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Generalized (P, ω)-partitions and generating functions for trees

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Abstract

We introduce (P, R)-partitions as a generalization of the (P, ω)-partitions of Stanley. When P is a Gaussian poset the generating function for P-partitions with largest part at most n factors as $\prod_{x \in P} \frac{1-q^{g(x)+n}}{1-q^{g(x)}}$ for certain integers $g(x)$. Although trees are not in general Gaussian posets, we show that if P is a tree then R can be chosen so that the generating function for (P, R)-partitions has a similar factorization.

Keywords: (P, R)-partition; Generating function; Edge labeling; Tree

1. Introduction

Throughout this article, let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{P}$ denote the set of integers, non-negative integers and positive integers, respectively. For a set S, we denote the cardinality of S by $|S|$. From now on, P is a partially ordered set (poset) and is assumed to be finite.

Let ω be a labeling of P, i.e. ω is a bijection from P to $\{1, 2, \ldots, |P|\}$. A (P, ω)-partition is a map φ from P to $\mathbb{N}$ satisfying the following conditions:

(i) $\varphi(x) - \varphi(y) \geqslant 0$ if $x < y$ in P, i.e. P is order reversing.
(ii) $\varphi(x) - \varphi(y) \geqslant 1$ if $x < y$ and $\omega(x) > \omega(y)$.

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For a (P,ω)-partition φ, the values $\varphi(x)$ $(x \in P)$ are called the parts of φ. We denote the set of (P,ω)-partitions by $\mathscr{A}(P,\omega)$. When π is a linear extension, a (P,π)-partition is simply called a P-partition. We can easily see that φ is a P-partition if and only if φ is an order-reversing map from P to $\mathbb{N}$.

We say that P has hook lengths if there exists a map h from P to $\mathbb{P}$ satisfying

$$\sum q^{|\varphi|} = \prod_{x \in P} \frac{1}{1-q^{h(x)}},$$

where the sum is over all P-partitions and where $|\varphi| := \sum_{x \in P} \varphi(x)$. A hook length poset is a poset which has hook lengths and we say that $h(x)$ is the hook length of x. A poset P is said to be Gaussian if there exists a map g from P to $\mathbb{P}$ such that, for any $n \in \mathbb{N}$,

$$\sum q^{|\varphi|} = \prod_{x \in P} \frac{1-q^{g(x)+n}}{1-q^{g(x)}},$$

where the sum on the left-hand side is over all P-partitions with largest part at most n. In particular, Gaussian posets are hook length posets.

Note that the order in the above definition of hook length posets is the dual of the order in the original definition by Sagan [4].

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, we define $P(\lambda) := \{(i,j) \in \mathbb{P}^2 : 1 \leqslant j \leqslant \lambda_i\}$ with an order given by $(i,j) \leqslant (i',j')$ if and only if $i \geqslant i'$ and $j \geqslant j'$. A poset of this form is known as a 'shape'. Moreover, $P(\lambda)$ is known to be a hook length poset with $h(i,j)$ defined by $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$, where $(\lambda'_1, \lambda'_2, \dots, \lambda'_k)$ is the conjugate of λ. A tree T is a finite connected poset with a maximum element such that every element except the maximum element is covered by exactly one element. Any tree T is also known to be a hook length poset with the hook length $h(x)$ defined by $h(x) = |\{y \in T : y \leqslant x\}|$.

Recently, in [3], Proctor and Peterson proved that d-Complete posets, which include shapes and trees, are hook length posets (see [2] for the definition of d-Complete posets). For $k \in \mathbb{P}$, the shape $P((k,k,\dots,k))$ is a Gaussian poset but we can easily see that shapes and trees are not always Gaussian posets (see [1] for more detailed information).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition and let ω_λ be the labeling of $P(\lambda)$ defined by $\omega_\lambda(i,j) := \sum_{k=1}^{i} \lambda_k - j + 1$ for $(i,j) \in P(\lambda)$. It is well known that for any $n \in \mathbb{P}$,

$$\sum q^{|\varphi|} = q^{\sum_{i \geqslant 1}(i-1)\lambda_i} \prod_{(i,j) \in P(\lambda)} \frac{1-q^{n+1+j-i}}{1-q^{h(i,j)}}, \tag{1}$$

where the sum on the left-hand side is over all $(P(\lambda),\omega_\lambda)$-partitions with largest part at most n (cf. [5]). However, an analogous formula for trees has hitherto not been found. Our aim is to find such a formula for trees by introducing generalized (P,ω)-partitions.

We now introduce some definitions and notation in order to state our main result explicitly.

