Let \( G \) be a connected graph with \( p \) vertices and \( q \) edges. Assume that \( G \) is not the complete graph \( K_3 \). Then

\[
s_{p-2}(G) \leq \binom{q - 1}{2}
\]

and equality holds if and only if \( G \) is the path \( P_{q+1} \).
Let $G$ and $H$ be two graphs whose chromatic polynomials are expressed in factorial basis. Then

$$P(G + H; \lambda) = P(G; \lambda) \oplus P(H; \lambda),$$

where the polynomial operator $\oplus$ denotes the operation, known as umbral multiplication, in which factorials are multiplied as powers (see [13, 15]).

Let $f(\lambda)$ and $g(\lambda)$ be two polynomials expressed in factorial basis. Then $g(\lambda)$ is said to be an umbral factor of $f(\lambda)$ if there exists a polynomial $h(\lambda)$, expressed in factorial basis, such that $f(\lambda) = g(\lambda) \oplus h(\lambda)$.

**Theorem 1.** Let $G = P_4 + P_4 + \cdots + P_4$ be the join of $m$ copies of the path $P_4$. Then $\chi(G) = \chi_m(P_4)$.

**Proof.** First note that $P(P_4; 2) = (\lambda)_4 + 3(\lambda)_3 + (\lambda)_2 = P(K_{1,3}; \lambda)$ and so $G$ has chromatic polynomial of the form

$$(\lambda)_4 + 3(\lambda)_3 + (\lambda)_2 \oplus (\lambda)_4 + 3(\lambda)_3 + (\lambda)_2 \oplus \cdots \oplus (\lambda)_4 + 3(\lambda)_3 + (\lambda)_2$$

with $m$ umbral factors $(\lambda)_4 + 3(\lambda)_3 + (\lambda)_2$.

Suppose there is a graph $Y$ such that $P(Y; \lambda) = P(G; \lambda) = \sum_{i=1}^{4m} s_i(G)(\lambda)_i$. Let $p$ and $q$ denote, respectively, the number of vertices and edges in $Y$. Then $p = 4m$ and $q = 8m^2 - 5m$. Moreover, $Y$ is uniquely $2m$-colorable as $G$ is so.

It is easier to look at $\bar{Y}$. Note that $s_{4m-2}(\bar{Y}) = m + 9\binom{m}{2} = \frac{1}{2}(9m^2 - 7m)$ because $s_{4m-2}(\bar{Y}) = s_{4m-2}(\bar{G})$.

Let $V_1, V_2, \ldots, V_{2m}$ be the color classes of the unique $2m$-coloring of $Y$. Let $V_{i,j}$ denote the subgraph induced by $V_i \cup V_j$, $i \neq j$. Call $V_{i,j}$ a 2-color subgraph of $Y$.

**Case (i):** Every $V_i$ has precisely two vertices.

In this case, $V_{i,j}$ is either a path $P_4$ or else a cycle $C_4$ because, by Theorem 12.16 of [10], $V_{i,j}$ is connected for $i \neq j$. By looking at the number of edges in $Y$, we see that exactly $m$ of the 2-color subgraphs $V_{i,j}$ are $P_4$ and the rest of the 2-color subgraphs are $C_4$. This means that in $\bar{Y}$, any two copies of $K_2$ are joined by at most one edge. Because $\bar{Y}$ has $3m$ edges, $m$ edges are used in joining up these $2m$ copies of $K_2$.

We assert that there are $m$ components in $\bar{Y}$. More precisely, we show that $\bar{Y}$ contains no $K_2$ as a component.

Suppose on the contrary that $\bar{Y}$ contains $K_2$ as a component. Then $Y = \bar{K}_2 + G_*$ where $\bar{G}_* = \bar{Y} - K_2$. Consequently,

$$P(Y; \lambda) = ((\lambda)_2 + (\lambda)_1) \oplus P(G_*; \lambda).$$

But this is a contradiction because $P(G; \lambda)$ does not contain $(\lambda)_2 + (\lambda)_1$ as an umbral factor.

As $\bar{Y}$ contains no $K_2$ as a component and as $\bar{Y}$ has $3m$ edges, it follows that $\bar{Y}$ is the union of $m$ copies of $P_4$.

**Case (ii):** Not every $V_i$ has precisely two vertices.
Then there is a \( j \) such that \( |V_j| = 1 \). Without loss of generality, let \( V_1, \ldots, V_r \) be such that \( |V_j| = 1 \) for \( j = 1, \ldots, r \), \( r \geq 1 \). Then \( Y = K_r + G_* \) for some graph \( G_* \). Let \( J_1, \ldots, J_t \) be the connected components of \( G_* \). Then \( G_* = J_1 + \cdots + J_t \).

