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NOTE

# A Massera Type Criterion for Linear Functional Differential Equations with Advance and Delay



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In this note, a Massera type criterion for the existence of periodic solutions for linear functional differential equations with advance and delay is established. Because of the presence of an advanced argument, the definition of the fundamental solution operator seems unknown. Hence a method different from the usual one is employed. Applications to periodic problems for nonlinear equations are also given. 

## 1. INTRODUCTION

In 1950, Massera [1] proved the following interesting result.

THEOREM A. *Consider the linear differential equation*

$$x' = a(t)x + b(t), \tag{1}$$

*where* $a: R \to R^{n\times n}$, $b: R \to R^n$ *are continuous and* $\omega$*-periodic for some* $\omega > 0$. *Then* (1) *admits an* $\omega$*-periodic solution if and only if it has a positively bounded solution*.

Here by a positively bounded solution we mean that for some $M > 0$, $|x(t)| \le M$, as $t \ge 0$.



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In 1973, Chow [2] extended the above mentioned result to linear functional differential equations with finite delay.

In this note, we shall generalize the criteria of Massera and Chow to functional differential equations with advance and delay of the form

$$x' = A(t, x_t) + B(t), \tag{2}$$

where $A: R \times C \to R^n$, $B: R \to R^n$ are continuous with respect to $(t, \phi)$ and $t$, respectively, and $\omega$-periodic in the time variable $t$ for some $\omega > 0$; and $A(t, \phi)$ is linear and homogeneous in the space variable $\phi \in C$; the space

$$C = C([-\Delta_1, \Delta_2], R^n), (\Delta_1, \Delta_2 \in R^+ \cup \{\infty\})$$

possesses the usual norm $\|\cdot\|$ for $\Delta_1, \Delta_2 \in R^+$ and suitable $C_g$-norm (listed below for details), otherwise; given $t \in R$, $x_t: [-\Delta_1, \Delta_2] \to R^n$ is defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [\Delta_1, \Delta_2]$.

For Eq. (1), the usual proof of Theorem A is strongly based on the variation of constants formula

$$x(t) = V(t)x_0 + \int_0^t V(t)V^{-1}(s)b(s)\, ds, \tag{3}$$

where $V(t)$ is the fundamental solution matrix satisfying $V(0) = I$ (the identical matrix).

For a delay-functional differential equation, a similar formula holds. However, in the presence of the advanced argument $\Delta_2 > 0$, to our knowledge, there is no similar formula. Hence, to present a similar result for the advanced-delay equations, we must utilize a different method, depending on Tychonoff's fixed point theorem.

For the study of periodic solutions of equations with advance and delay, to our knowledge, there have been few works; a recent one can be found in [3]. For some recent related works, also refer to papers [4–11].

This paper is organized as follows. In Section 2, we state and prove our main result, which shows that for Eq. (2), the existence of periodic solutions is equivalent to that of positively bounded solutions. As applications, in Section 3, we discuss the periodic problems for nonlinear equations, with the use of Fan's fixed point theorem concerning set-valued maps.

## 2. MAIN RESULT

First, let us give details of the norms or $C_g$-norm employed in this paper. Let $g: R \to [1, \infty)$ be a continuous function such that it is nonincreasing on $R^-$ and nondecreasing on $R^+$; moreover, $g(0) = 1$, $g(\pm\infty) = \infty$.

As in [12], we may define a norm space $C_g$ as

$$C_g = \left\{\phi \in C\big([-r_1, r_2], R^n\big):\ \sup_{s\in[-r_1,r_2]} |\phi(s)|/g(s) \equiv |\phi|_g < \infty\right\},$$

with the norm $|\cdot|_g$. Write

$$C_{g^-} = \left\{\phi \in C\big([-r_1, 0], R^n\big):\ \sup_{s\in[-r_1,0]} |\phi(s)|/g(s) \equiv |\phi|_{g^-} < \infty\right\}.$$

For simplicity, we shall assume that $\Delta_1 = \Delta_2 = \infty$ in the sequel.