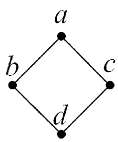
For $x, y \in P$, we use the notation $x \lessdot y$ if and only if x is covered by y, i.e. $x < y$ and no element $z \in P$ satisfies $x < z < y$. We put

$$\mathrm{Cov}(P) := \{(x, y) \in P \times P : x \lessdot y\},$$

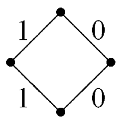
$$C^{-}(x) := \{y \in P : y \lessdot x\}.$$

An edge labeling of P is a map from $\mathrm{Cov}(P)$ to $\mathbb{Z}$ and we denote the set of edge labelings of P by $\mathrm{EL}(P)$. Let $R \in \mathrm{EL}(P)$. A (P, R)-partition is a map from P to $\mathbb{N}$ such that $\varphi(x) - \varphi(y) \geqslant R(x, y)$ for all $(x, y) \in \mathrm{Cov}(P)$. We denote the set of (P, R)-partitions by $\mathscr{A}(P, R)$.

For example, let P be a finite poset defined by

$P =$  .

We have $\mathrm{Cov}(P) = \{(d, b), (d, c), (b, a), (c, a)\}$ and a map R from $\mathrm{Cov}(P)$ to $\mathbb{Z}$ defined by $R(d, b) := 1$, $R(d, c) := 0$, $R(b, a) := 1$ and $R(c, a) := 0$ is an edge labeling of P. From now on, we express $R \in \mathrm{EL}(P)$ by putting $R(x, y)$ beside the edge (x, y) of the Hasse diagram of P for all $(x, y) \in \mathrm{Cov}(P)$. Hence, in our example, R is expressed as follows:

$R =$  .

Note that for a labeling ω of P, there always exists $R \in \mathrm{EL}(P)$ satisfying $\mathscr{A}(P, R) = \mathscr{A}(P, \omega)$, but for $R \in \mathrm{EL}(P)$ satisfying $R(\mathrm{Cov}(P)) \subseteq \{0, 1\}$, there does not always exist a labeling ω of P such that $\mathscr{A}(P, R) = \mathscr{A}(P, \omega)$. For example, for R in the figure above, no labeling ω of P satisfies that $\mathscr{A}(P, R) = \mathscr{A}(P, \omega)$.

For $n \in \mathbb{Z}$, we define

$$\mathscr{A}(P, R, n) := \{\varphi \in \mathscr{A}(P, R) : \varphi(x) \leqslant n \text{ for all } x \in P\},$$

$$U_n(P, R; q) := \sum_{\varphi \in \mathscr{A}(P, R, n)} q^{|\varphi|},$$

where we put $|\varphi| := \sum_{x \in P} \varphi(x)$.

For $R \in \mathrm{EL}(P)$, we can easily see that there exists a unique $\varphi_0 \in \mathscr{A}(P, R)$ satisfying

$$|\varphi_0| \leqslant |\varphi| \quad \text{for any } \varphi \in \mathscr{A}(P, R).$$

We denote the above φ_0 by φ_0^R.

We denote the set of maximal elements of P (resp. the set of minimal elements of P) by $\mathrm{Max}(P)$ (resp. $\mathrm{Min}(P)$). For $x, y \in P$, we put

$$C(x, y) := \bigcup_{r \geqslant 0} \{(z_0, z_1, \ldots, z_r) \in P^{r+1} : x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_r = y\}.$$

For $R \in \mathrm{EL}(P)$ and $C = (z_0, z_1, \ldots, z_r) \in C(x, y)$, we put

$$w_R(C) := \begin{cases} \sum_{i=0}^{r-1} R(z_i, z_{i+1}) & \text{if } r \geqslant 1, \\ 0 & \text{if } r = 0. \end{cases}$$

For $R \in \mathrm{EL}(P)$, we define two conditions (RX) and (RN) as follows:

(RX) $w_R(C) \geqslant 0$ for all $C \in \bigcup_{x \in P, y \in \mathrm{Max}(P)} C(x, y)$,
(RN) $w_R(C) \geqslant 0$ for all $C \in \bigcup_{x \in \mathrm{Min}(P), y \in P} C(x, y)$.

We denote the set of edge labelings of P satisfying condition (RX) (resp. condition (RN)) by $\mathrm{RX}(P)$ (resp. $\mathrm{RN}(P)$).

From now on, T is a tree with $|T| \geqslant 2$. For $R \in \mathrm{EL}(T)$, we define a map z_R from T to $\mathbb{Z}$ by

$$z_R(x) := \begin{cases} 1 & \text{if } x \in \mathrm{Min}(T), \\ \max\{z_R(y) - R(y, x) + 1 : y \in C^-(x)\} & \text{otherwise,} \end{cases}$$

where for a finite subset A of $\mathbb{Z}$, $\max A$ means the maximum element in A. For $x \in T$, we put $T_x := \{y \in T : y \leqslant x\}$, i.e. T_x is the principal order ideal of T generated by x.