If for some \( i \), \( J_i = K_3 \), then \( Y \) contains a subgraph \( K_3 \cup K_1 \). This means that \( Y = K_{1,3} + G_1 \) for some graph \( G_1 \) and so

\[
P(Y; \lambda) = ((\lambda)_4 + 3(\lambda)_3 + (\lambda)_2) \oplus P(G_1; \lambda).
\]

Since \( P(G_1; \lambda) = ((\lambda)_4 + 3(\lambda)_3 + (\lambda)_2) \oplus \cdots \oplus ((\lambda)_4 + 3(\lambda)_3 + (\lambda)_2) \) with \( m - 1 \) umbral factors \((\lambda)_4 + 3(\lambda)_3 + (\lambda)_2\), by induction on \( m \), we have \( G_1 \in \mathcal{F}_{m-1}(P_4) \) and this implies that \( Y \in \mathcal{F}_m(P_4) \) and consequently, \( \ell(G) = \mathcal{F}_m(P_4) \).

Similarly, if for some \( i \), \( J_i = P_4 \), then \( Y = P_4 + G_1 \). By induction on \( m \), again we have \( Y \in \mathcal{F}_m(P_4) \).

So assume that \( J_i \) is neither \( K_3 \) nor \( P_4 \) for any \( i = 1, \ldots, t \). Let \( p_i \) and \( q_i \) denote, respectively, the number of vertices and edges in \( J_i \). Then \( \sum_{i=1}^t q_i \) is equal to \( 3m \), the number of edges in \( Y \).

If \( p_i \leq 5 \), then \( J_i \) is either \( P_2 \) or \( P_3 \). However, neither case is possible because \( P(G; \lambda) \) does not contain \((\lambda)_2 + (\lambda)_1\) or \((\lambda)_2 + 2(\lambda)_2\) as an umbral factor. Hence \( p_i \geq 4 \).

The above observation implies that \( 4m = |V(Y)| = r + \sum_{i=1}^t p_i \geq r + 4t \) and so \( t < m \) because \( r \geq 1 \).

Because \( Y = K_r + G_* \), we have \( P(Y; \lambda) = (\lambda)_r \oplus P(G_*; \lambda) \). It follows that \( s_{p-2}(Y) = s_{p-2}(\tilde{G}_*) \) where \( p_* \) is the number of vertices in \( G_* \).

But

\[
P(G_*; \lambda) = \sum_{j=1}^{p_*} s_j(\tilde{G}_*)(\lambda)_j
\]

\[= P(J_1; \lambda) \oplus \cdots \oplus P(J_t; \lambda), \tag{1}
\]

where

\[
P(J_i; \lambda) = \sum_{j=1}^{p_i} s_j(J_i)(\lambda)_j
\]

\[= (\lambda)_{p_i} + q_i(\lambda)_{p_i-1} + s_{p_i-2}(J_i)(\lambda)_{p_i-2} + \cdots,
\]

\( i = 1, \ldots, t \).

By taking the umbral multiplication and by equating the coefficient of \((\lambda)_{p_*-2}\) in (1), we have

\[
s_{p_*-2}(\tilde{G}_*) = \sum_{1 \leq i < j \leq t} q_i q_j + \sum_{i=1}^t s_{p_*-2}(J_i)
\]

\[\leq \sum_{i<j} q_i q_j + \sum_{i=1}^t \left( \frac{q_i - 1}{2} \right) \]
by Lemma 1. Consequently,
\[ s_{p-2}(G) \leq \frac{\sum_{i<j} 2q_i q_j + \sum_{i=1}^t (q_i^2 - 3q_i + 2)}{2} \]
\[ = \frac{(\sum_{i=1}^t q_i)^2 - 3\sum_{i=1}^t q_i + 2t}{2} \]
\[ = \frac{9m^2 - 9m + 2t}{2} \]
\[ < \frac{9m^2 - 9m}{2} \]
because \( t < m \). But this is a contradiction because \( s_{p-2}(Y) = \frac{1}{2}(9m^2 - 7m) \).