Throughout this paper, we assume:

(H1) $A: R \times C_g \to R^n$, $B: R \to R^n$ are continuous with respect to $(t, \phi)$ and $t$, respectively, and $\omega$-periodic in $t$ for some $\omega > 0$;

(H2) $A(t, \phi)$ is linear and homogeneous with respect to $\phi$, and there exists $M > 0$ such that

$$|A(t, \phi)| \le M|\phi|_g, \qquad \text{for } t \in R,\ \phi \in C_g.$$

Let us recall that a map $x: R^+ \to R^n$ is said to be a solution of (2) with the initial value

$$(x_0)^- = \phi, \qquad \text{for } \phi \in C_{g^-}, \tag{4}$$

if $x(t)$ satisfies Eq. (2) on $R^+$ and (4), where $\phi^- = \phi|_{R^-}$. When a solution $x(t)$ of (2) defined on $R$ satisfies $x(t+\omega) = x(t)$, for $t \in R$, we call it an $\omega$-periodic solution of (2).

We are in a position to state our main result.

THEOREM B. *Equation* (2) *admits an $\omega$-periodic solution if and only if it admits a positively bounded solution with a bounded initial value*.

*Remark*. When $\Delta_1 = \Delta_2 = 0$, Theorem B is just Theorem A; when $\Delta_1 \in R^+$ and $\Delta_2 = 0$, Theorem B is the well known result of Chow.

*Proof of Theorem* B. The necessity is obvious. It suffices to prove the sufficiency. Let $x(t)$ be a positively bounded solution of (2) satisfying the initial value $(x_0)^- = \phi$, for some bounded $\phi \in C_{g^-}$. Write $K_0 = \sup_{R^-}|\phi(s)|$. Hence there exists $K_1 > 0$ such that

$$|x(t)| \le K_1, \qquad \text{for } t \ge 0. \tag{5}$$

Set

$$x_k(t) = x(t + (k-1)\omega), \qquad t \in R,\ k = 1, 2, \dots .$$

From (5) and the boundedness of $\phi$ we see that $(x_k)_t \in C_g$ for $k = 1, 2, \ldots$ and $t \geq 0$. Indeed, for each $k$, and $t \geq 0$,

$$
\begin{aligned}
|(x_k)_t|_g &= \sup_{s \in R} |x(t + (k-1)\omega + s)|/g(s) \\
&\leq \max\Big\{ \sup_{s \in (-\infty, -t-(k-1)\omega]} |x(t + (k-1)\omega + s)|/g(s), \\
&\qquad \sup_{s \in [-t-(k-1)\omega, \infty)} |x(t + (k-1)\omega + s)|/g(s) \Big\} \\
&\leq \max\Big\{ \sup_{R^-} |x(s)|/g(s - t - (k-1)\omega), \\
&\qquad \sup_{R^+} |x(s)|/g(s - t - (k-1)\omega) \Big\} \\
&\leq \max\Big\{ \sup_{R^-} |\phi(s)|/g(s), K_1 \Big\} \leq \max\{K_0, K_1\} \leq K_0 + K_1. \qquad (6)
\end{aligned}
$$

By the periodicity of (2), we have that for each $t \in R$, as $t + (k-1)\omega \geq 0$, $x_k(t)$ satisfies Eq. (2), and consequently from (H2) we have that there exists $M > 0$ such that

$$|x_k'(t)| \leq M|(x_k)_t|_g + \|B\|_\infty, \qquad (7)$$

where $\|B\|_\infty = \sup_R |B(t)|$. Therefore from (6) and (7) it follows that for each $k$, as $t + (k-1)\omega \geq 0$,

$$|x_k'(t)| \leq M(K_0 + K_1) + \|B\|_\infty.$$

This shows that $x_k(t)$ $(k = 1, 2, \ldots)$ are equicontinuous. It follows from (6) that for $t \geq 0$, $(x_k)_0$ $(k = 1, 2, \ldots)$ are bounded in $C_g$. We claim that $\{(x_k)_0\}_k$ is precompact in $C_g$. For simplicity of notation, without loss of generality, write any subsequence of $\{(x_k)_0\}_k$ by itself. Utilizing the Arzela–Ascoli theorem yields that $\{x_k(t)\}_k$ has a uniformly convergent subsequence on $[-m, m]$ for every positive integer $m$. By a diagonal process, we may assume for some $x_* \in C(R, R^n)$ that $x_k(t)$ is uniformly convergent to $x_*(t)$ on every $[-m, m]$. Then