For $R \in \mathrm{EL}(T)$, we define a condition (T) as follows:

(T) For each $x \in T \backslash \mathrm{Min}(T)$, if $C^-(x) = \{y_1, y_2, \ldots, y_r\}$ and

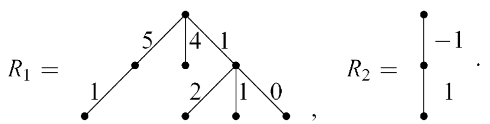
$$R(y_k, x) - z_R(y_k) \leqslant R(y_{k+1}, x) - z_R(y_{k+1})$$

for any $k \in \{1, 2, \ldots, r-1\}$, then

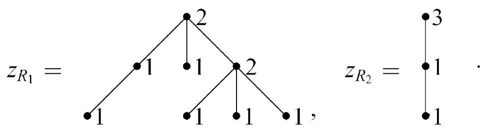
$$R(y_k, x) - z_R(y_k) + |T_{y_k}| = R(y_{k+1}, x) - z_R(y_{k+1}) \qquad (2)$$

for any $k \in \{1, 2, \ldots, r-1\}$.

For example, let R_1 and R_2 be edge labelings given by



Then,



Note that for $R \in \mathrm{EL}(T)$, we expressed the map z_R from T to $\mathbb{Z}$ by replacing x with $z_R(x)$ for all $x \in T$ in the Hasse diagram of T. We can easily check that R_1 and R_2 satisfy condition (T). In particular, if T is a chain, any edge labeling of T satisfies condition (T).

Finally, we can state our main result.

Theorem 1.1. *For $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$ satisfying condition (T), we have*

$$U_n(T, R; q) = \frac{q^{|\varphi_0^R|} \prod_{i=1}^{|T|} (1 - q^{n+z_R(x_0)+1-i})}{\prod_{x \in T} (1 - q^{h(x)})} \tag{3}$$

for any $n \geqslant \max\{\varphi_0^R(x) : x \in T\}$, where x_0 is the maximum element in T, and where $h(x) = |\{y \in T : y \leqslant x\}|$.

Let A be a set and let B be an additive group. For maps f and g from A to B, we define a map $f+g$ (resp. $f-g$) from A to B by $(f+g)(x) := f(x) + g(x)$ (resp. $(f-g)(x) := f(x) - g(x)$) for $x \in A$.

This article is organized as follows: In Section 2, we define compatible edge labelings and we show that $U_n(P, R+R'; q)$ can be expressed in terms of $U_m(P, R; q)$ if R' is a compatible edge labeling of P, $R \in \mathrm{RX}(P) \cap \mathrm{RN}(P)$, and $R + R' \in \mathrm{RN}(P)$. We devote Section 3 to proving Theorem 1.1. In Section 4, productive edge labelings and hook edge labelings are defined as edge labelings R whose generating function $U_n(P, R; q)$ is analogous to (1). For a tree T and $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$, we show that R is a productive edge labeling of T if and only if R is a hook edge labeling of T. Moreover, we give a partial result about our conjecture that, for $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$, R satisfies condition (T) if and only if R is a hook edge labeling of T.

2. Compatible edge labelings

For $x, y \in P$, we put

$$C(P) := \bigcup_{x \in \mathrm{Min}(P), y \in \mathrm{Max}(P)} C(x, y).$$

For $R \in \mathrm{RX}(P)$, if R satisfies

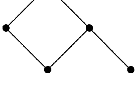
$$w_R(C_1) = w_R(C_2) \quad \text{for all } C_1, C_2 \in C(P),$$

we say that R is a compatible edge labeling of P and we denote the set of compatible edge labelings of P by $\mathrm{CEL}(P)$. In particular, for $R \in \mathrm{CEL}(P)$, we can define

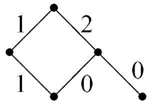
$$m_R := w_R(C) \quad \text{for } C \in C(P).$$

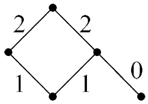
Also, for $R \in \mathrm{EL}(P)$, if $R(x, y) \geqslant 0$ for all $(x, y) \in \mathrm{Cov}(P)$, we say that R is a non-negative edge labeling of P and we denote the set of non-negative edge labelings of P by $\mathrm{NEL}(P)$. Note that $\mathrm{NEL}(P) \subseteq \mathrm{RX}(P) \cap \mathrm{RN}(P)$.

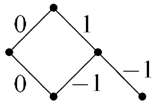
Example 2.1. Let

$P =$  .

For edge labelings R_1, R_2 and R_3 defined by

$R_1 =$ 

, $R_2 =$ 

, $R_3 =$ 

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we have $R_1 \in \mathrm{CEL}(P) \cap \mathrm{NEL}(P)$, $R_2 \in \mathrm{NEL}(P) \backslash \mathrm{CEL}(P)$, $R_3 \in \mathrm{CEL}(P) \backslash \mathrm{RN}(P)$.

By the definitions of compatible edge labelings and non-negative edge labelings, we can easily obtain the following and the proof is therefore omitted.

Lemma 2.2. *For $R \in \mathrm{EL}(P)$, there exists $R' \in \mathrm{NEL}(P) \cap \mathrm{CEL}(P)$ such that $R + R' \in \mathrm{NEL}(P)$.*

Moreover, we have the following.