This completes the proof. \( \Box \)

3. A generalization

Let \( V[n] \) denote the set of all graphs \( G \) whose vertex set can be partitioned into \( n \) subsets \( V_1, \ldots, V_n \) such that, for any two vertices \( u \) and \( v \) in \( G \), there is an automorphism \( \sigma \) of \( G \) with \( \sigma(u) = v \) if and only if \( u \) and \( v \) are in \( V_i \) for some \( 1 \leq i \leq n \). In this case, the vertices are said to be similar if and only if they belong to the same \( V_i \). Note that \( V[1] \) is the set of all vertex-transitive graphs and that \( K_{n_1, n_2} \in V[2] \) if \( n_1 \neq n_2 \).

Let \( X_1 \cup_m X_2 \) denote any graph \( G \) obtained by overlapping \( X_1 \) and \( X_2 \) at a \( K_m \). Such a \( K_m \) is called a separating complete subgraph of \( G \). In particular, it is called a cut vertex if \( m = 1 \) and a separating edge if \( m = 2 \). Note that if \( m = 0 \), then \( G \) is a disconnected graph.

Suppose that, for each \( i = 1, 2 \), there is a vertex \( x_i \) in \( X_i \) such that \( x_i \) is adjacent to all the vertices of \( X_1 \cap X_2 = K_m \), and that \( N(x_i) \cap K_m = A_i \) is non-empty. Here \( N(x) \) denotes the neighborhood of a vertex \( x \).

Let \( \varphi_m \) denote the operation: remove from \( G \) all edges joining \( x_1 \) to \( A_1 \), and then join \( x_2 \) to all the vertices in \( A_1 \). Then the resulting graph is not isomorphic to \( G \) but they have the same chromatic polynomial (see [4, Eq. (3)]).

Now consider the graph \( Y_1 \cup_{m+1} Y_2 \). Let \( y \) be a vertex in \( Y_1 \cap Y_2 = K_{m+1} \) such that \( A_1 = N(y) \setminus Y_2 \cap N(y) \) and \( A_2 = N(y) \setminus Y_1 \cap N(y) \) are both non-empty. Let \( H \) be the graph obtained by joining a new vertex \( x \) to all the vertices in \( Y_1 \cap Y_2 - \{y\} \).

Let \( \varphi_m^{-1} \) denote the operation: remove from \( H \) all edges joining \( y \) to \( A_1 \), and then join \( x \) to all the vertices in \( A_1 \). Then as before, the resulting graph is not isomorphic to \( H \) but they have the same chromatic polynomial \( P(H; \lambda) \). Note that \( \varphi_m^{-1} \) is the reverse of \( \varphi_m \), and vice versa. Note also that these operations work for \( m = 0 \) as well. In this case \( K_m \) is the empty graph, a graph having no vertices and edges (see [11] for this concept).

Let \( \pi_m \) denote the operation: join a new vertex to all the vertices of a \( K_m \) in a graph \( G \). The resulting graph has chromatic polynomial \((\lambda - m)P(G; \lambda)\).
Let \( G \in V[n] \). If \( m = 0 \), the operation \( \pi_m \) produces one such graph because it amounts to adding an isolated vertex to \( G \). If \( m = 1 \), the operation \( \pi_m \) produces \( n \) non-isomorphic graphs. If the new vertex is joined to a \( K_m \) that is contained in a separating complete subgraph \( K_{m+1} \) (if \( m = 0 \), this means that the resulting graph contains a cut vertex), apply the operation \( \phi_m^{-1} \) to this \( K_{m+1} \) in as many ways as is possible. To obtain the chromatic equivalence class, apply the two operations \( \pi_m \) and \( \phi_m^{-1} \) to every graph in \( \mathcal{C}(G) \), taking into account the possible isomorphic cases that arise. We shall illustrate this in Example 1 with \( m = 1 \).

To see that this in fact generates the chromatic equivalence class, let \( Y \) be such that \( P(Y; \lambda) = (\lambda - m)P(G; \lambda) \). Assume first that \( m = 0 \). Then \( Y \) has one component more than those of \( G \) (see [13, Corollary, p. 65]). If \( Y \) contains an isolated vertex \( x \), then the graph \( Y - \{x\} \) is in \( \mathcal{C}(G) \), and this means that \( Y \) is obtained by applying the operation \( \pi_m \). If \( Y \) contains no isolated vertices, then apply the operation \( \phi_m \) to \( Y \) to get a graph \( X \) which contains an isolated vertex \( x \). Now, the graph \( X - \{x\} \) is in \( \mathcal{C}(G) \) and this means that \( Y \) is obtained by applying the operation \( \phi_m^{-1} \) to \( X \).