$$|x_*(t)| \leq K_0 + K_1,$$

for $t \in R$, and hence

$$|(x_*)_0|_g \leq K_0 + K_1.$$

Given $\epsilon > 0$, there exists a positive integer $m_0$ such that

$$\frac{K_{\text{-}} + K_1}{g(s)} < \frac{\epsilon}{2}$$

whenever $|s| \geq m_0$. Also, there exists a positive integer $k_0$ such that

$$|x_k(t) - x_*(t)| < \epsilon,$$

for $t \in [-m_0, m_0]$, whenever $k \geq k_0$. Thus

$$\begin{aligned}
&|(x_k)_0 - (x_*)_0|_g \\
&\quad \leq \max\left\{ \sup_{s \in (-\infty, -m_0] \cup [m_0, \infty)} \frac{|x_k(s) - x_*(s)|}{g(s)}, \right. \\
&\qquad\qquad \left. \sup_{s \in [-m_0, m_0]} \frac{|x_k(s) - x_*(s)|}{g(s)} \right\} \\
&\quad \leq \max\left\{ \frac{2(K_0 + K_1)}{g(-m_0)}, \frac{2(K_0 + K_1)}{g(m_0)}, \right. \\
&\qquad\qquad \left. \sup_{s \in [-m_0, m_0]} |x_k(s) - x_*(s)| \right\} < \epsilon.
\end{aligned}$$

This shows that

$$|(x_k)_0 - (x_*)_0|_g \to 0 \qquad (k \to \infty), \tag{8}$$

and thereby, the claim holds.

Set

$$S = \overline{\mathrm{Co}}\{(x_k)_0\},$$

where $\overline{\mathrm{Co}}$ denotes the closed convex hull of the set with respect to $|\cdot|_g$. Then $S$ is compact. Since for each $k$, $(x_k)_\omega \equiv (x_{k+1})_0 \in S$, we have

$$y_\omega \in S, \qquad \text{for } y \in S.$$

Define an operator $T: S \to C_g$ by

$$Ty = y_\omega.$$

Clearly, $T: S \to S$. Let $y_k \to y$ in $S$. Then

$$\|y_k\|_\infty, \|y\|_\infty < K_0 + K_1.$$

Similar to the argument of (8), we have

$$|Ty_k - Ty|_g \to 0 \qquad (k \to \infty),$$

and hence $T$ is continuous on $S$. By Tychonoff's fixed point theorem [13, p.

90, Theorem 10.1], $T$ has a fixed point $x_0$ in $S$, i.e.,

$$Tx_0 = x_0,$$

which implies

$$x_0(t) = x_0(t+\omega), \qquad t \in R.$$

Hence, $x_0(t)$ is $\omega$-periodic.

Now let us show that $x_0(t)$ is an $\omega$-periodic solution of (2).

From the periodicity of (2) it follows that for each $k$, as $t + (k-1)\omega \geq 0$, $x_k(t)$ satisfies (2), and hence it is still one solution of (2). Note that (2) is linear. Therefore, for each $\lambda \in [0,1]$ and positive integers $k, k'$, the function

$$y(t) \equiv \lambda x_k(t) + (1-\lambda) x_{k'}(t)$$

satisfies (2), whenever

$$t + (k-1)\omega \geq 0, \qquad t + (k'-1)\omega \geq 0.$$

Consequently, $y(t)$ is a solution of (2). From the continuity of $A(t,\phi)$ relative to $|\cdot|_g$ for $\phi$ and the construction of $S$, it follows that every $y(t)$ lying in $S$ satisfies (2) for $t \geq 0$ and hence is a solution of (2). This proves the desired conclusion and completes the proof.

## 3. APPLICATIONS FOR NONLINEAR EQUATIONS

In this section, we shall use Theorem B to prove the existence of periodic solutions for nonlinear equations.

Consider the functional differential equation

$$x'(t) = a(t, x_t, x_t) + b(t, x_t), \tag{9}$$

where $a\colon R \times C_g \times C_g \to R^n$, $b\colon R \times C_g \to R^n$ are continuous with respect to $(t, \phi, \varphi)$ and $(t, \phi)$, respectively, are $\omega$-periodic in $t$ for some $\omega > 0$, and take bounded sets to bounded ones; moreover $a(t, \phi, \varphi)$ is linear and homogeneous with respect to $\varphi$.

Set $C_\omega = \{x\colon R \to R^n$ is continuous and $\omega$-periodic$\}$ with the usual supremum norm $|\cdot|_\infty$.

The main result of this section is the following.

THEOREM C. *Assume that there exists a constant $r > 0$ such that for each $y \in C_\omega$ with $|y|_\infty \leq r$, the linear equation*

$$x'(t) = a(t, y_t, x_t) + b(t, y_t) \tag{10}$$

*has a positively bounded solution $x(t)$ with a bounded initial value $\phi$ such that for some $t_x > 0$,*

$$|x(t)| \le r, \qquad \text{for } t \ge t_x. \tag{11}$$

*Then* (9) *admits an $\omega$-periodic solution $x_0(t)$ with $|x_0|_\infty \le r$.*

*Proof.* Obviously, Eq. (10) is a linear periodic system. By Theorem B and (11), (10) admits an $\omega$-periodic solution $x_1(t)$ with $|x_1|_\infty \le r$.