Lemma 2.3. *For $R \in \mathrm{RX}(P)$ and $R' \in \mathrm{CEL}(P)$, we have*

(i) $\varphi_0^{R'}(x) = w_{R'}(C)$ *for all* $x \in P$ *and* $C \in \bigcup_{u \in \mathrm{Max}(P)} C(x,u)$,
(ii) $\varphi_0^R + \varphi_0^{R'} = \varphi_0^{R+R'}$.

By noting that

$$\varphi_0^R(x) = \max\left\{ w_R(C) : C \in \bigcup_{u \in \mathrm{Max}(P)} C(x,u) \right\}$$

for $R \in \mathrm{RX}(P)$ and $x \in P$, this lemma is easily obtained and the proof is therefore omitted.

For the generating function of (P,R)-partitions, we have the following.

Proposition 2.4. *Let $R \in \mathrm{RX}(P) \cap \mathrm{RN}(P)$ and $R' \in \mathrm{CEL}(P)$. If $R + R' \in \mathrm{RN}(P)$, for any $n \in \mathbb{Z}$,*

$$U_n(P, R+R'; q) = q^{|\varphi_0^{R'}|} U_{n-m_{R'}}(P, R; q). \tag{4}$$

Proof. By Lemma 2.3, we have $\varphi_0^{R+R'}(x) = \varphi_0^R(x) + m_{R'}$ for any $x \in \mathrm{Min}(P)$. We can easily see that for $\tilde{R} \in \mathrm{RN}(P)$ and $\varphi \in \mathscr{A}(P, \widetilde{R})$,

$$\max\{\varphi(x) : x \in P\} = \max\{\varphi(x) : x \in \mathrm{Min}(P)\}. \tag{5}$$

Therefore, we can obtain that $U_{n-m_{R'}}(P, R; q) = 0$ if and only if $U_n(P, R+R'; q) = 0$. Hence, we show (4) in the case that the both sides of (4) are not equal to 0. By Lemma 2.3(i) and our assumptions, we can define the map f from $\mathscr{A}(P, R, n - m_{R'})$ to $\mathscr{A}(P, R+R', n)$ by $f(\varphi) := \varphi + \varphi_0^{R'}$. Also we can see that f is a bijection such that

$|f(\varphi)| = |\varphi| + |\varphi_0^{R'}|$ for any $\varphi \in \mathscr{A}(P, R, n - m_{R'})$. Therefore, we obtain

$$\begin{aligned} U_n(P, R + R'; q) &= \sum_{\varphi \in \mathscr{A}(P,R,n-m_{R'})} q^{|f(\varphi)|} \\ &= q^{|\varphi_0^{R'}|} U_{n-m_{R'}}(P, R; q) \end{aligned}$$

and this completes the proof of the proposition. □

3. Proof of main result

For the map z_R, the following lemma holds.

Lemma 3.1. *Let* $R \in \mathrm{EL}(T)$.

(i) *If* $R \in \mathrm{RN}(T)$, *we have* $z_R(x) \leqslant |T_x|$ *for all* $x \in T$.
(ii) *For* $R' \in \mathrm{CEL}(T)$, $x \in T$ *and* $C_x \in \bigcup_{u \in \mathrm{Min}(T)} C(u, x)$, *we have*

$$z_{R+R'}(x) = z_R(x) - w_{R'}(C_x). \tag{6}$$

Proof. (i) If $x \in \mathrm{Min}(T)$, since $z_R(x) = 1$, we have $z_R(x) = |T_x|$. We show (i) in the case $x \in T \backslash \mathrm{Min}(T)$. By the definition of the map z_R, there exists a chain $C = (w_0, w_1, \ldots, w_r) \in \bigcup_{u \in \mathrm{Min}(T)} C(u, x)$ such that $z_R(w_{i+1}) = z_R(w_i) - R(w_i, w_{i+1}) + 1$ for all $i \in \{0, 1, \ldots, r-1\}$. Hence, we have $z_R(x) = r - w_R(C) + 1$. Thus, by condition (RN), we have $z_R(x) \leqslant |T_x|$ and (i) holds. (ii) If $x \in \mathrm{Min}(T)$, since $w_{R'}(C_x) = 0$, we can easily obtain (6). For $x \in T \backslash \mathrm{Min}(T)$, we suppose that (6) holds for all $y \in T_x \backslash \{x\}$ and we show (6) for x. By our induction hypothesis and the fact that $w_{R'}(C_x) - w_{R'}(C_y) = R'(y, x)$ for any $y \in C^-(x)$, we can obtain (6). □

For condition (T), we have the following.

Lemma 3.2. *Let* $R \in \mathrm{EL}(T)$ *satisfy condition* (T) *and let* $R' \in \mathrm{CEL}(T)$. *Then* $R + R'$ *and* $R - R'$ *satisfy condition* (T).

This result is easy to prove using Lemma 3.1(ii) and the proof is therefore omitted.

For $R \in \mathrm{EL}(T)$ and $x \in T$ with $\mathrm{Cov}(T_x) \neq \emptyset$, we denote the restriction of R to $\mathrm{Cov}(T_x)$ by R_x, i.e. $R_x \in \mathrm{EL}(T_x)$.