Assume now that \( m = 1 \). Then \( Y \) has one non-trivial block more than those of \( G \) (see [19, Theorem 1]). If \( Y \) contains an end-vertex \( x \), then the graph \( Y - \{x\} \) is in \( \mathcal{C}(G) \) and this means that \( Y \) is obtained from \( Y - \{x\} \) by applying the operation \( \pi_m \). If \( Y \) contains no end-vertices, then apply the operation \( \phi_m \) to \( Y \) to get a graph \( X \) which contains an end-vertex \( x \). Now the graph \( X - \{x\} \) is in \( \mathcal{C}(G) \) and this means that \( Y \) is obtained by applying the operation \( \phi_m^{-1} \) to \( X \).

The above observations are now summarized below.

**Theorem 2.** Let \( G \in V[n] \) where \( n \geq 1 \). For \( m = 0, 1 \), the set of all graphs with the same chromatic polynomial \( (\lambda - m)P(G; \lambda) \) is obtained by applying the operations \( \pi_m \) and \( \phi_m^{-1} \) to every graph in \( \mathcal{C}(G) \).

**Example 1.** Let \( G \) be the graph in Fig. 1. Then \( G \in V[2] \). This is because \( V(G) = V_1 \cup V_2 \) where \( V_1 = \{A_1, A_2, B_1, B_2\} \) and \( V_2 = \{y, z\} \) and it can be verified that for any two vertices \( u \) and \( v \) in \( G \) such that \( \sigma(u) = v \) for some automorphism \( \sigma \) of \( G \), either \( u, v \in V_1 \) or else \( u, v \in V_2 \).

Note that \( P(G; \lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^2 \) and that \( G \) is chromatically unique. So \( \mathcal{C}(G) = \{G\} \). We wish to obtain all non-isomorphic graphs having the same chromatic polynomial \( (\lambda - 1)P(G; \lambda) \).

Applying the operation \( \pi_1 \) to every graph in \( \mathcal{C}(G) \), we obtain two non-isomorphic graphs \( G_1 \) and \( G_2 \) (see Fig. 1). Any other graph obtained by applying \( \pi_1 \) is isomorphic to either \( G_1 \) or \( G_2 \) because vertices that are similar are either in \( V_1 \) or \( V_2 \).

In the graph \( G_2 \), since the vertex \( x \) is joined to the vertex \( z \) (take this as a \( K_1 \)) that is contained in a separating edge \( yz \) (take this as a \( K_2 \)), apply the operation \( \phi_1^{-1} \) to obtain the graph \( G_3 \) of Fig. 1. Since the vertices \( A_1 \) and \( A_2 \) are similar, only one such graph is produced under the operation \( \phi_1^{-1} \). Thus the chromatic equivalence class of a graph whose chromatic polynomial is \( \lambda(\lambda - 1)^2(\lambda^2 - 3\lambda + 3)^2 \) consists of three graphs \( G_1, G_2 \), and \( G_3 \). \( \square \)
Note that [3, Theorem 4] (also [14, Theorem 6]) is a special case of the above theorem when \( m = 1, n = 1 \) and when \( G \) is chromatically unique.

Note also that Theorem 2 enables one to determine the chromatic equivalence class of a graph having chromatic polynomial of the form \( \lambda^{x_1}(\lambda - 1)^{x_2}P(G; \lambda) \) where \( G \) is a graph and \( x_1 \) and \( x_2 \) are non-negative integers.

Remark. Let \( G = K_{n_1, n_2, \ldots, n_t} \). If \( n_1 < n_2 < \cdots < n_t \), then \( G \notin \mathbb{V}^r \). In [16], it is shown that the complete bipartite graph \( K_{n_1, n_2} \) is chromatically unique if \( n_2 > n_1 > 2 \). The complete tripartite case is considered in [6]. Applying Theorem 2 to these results, we see that there is an abundance of graphs \( X \) for which \( |\mathcal{E}(X)| = 2 \), or for which \( |\mathcal{E}(X)| = 3 \).

4. A characterization for \( T \)-chromatically unique graphs

A graph \( G \) is said to be \( T \)-chromatically unique if for any positive integer \( n \) and any graph \( H \) such that \( P(H; \lambda) = (\lambda - 1)^nP(G; \lambda) \), \( H \) contains \( G \) as a subgraph. In [9], Guo introduces the concept of \( T \)-chromatically unique graphs and proves that all \( T \)-chromatically unique graphs are chromatically unique. Furthermore, he proves that all chromatically unique graphs that are vertex-transitive are \( T \)-chromatically unique.