Given $y \in C_\omega$, with $|y|_\infty \le r$, set

$$K(y) = \{u : u(t) \text{ is an } \omega\text{-periodic solution of } (10) \text{ with } |u|_\infty \le r.\}$$

Then the above argument implies that

$$K(y) \ne \emptyset$$

Since (10) is linear, $K(y)$ is a closed convex set in $C_\omega$. From the Arzela−Ascoli theorem it easy to see that $K(y)$ is compact.

Since $a(t, \phi, \varphi)$ and $b(t, \phi)$ take bounded sets to bounded ones, there exists $L > 0$ such that

$$|a(t, \phi, \varphi)| + |b(t, \phi)| \le L, \qquad \text{for } t \in [0, \omega], \phi, \varphi \in C_g \text{ with } \|\phi\|_\infty \le r \text{ and } \|\varphi\|_\infty \le r. \tag{12}$$

Set

$$D = \{y \in C_\omega : |y|_\infty \le r, |y(t_1) - y(t_2)| \le (L+1)|t_1 - t_2|, \text{ for } t_1, t_2 \in [0, \omega]\}.$$

Then by the Arzela−Ascoli theorem, $D$ is compact and convex. Define an operator $T: D \to 2^{C_\omega}$ by

$$T(y) = K(y), \qquad \text{for } y \in D.$$

From (12) and the definition of $K(y)$ it follows that $T: D \to 2^D$. By a well known result (see, for example, [14, Lemma 2.3]), $T$ is upper semicontinuous on $D$. Note that $D$ is compact. Therefore $T: D \to 2^D$ is completely continuous. According to Fan's fixed point theorem [15], $T$ has a fixed point $x_0$ in $D$, that is,

$$x_0 \in T(x_0),$$

which show that $x_0(t)$ is the desired $\omega$-periodic solution of (9). This proves the theorem.

Finally, let us consider a simple example.

Consider the equation

$$x'(t) = a_1(t)x(t) + a_2(t)x(t-r_1) + a_3(t)x(t+r_2) + e(t), \quad (13)$$

where $a_i\,(i = 1, 2)$: $R \to R$, $e$: $R \to R$ are continuous and $\omega$-periodic; $r_1, r_2$ are positive constants.

We have the following result.

COROLLARY 1. *Assume the following condition holds*:

$$a_1(t) + |a_2(t)| + |a_3(t)| \le -l < 0, \qquad \text{for } t \in R. \quad (14)$$

*Then* (13) *admits a unique $\omega$-periodic solution.*

*Proof.* Let $r = |e|_\infty / l + 1$. Then we claim that for each $y \in C_\omega$ with $|y| \le r$, the solution $x(t)$ of the problem

$$x'(t) = a_1(t)x(t) + a_2(t)y(t-r_1) + a_3(t)y(t+r_2) + e(t),$$
$$x(0) = x_0, \qquad |x_0| \le r \quad (15)$$

satisfies

$$|x(t)| \le r, \qquad \text{for } t \ge 0,$$

and hence is positively bounded. If this fails, then there are $t_0 \in [0, \infty)$ and $\delta > 0$ such that

$$|x(t_0)| = r, \qquad |x(t)| > r, \qquad \text{for } t \in (t_0, t_0 + \delta). \quad (16)$$

For definiteness, we assume $x(t_0) > 0$. Set $u(t) \equiv r$. Then from (14) and (15), we derive

$$\begin{aligned} x'(t_0) - u'(t_0) &= a_1(t_0)x(t_0) + a_2(t_0)y(t_0 - r_1) \\ &= a_s(t_0)y(t_0 + r_2) + c(t_0) \\ &= a_1(t_0)r + |a_2(t_0)|r + |a_s(t_0)|r + |e|_\infty \\ &= -rl + |e|_\infty = -\left(\frac{|e|_\infty}{l} + 1\right)l + |e|_\infty) = -l < 0, \end{aligned}$$

which implies that there is $\eta > 0$ such that

$$x(t) < u(t) = r, \qquad \text{for } t \in (t_0, t_0 + \eta),$$

a contradiction. This proves the claim. Therefore the existence conclusion follows from Theorem C. The uniqueness is clear. This completes the proof.

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