For $a, b, n \in \mathbb{Z}$, we put

$$[n] := \frac{1 - q^n}{1 - q},$$

$$[n]! := \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ [n][n-1]! & \text{if } n > 0, \end{cases}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} := \begin{cases} \dfrac{[a]!}{[a-b]![b]!} & \text{if } 0 \leqslant b \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We can easily see that (3) is equivalent to

$$U_n(T,R;q) = \frac{q^{|\varphi_0^R|}[|T|]!}{\prod_{x\in T}[h(x)]} \begin{bmatrix} n + z_R(x_0) \\ |T| \end{bmatrix} \tag{7}$$

for any $n \in \mathbb{Z}$. Hence, we prove (7). First, we show (7) in the case $R \in \mathrm{NEL}(T)$. Note that if $n<0$, by Lemma 3.1(i), (7) is easily obtained. We show the theorem by induction on $|T|$. In the case $|T|=2$, by direct calculation, (7) holds. We suppose that (7) holds when $|T| \leqslant j-1$ and we show the equality in the case $|T|=j$ $(j\geqslant 3)$. Since R satisfies condition (T), we may assume that $C^-(x_0)=\{y_1,y_2,\dots,y_r\}$ satisfy (2). By noting that, for $\varphi \in \mathscr{A}(T,R,n)$, if $\varphi(x_0)=a$ then $a+R(y_k,x_0) \leqslant \varphi(x) \leqslant n$ for all $x \in T_{y_k}$, $U_n(T,R;q)$ can be thus written as follows:

$$\begin{aligned} &U_n(T,R;q) \\ &\quad = \sum_{a\geqslant 0} q^a \prod_{1\leqslant k\leqslant r, y_k \notin \mathrm{Min}(T)} U_{n-a-R(y_k,x_0)}(T_{y_k},R_{y_k};q)\, q^{|T_{y_k}|(a+R(y_k,x_0))} \\ &\qquad \times \prod_{1\leqslant k\leqslant r, y_k\in \mathrm{Min}(T)} \sum_{i=a+R(y_k,x_0)}^{n} q^i. \end{aligned}$$

If $y_k \notin \mathrm{Min}(T)$, by our induction hypothesis, we have

$$\begin{aligned} &U_{n-a-R(y_k,x_0)}(T_{y_k},R_{y_k};q) \\ &\quad = \frac{q^{|\varphi_0^{R_{y_k}}|}[|T_{y_k}|]!}{\prod_{x\in T_{y_k}}[h(x)]} \begin{bmatrix} n-a-R(y_k,x_0)+z_R(y_k) \\ |T_{y_k}| \end{bmatrix}. \end{aligned}$$

On the other hand, if $y_k \in \mathrm{Min}(T)$, we can calculate

$$\sum_{i=a+R(y_k,x_0)}^{n} q^i = \frac{q^{|T_{y_k}|(a+R(y_k,x_0))}[|T_{y_k}|]!}{\prod_{x\in T_{y_k}}[h(x)]} \begin{bmatrix} n-a-R(y_k,x_0)+z_R(y_k) \\ |T_{y_k}| \end{bmatrix}.$$

Therefore, we can obtain

$$\begin{aligned} U_n(T,R;q) = &\frac{q^{|\varphi_0^R|}}{\prod_{x\in T\setminus\{x_0\}}[h(x)]} \sum_{a\geqslant 0} q^{|T|a} \\ &\times \prod_{k=1}^{r} \begin{bmatrix} n-a-R(y_k,x_0)+z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]!. \end{aligned}$$

By (2), we can calculate that

$$\prod_{k=1}^{r} \begin{bmatrix} n-a-R(y_k,x_0)+z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]! = \begin{bmatrix} n-a-R(y_1,x_0)+z_R(y_1) \\ |T|-1 \end{bmatrix} [|T|-1]!.$$

Note that if $n-a-R(y_k,x_0)+z_R(y_k)<|T_{y_k}|$, we can easily obtain $n-a-R(y_1,x_0)+z_R(y_1)<|T|-1$. Hence, by the well-known formula that

$$\sum_{k\geqslant 0} q^{mk} \begin{bmatrix} n-k \\ m-1 \end{bmatrix} = \begin{bmatrix} n+1 \\ m \end{bmatrix}$$

for $m\in\mathbb{P}$ and $n\in\mathbb{Z}$, we have

$$U_n(T,R;q) = \frac{q^{|\varphi_0^R|}[|T|]!}{\prod_{x\in T}[h(x)]} \begin{bmatrix} n-R(y_1,x_0)+z_R(y_1)+1 \\ |T| \end{bmatrix}.$$