In this section, a characterization for \( T \)-chromatically unique graphs is given.

**Theorem 3.** Let \( G \) be a chromatically unique graph. Then \( G \) is \( T \)-chromatically unique if and only if \( G \) contains no separating edge.

**Proof.** The proof for the necessity part is by contradiction. Suppose \( G \) is \( T \)-chromatically unique. If \( G \) contains a separating edge, then \( G = X \cup_2 Y \) for some
graphs $X$ and $Y$ and

$$P(G; \lambda) = \frac{P(X; \lambda)P(Y; \lambda)}{\lambda(\lambda - 1)}$$

by [13, Theorem 3].

Let $H = X \cup_1 Y$. Then

$$P(H; \lambda) = \frac{P(X; \lambda)P(Y; \lambda)}{\lambda} = (\lambda - 1)P(G; \lambda)$$

but $H$ does not contain $G$ as a subgraph. Hence $G$ is not $T$-chromatically unique.

To prove the sufficiency part, assume that $G$ contains no separating edge.

Let $H_n$ be any graph such that $P(H_n; \lambda) = (\lambda - 1)^nP(G; \lambda)$. We wish to show that, for any $n \geq 1$, $H_n$ always contains $G$ as a subgraph. We proceed this by induction on $n$.

When $n = 1$, we have $P(H_1; \lambda) = (\lambda - 1)P(G; \lambda)$. By Theorem 2, $H_1$ is obtained by applying the operations $\pi_1$ and $\varphi^{-1}_1$ to every graph in $\mathcal{H}(G)$. Since $G$ is chromatically unique, $\mathcal{H}(G) = \{G\}$. As $G$ contains no separating edge, only the operation $\pi_1$ is carried out. Hence $H_1$ is obtained by joining a new vertex to some vertex of $G$. Clearly $H_1$ contains $G$ as a subgraph.

Assume that for every graph $H_k$ with $P(H_k; \lambda) = (\lambda - 1)^kP(G; \lambda)$, $H_k$ always contains $G$ as a subgraph where $k \geq 1$. Now

$$P(H_{k+1}; \lambda) = (\lambda - 1)^{k+1}P(G; \lambda) = (\lambda - 1)P(H_k; \lambda)$$

By Theorem 2, the graph $H_{k+1}$ is one of the graphs obtained by applying the operations $\pi_1$ and $\varphi^{-1}_1$ to every graph in $\mathcal{H}(H_k)$.

Let $J \in \mathcal{H}(H_k)$. Then $P(J; \lambda) = (\lambda - 1)^kP(G; \lambda)$. By the induction hypothesis, $J$ contains $G$ as a subgraph.

Since $G$ is chromatically unique, $G$ has at most two blocks by Proposition 2 of [3]. There are two cases to consider.

Case (i): $G$ is itself a block.

Then $J$ has $k + 1$ blocks $G, B_1, \ldots, B_k$. Since $|E(J)| = k + |E(G)|$, it must be the case that each $B_i$ is a $K_2$.

Applying the operation $\pi_1$ to $J$, we have a graph consisting of $k + 2$ blocks $G, B_1, \ldots, B_k, B_{k+1}$ where $B_{k+1}$ is a $K_2$. Since $G$ contains no separating edge, the operation $\varphi_1^{-1}$ is applied only to a separating edge of $J$ which is a $B_i$ for some $1 \leq i \leq k$. But this will only produce a graph isomorphic to some graph obtained by applying the operation $\pi_1$ to the graphs in $\mathcal{H}(H_k)$.

This shows that $H_{k+1}$ contains $G$ as a subgraph.

Case (ii): $G$ has two blocks.

Then by Propositions 2 and 3 of [3], $G = B \cup K_2$ for some block $B$ which is chromatically unique and vertex-transitive. By Theorem 8 of [14], $B$ has at most one separating edge. Moreover, in the event that $B$ contains a separating edge, the necessary conditions for $B$ to be chromatically unique obtained in [4, 14] show that $B$ is not vertex-transitive. This contradiction proves that $B$ contains no separating edge.
Now $G$ has $k + 2$ blocks $B, K_2, B_1, \ldots, B_k$. Following the same argument as in Case (i), it is seen that each $B_i$ is a $K_2$. When the operations $\pi_1$ and $\varphi_1^{-1}$ are applied to the graph $J$, the situation is similar to Case (i) and will produce a graph containing $B$ as a subgraph. Since the other blocks of this graph are all $K_2$, it is clear that this graph contains $G$ as a subgraph.

This completes the proof. □

References