By (2) and the definition of z_R, we can easily see that $z_R(x_0)=z_R(y_1)-R(y_1,x_0)+1$ and we get (7) in the case $R\in\mathrm{NEL}(T)$. Next, we show the theorem in general. By Lemma 2.2, there exists $R'\in\mathrm{CEL}(T)\cap\mathrm{NEL}(T)$ such that $R+R'\in\mathrm{NEL}(T)$. By Lemma 3.2, $R+R'$ also satisfies condition (T). Hence, by our above result, Lemma 2.3(ii), Proposition 2.4 and Lemma 3.1, we can obtain (7). □

Remark 3.3. Let $R\in\mathrm{RX}(T)\cap\mathrm{RN}(T)$ satisfy condition (T). By (7), we have

$$U_{|T|-z_R(x_0)}(T,R;q) = q^{|\varphi_0^R|}\frac{[|T|]!}{\prod_{x\in T}[h(x)]}.$$

4. Productive edge labelings

For $R\in\mathrm{EL}(P)$, if there exist maps c_R and h_R from P to $\mathbb{Z}$ such that

$$U_n(P,R;q) = q^{|\varphi_0^R|}\prod_{x\in P}\frac{1-q^{n+c_R(x)}}{1-q^{h_R(x)}}$$

for any $n\geqslant\max\{\varphi_0^R(x): x\in P\}$, we say that R is a productive edge labeling of P and we denote the set of productive edge labelings of P by $\mathrm{PEL}(P)$. In particular, for $R\in\mathrm{PEL}(P)$, if P is a hook length poset and $\{h_R(x): x\in P\}=\{h(x): x\in P\}$ as multisets, where $h(x)$ is the hook length of x, we say that R is a hook edge labeling of P and we denote the set of hook edge labelings of P by $\mathrm{HEL}(P)$.

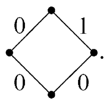
Using this terminology, a Gaussian poset P is a poset whose edge labeling R defined by $R(x,y)=0$ for all $(x,y)\in\mathrm{Cov}(P)$ is a hook edge labeling of P with $c_R=h_R$. Therefore, $\mathrm{HEL}(P)\neq\emptyset$ for a Gaussian poset P.

Example 4.1. (i) For a shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, we define $R_\lambda \in \mathrm{EL}(P(\lambda))$ by

$$R_\lambda((i,j),(i',j')) := \begin{cases} 0 & \text{if } i = i', \\ 1 & \text{if } j = j' \end{cases} \quad \text{for } ((i,j),(i',j')) \in \mathrm{Cov}(P(\lambda)).$$

We can easily see that $\varphi \in \mathscr{A}(P(\lambda), R_\lambda)$ if and only if $\varphi \in \mathscr{A}(P(\lambda), \omega_\lambda)$. Therefore, it follows from (1) that $R_\lambda \in \mathrm{HEL}(P(\lambda))$.

(ii) We define $R \in \mathrm{EL}(P((2,2)))$ as follows:

$R =$ 

By direct calculation, we can obtain

$$U_n(P, R; q) = \frac{q^2(1-q^n)(1-q^{n+1})(1-q^{n+2})(1-q^{n+3})}{(1-q^4)(1-q^3)(1-q)^2}$$

for any $n \geqslant 1$. Hence, $R \in \mathrm{PEL}(P((2,2))) \backslash \mathrm{HEL}(P((2,2)))$.

Using the following proposition, we can make new productive edge labelings from known productive edge labelings.

Proposition 4.2. *For $R \in \mathrm{PEL}(P) \cap \mathrm{RX}(P) \cap \mathrm{RN}(P)$ and $R' \in \mathrm{CEL}(P)$, if $R + R' \in \mathrm{RN}(P)$, we have*

$$U_n(P, R + R'; q) = q^{|\varphi_0^{R+R'}|} \prod_{x \in P} \frac{1 - q^{n + c_R(x) - m_{R'}}}{1 - q^{h_R(x)}}$$

for any $n \geqslant \max\{\varphi_0^{R+R'}(x) : x \in P\}$. In particular, $R + R' \in \mathrm{PEL}(P)$.

By Lemma 2.3(ii), Proposition 2.4 and (5), this proposition is easily obtained and the proof is therefore omitted.

Note that the proposition obtained by replacing $\mathrm{PEL}(P)$ with $\mathrm{HEL}(P)$ in Proposition 4.2 is also valid.

For example, for $\lambda = (3,2)$, let R_λ be an edge labeling defined in Example 4.1(i) and let R_1, R_2 be edge labelings as given in Example 2.1. We can see that $R_\lambda + R_1 = R_2$ and we can easily check that $R_\lambda \in \mathrm{RX}(P(\lambda)) \cap \mathrm{RN}(P(\lambda))$, $R_1 \in \mathrm{CEL}(P(\lambda))$ and $R_2 \in \mathrm{RN}(P(\lambda))$. Also, by Example 4.1, $R_\lambda \in \mathrm{HEL}(P(\lambda))$. Therefore, by Proposition 4.2, $R_2 \in \mathrm{HEL}(P(\lambda))$.

For trees, we have the following.

Theorem 4.3. *For a tree T, we have*

$$\mathrm{HEL}(T) \cap \mathrm{RX}(T) \cap \mathrm{RN}(T) = \mathrm{PEL}(T) \cap \mathrm{RX}(T) \cap \mathrm{RN}(T).$$

For the proof of this theorem we need two lemmas.

Lemma 4.4. *For* $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$, *we have*

$$\sum_{\varphi \in \mathscr{A}(T,R)} q^{|\varphi|} = \frac{q^{|\varphi_0^R|}}{\prod_{x \in T}(1 - q^{h(x)})}. \tag{8}$$

Proof. We can easily prove that (8) holds when $R \in \mathrm{NEL}(T)$. By Lemma 2.2, there exists $R' \in \mathrm{CEL}(T)$ such that $R + R' \in \mathrm{NEL}(T)$. Hence, by Lemma 2.3(ii) and Proposition 2.4, (8) holds. □

Lemma 4.5. *Suppose* $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r \in \mathbb{P}$ *satisfy*

$$[a_1][a_2]\cdots[a_r] = [b_1][b_2]\cdots[b_r].$$

Then

$$\{a_1, a_2, \ldots, a_r\} = \{b_1, b_2, \ldots, b_r\} \text{ as multisets.}$$

We can prove the result by induction on r and by considering the lowest degree non-constant term. Therefore the proof is omitted.

Proof of Theorem 4.3. For $R \in \mathrm{PEL}(T)$, we obtain

$$\sum_{\varphi \in \mathscr{A}(T,R)} q^{|\varphi|} = \lim_{n \to \infty} q^{|\varphi_0^R|} \prod_{x \in T} \frac{1 - q^{n + c_R(x)}}{1 - q^{h_R(x)}} = \frac{q^{|\varphi_0^R|}}{\prod_{x \in T}(1 - q^{h_R(x)})}.$$

Hence, by Lemmas 4.4 and 4.5, we have $\{h_R(x) : x \in T\} = \{h(x) : x \in T\}$ as multisets. Thus, Theorem 4.3 holds. □

For a tree T, by Theorem 1.1, $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$ satisfying (T) is a hook edge labeling of T. Let T be a tree with the maximum element x_0 and $C^-(x_0) = \{y_1, y_2, \ldots, y_r\}$. For $k \in \{1, 2, \ldots, r\}$, let R_k be an edge labeling of T_{y_k} and let R be an edge labeling of T satisfying that $R_{y_k} = R_k$ for $k \in \{1, 2, \ldots, r\}$ and $R(y_k, x_0) - z_{R_k}(y_k) + |T_{y_k}| = R(y_{k+1}, x_0) - z_{R_{k+1}}(y_{k+1})$ for $k \in \{1, 2, \ldots, r-1\}$. Then, if every R_k satisfies condition (T), we can easily see that R also satisfies condition (T). Therefore, for a tree T, by proceeding inductively on $|T|$, we can make an edge labeling of T satisfy condition (T). Hence, by Lemmas 2.2, 3.2 and Theorem 1.1, $\mathrm{HEL}(T) \neq \emptyset$ for an arbitrary tree T. We conjecture the following.

Conjecture 4.6. *For* $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$, *the following are equivalent*:

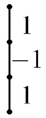
(i) R *satisfies condition* (T).
(ii) $R \in \mathrm{HEL}(T)$.

In Theorem 1.1, we proved that if (i) holds, then (ii) is valid. We have the following result about the converse direction.

Theorem 4.7. *Let* $R \in \mathrm{RX}(T) \cap \mathrm{RN}(T)$. *If* $R_x \in \mathrm{HEL}(T_x)$ *for any* $x \in T \backslash \mathrm{Min}(T)$, R *satisfies condition* (T).

Note that for $R \in \mathrm{HEL}(T) \cap \mathrm{RX}(T) \cap \mathrm{RN}(T)$ and $x \in T \backslash \mathrm{Min}(T)$, R_x is not always a hook edge labeling of T_x. For example, let

$T =$ 

, $R =$ 

.

We can easily see that R satisfies condition (T) and $R \in \mathrm{HEL}(T) \cap \mathrm{RX}(T) \cap \mathrm{RN}(T) \backslash \mathrm{NEL}(T)$. But R_c is not a hook edge labeling of T_c.

Proof of Theorem 4.7. First, we prove the theorem in the case $R \in \mathrm{NEL}(T)$ by induction on $|T|$. In the case $|T| = 2$, we can easily see that R satisfies condition (T). We suppose that this theorem holds when $|T| \leqslant j - 1$ and we show the theorem in the case $|T| = j$ $(j \geqslant 3)$. For any $x \in T \backslash (\{x_0\} \cup \mathrm{Min}(T))$, by our induction hypothesis, $R_x \in \mathrm{NEL}(T_x)$ and R_x satisfies condition (T). It follows from (7) that for any $n \in \mathbb{Z}$,

$$U_n(T_x, R_x; q) = \frac{q^{|\varphi_0^{R_x}|}[|T_x|]!}{\prod_{y \in T_x} [h(y)]} \begin{bmatrix} n + z_R(x) \\ |T_x| \end{bmatrix}.$$

Let $C^-(x) = \{y_1, y_2, \ldots, y_r\}$ satisfy

$$R(y_k, x_0) - z_R(y_k) + |T_{y_k}| \leqslant R(y_{k+1}, x_0) - z_R(y_{k+1}) + |T_{y_{k+1}}| \tag{9}$$

for all $k \in \{1, 2, \ldots, r-1\}$. We can obtain that for any $n \in \mathbb{Z}$,

$$\begin{aligned} U_n(T, R; q) = & \frac{q^{|\varphi_0^R|}}{\prod_{x \in T \backslash \{x_0\}} [h(x)]} \sum_{a=0}^{n - R(y_r, x_0) + z_R(y_r) - |T_{y_r}|} q^{|T|a} \\ & \times \prod_{k=1}^{r} \begin{bmatrix} n - a - R(y_k, x_0) + z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]!. \end{aligned} \tag{10}$$

Since $R \in \mathrm{HEL}(T)$, there exists a map c_R from T to $\mathbb{Z}$ such that

$$U_n(T, R; q) = q^{|\varphi_0^R|} \prod_{x \in T} \frac{1 - q^{n + c_R(x)}}{1 - q^{h(x)}} \tag{11}$$

for any $n \geqslant \max\{\varphi_0^R(x) : x \in T\}$. Let $s = R(y_r, x_0) - z_R(y_r) + |T_{y_r}|$. By (9) and (10), we can see that $U_s(T, R; q) \neq 0$ and we have $s \geqslant \max\{\varphi_0^R(x) : x \in T\}$. Hence, by (10) and (11), we can obtain

$$\prod_{x \in T} [s + c_R(x)] = [|T|] \prod_{k=1}^{r} \begin{bmatrix} s - R(y_k, x_0) + z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]!.$$

Therefore, using Lemma 4.5 and the equality $[|T|+1] = [|T|] + q^{|T|}$, we have

$$\prod_{x \in T} [s + c_R(x) + 1] = ([|T|] + q^{|T|}) \prod_{k=1}^{r} \begin{bmatrix} s + 1 - R(y_k, x_0) + z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]!. \tag{12}$$

On the other hand, by (10) and (11), we can obtain

$$\prod_{x \in T} [s + c_R(x) + 1] = [|T|] \prod_{k=1}^{r} \begin{bmatrix} s + 1 - R(y_k, x_0) + z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]! + [|T|] q^{|T|} \prod_{k=1}^{r} \begin{bmatrix} s - R(y_k, x_0) + z_R(y_k) \\ |T_{y_k}| \end{bmatrix} [|T_{y_k}|]!. \tag{13}$$

Comparing (12) and (13) we have

$$[1] \prod_{k=1}^{r} [s + 1 - R(y_k, x_0) + z_R(y_k)] = [|T|] \prod_{k=1}^{r} [s + 1 - R(y_k, x_0) + z_R(y_k) - |T_{y_k}|]. \tag{14}$$

Let $a_k := s + 1 - R(y_k, x_0) + z_R(y_k) - |T_{y_k}|$ for $k \in \{1, 2, \ldots, r\}$. By (9), we have

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r = 1. \tag{15}$$

By (14) and Lemma 4.5, we have

$$\{a_1 + |T_{y_1}|, a_2 + |T_{y_2}|, \ldots, a_r + |T_{y_r}|\} = \{|T|, a_1, a_2, \ldots, a_{r-1}\}$$

as multisets. Hence, by (15), for any $k \in \{1, 2, \ldots, r-1\}$, we can obtain that $a_k = a_{k+1} + |T_{y_{k+1}}|$ and we have (2). Therefore, we have proved the theorem in the case $R \in \mathrm{NEL}(T)$. Next we prove the theorem in general. By Lemma 2.2, there exists $R' \in \mathrm{CEL}(T) \cap \mathrm{NEL}(T)$ such that $R + R' \in \mathrm{NEL}(T)$. We can easily see that $R'_x \in \mathrm{CEL}(T_x) \cap \mathrm{NEL}(T_x)$ and $(R + R')_x \in \mathrm{NEL}(T_x)$ for any $x \in T \backslash \mathrm{Min}(T)$. Hence, by Proposition 4.2, we can obtain that $(R + R')_x \in \mathrm{HEL}(T_x) \cap \mathrm{NEL}(T_x)$ for any $x \in T \backslash \mathrm{Min}(T)$. It follows from our above result that $R + R'$ satisfies condition (T). Thus, by Lemma 3.2, we can see that R also satisfies condition (T). This completes the proof of the theorem. □

Note that by the proof of Theorem 4.7, if the following can be proved, then our conjecture is true. If $R \in \mathrm{NEL}(T) \cap \mathrm{HEL}(T)$, then $R_x \in \mathrm{HEL}(T_x)$ for any $x \in T \backslash \mathrm{Min}(T)$.

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